MECHANICS APPLIED TO VIBRATIONS AND BALANCING

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PREFACE

The purpose of this book is to present the general theory of vibrations in its various aspects, as a subject having important relations to several spheres of technology. Apart from their intrinsic value, the worked examples and applications of the theory embodied in each chapter are an essential part of the treatment because they give meaning to the abstractions of formal dynamics. Opportunities have been taken to suggest matters for further research arising from some of the conclusions reached.

Although written primarily for engineers, and therefore in terms of appropriate units, it is hoped that this book may also be of service to students of physics and others who wish to acquire a working knowledge of the principal equations of that science. The attempt has therefore been made to develop each chapter from the simplest beginnings and to expound the nature of fundamental principles as clearly as possible. Acquaintance with the elements of the calculus and analytical geometry up to the standard of the associate membership examinations of the senior engineering institutions has been assumed. Where any additional mathematical apparatus is required it is given in explanatory form in the text. Since the attainment of these aims in a book of moderate size has rendered conciseness necessary, the vibration of complex systems has been described by summarizing preceding results with reference to aircraft and airscrews in flight, and to other typical cases.

As the endeavour has been to give a description of the subject which an engineer would regard as fairly complete, the disturbing agencies which are commonly met with in the drawing office and in the field have been discussed. The unbalanced dynamical forces and couples associated with engines and locomotives are perhaps the most important sources of disturbance of mechanical origin. In the two chapters devoted to these questions Lagrange's equations have been enunciated and employed, partly on account of their fitness and utility, and partly as an introduction to the idea of generalized co-ordinates involved in the more advanced theory of Chapter III.

One practical advantage of adopting Lagrange's method is worth pointing out, namely, the ease with which the student can consider in succession the main parts of a specified engine, and so exhibit the salient factors in the design not only of the engine as a

whole, but also of such components as the governor and valve gear. A combination of this information and that of Chapter IV on the transmission of stress through elastic materials offers a ready means of tracing the relative motion between the engine and its governor which would be caused by a known change in the load. Alternatively, it is in this way practicable to calculate the stresses on the pin-joints of the valve mechanism of a locomotive under varying conditions of speed or draw-bar pull.

Attention has been drawn to the physical phenomena through which the implied wave-energy may be dissipated and dispersed, for many of the vibrational problems of the moment stand most of all in need of investigations into damping. With the help of results thus obtained an insight may be gained into the consequences of a periodic load on composite materials, of which reinforced concrete is a notable example, though masonry and wood are also of some interest in this sense, for reasons stated in the text. The analysis is, moreover, brought into direct relation with certain methods of testing materials, and with the analogous method used to detect the presence of rock for foundations. It is a simple matter to apply the same principles in prospecting for deposits of oil, and measuring the depth of the ocean.

The reader who follows Chapter III will realize how an extremely powerful means of investigating the vibration of structural systems having a finite number of degrees of freedom may be derived from Lagrange's formulæ. Lord Rayleigh's theorem on the stationary property of the normal modes of vibration is, in certain circumstances, of still greater utility, in that it can be adapted also to the case of complicated continuous systems. From the view presented here Chapters V and VI are special extensions of previous considerations. Into Chapter V has been incorporated an outline of a method which the author has found of great value in the design of suspension bridges of large span, but the argument holds good for any system of the prescribed type, owing to the absence of 'privileged' co-ordinates in the equations that matter.

The exemplification of the theory does not, however, claim to be exhaustive. The ramifications of the subject are numerous, and the examples here given are naturally taken from those branches which have occurred in the author's own engineering practice; this accounts, in particular, for the discussion on structures designed to withstand disturbances in the earth, in connection with which subject he has enjoyed the questionable advantage of having witnessed several earthquakes, including a few characterised by the numbers 7 and 8 on the Rossi-Forel scale of intensity.

Articles 5–10, 26, 27 and 106 have been based on methods due to Professor C. E. Inglis, and Professor E. T. Whittaker's compre-

hensive *Treatise on the Analytical Dynamics of Particles and Rigid Bodies* has proved to be an invaluable aid in the preparation of sections dealing with the mathematical side of the subject.

Dr. H. Jeffreys, F.R.S., has placed the author in his debt by reading what has been said about his theory of earthquakes in Article 78. This also applies to Mr. E. Latham, M.Inst.C.E., and the Royal Astronomical Society for permission to reproduce Figures 113 and 187, respectively.

In a work such as this, it is probably useless to hope that no inaccuracies have remained undetected, but the number of errors and obscurities has been diminished by the vigilance and criticism of Mr. G. F. Herrenden Harker, M.A., who has read the complete book in proof. To him and to the Publishers, who have given every assistance with unfailing patience, the author's grateful thanks are offered.

D. L. T.

WESTMINSTER.
September, 1939.

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CHAPTER I

BALANCING OF ENGINES

1. The theoretical treatment of this subject involves investigations into the unbalanced forces and couples produced by the moving parts of engines in particular, and of machinery in general. These forces and couples will be referred to as inertia effects, to distinguish them from the other disturbing agencies which may act on specified mechanisms. In practical applications of the theory the aim is that of designing, given mechanical systems, so as to result in the unbalanced inertia effects being either partially or completely neutralized, since the procedure leads to minimum values of the effective stresses on the structures concerned.

Very slight observation is enough to convince us that these sources of disturbance tend to initiate vibrations in the elastic materials used in the construction of engines on the one hand, and the foundations on the other. The practical significance of this disturbed motion is to be found in the stresses associated with the 'waves' which are in consequence propagated through the structure in a manner that will be explained in Chapter IV. Owing to the earth itself being more or less an elastic sphere, the 'stress-waves' may be transmitted to buildings situated some distance away, when in certain circumstances the structures are liable to objectionable oscillatory motion. Moreover, with systems in which the foundations consist of slender members, as is exemplified by the fuselage of aircraft, the unbalanced effects of an engine may induce appreciable stresses in the supporting structure.

The matter has a bearing also on the trouble which is sometimes experienced with sets of alternators working in parallel, where some of the machines are driven by reciprocating engines, and the others by turbines. Heavy interchange currents may pass between machines operated in this way, due to the non-uniform speed of the engines compared with that of the turbines.

These introductory remarks suggest the type of problem to be considered under this heading in the present work, in the course of which the analysis will be generalized so as to apply to the unbalanced effects of the principal parts of engines, such as the valve-gear, governor, piston and its attachments, connecting rod, crankshaft and flywheel, foundations, and so on.

2. Newton's Laws of Motion. Since the mechanism of reciprocating engines includes masses that undergo periodic variations in velocity, attention may at the outset be drawn to the fundamental laws of motion.

According to the first of these laws: Every body continues in its state of uniform motion, including rest as a particular case, except so far as it is compelled to change that state by external forces.

The magnitude of the force is defined by the second law: The rate of change of momentum is proportional to the effective force, in a direction parallel to that of the force. This statement is implied in the familiar relation

$$\text{force} = \text{mass} \times \text{acceleration}.$$

A force is accompanied by one of equal magnitude and opposite sense, in accordance with the third law: Corresponding to every action there is always an equal and opposite reaction; or between two bodies acting on one another, the action of the one is equal in magnitude and opposite in direction to the action of the other.

Internal forces such as, for example, those caused by the pressure of the working fluid used in engines, are counteracted by stresses on the structural framework.

We may, from the dynamical point of view, therefore regard an engine as equivalent to a system of inertia forces consisting of two main types, due to the motion of:

(i) Rotating masses, as are exemplified by the crankshaft and flywheel;

(ii) Reciprocating masses, as are exemplified by the piston and crosshead.

3. Rotating Masses. (*a*) *In One Plane.* In Fig. 1 let AB represent a shaft revolving with angular velocity ω, and to

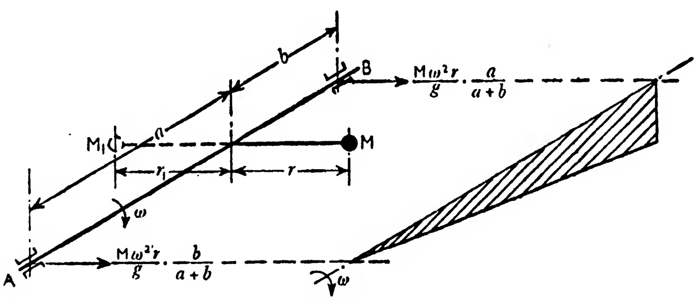


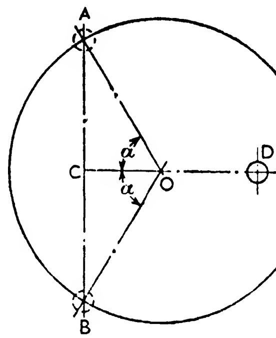
Fig. 1.

which is fixed, at the end of a crank of radius r, a mass of weight M. The inertia effect is equivalent to a centrifugal force acting in the plane of rotation, whence on this account reactions amounting to

$\frac{Mr\omega^2}{g}\cdot\frac{b}{a+b}$ and $\frac{Mr\omega^2}{g}\cdot\frac{a}{a+b}$ are imposed on the bearings at A and B, respectively. The alternating loads thus induced may be examined with the aid of the bending-moment diagram for the shaft, indicated by the hatched figure, which revolves about the axis with angular velocity ω.

It is easily seen that balance may be secured by adding to the system a mass of weight M_1 at a radius r_1 in the same plane and diametrically opposite M, and at the same time ensuring that the condition $M_1r_1 = Mr$ is satisfied. If, for instance, $M_1 = M$, then $r_1 = r$, and the necessary balance-weight consists of the mass M_1 placed at radius r_1 opposite the original mass, as indicated by the dotted circle in the figure.

A single mass may, alternatively, be balanced by any number of counter-weights rotating in the same plane, provided the centre of gravity of the system thus formed lies on the axis of rotation.



Ex. 1. Suppose Fig. 2 to represent a thin disc of 3 ft. diameter, rotating with a peripheral speed of 2,000 ft. per min., and subject to the unbalanced effect of 0·5 lb. acting at a distance of 1 ft. from the axis of rotation, at the point D. Find the angular position for two balance-weights, each weighing 0·4 lb., which will secure complete balance.

Since $\omega = 22{\cdot}22$ rad. per sec., the unbalanced force is $\frac{0{\cdot}5(22{\cdot}22)^2}{g}$ lb., that is 7·70 lb. Hence if A and B in the figure are the positions of the specified counter-weights, with O as the centre of the disc, then the line DO produced must bisect the angle AOB, to satisfy the condition that the centres of gravity of the balance-weights and the unbalanced mass at D shall be diametrically opposite. Therefore if DO produced bisect the line AB in C, on taking moments about O, we have

$$0{\cdot}5 \times OD = 2 \times 0{\cdot}4 \times OC,$$

whence $$OC = 0{\cdot}625 \text{ ft.},$$

and $$AOC = \cos^{-1} 0{\cdot}4166, \text{ or } AOC = 65\tfrac{1}{3} \text{ deg.},$$

approximately. This arrangement of counter-weights, having a centre of gravity lying in the plane of the disc, would accordingly result in complete balance.

When any number of unbalanced masses rotate with the same angular velocity in one plane and about a common axis,

the total effect is readily found by the aid of the polygon of forces.

The point may be illustrated with reference to the three masses M_1, M_2, M_3 situated respectively at the radii r_1, r_2, r_3 shown in Fig. 3, for which the inertia forces taken in turn are $\frac{M_1 r_1 \omega^2}{g}$, $\frac{M_2 r_2 \omega^2}{g}$, $\frac{M_3 r_3 \omega^2}{g}$. If the balance-weight rotates with the same angular velocity and in the same plane as that of the masses, then ω and g are common factors in the calculations, so that unity may for simplicity be assigned to these quantities in the process of drawing the polygon of mass-arm products.

To solve the problem thus presented, draw to a convenient scale Oa in Fig. 3 parallel to r_1 and equal in length to the product $M_1 r_1$;

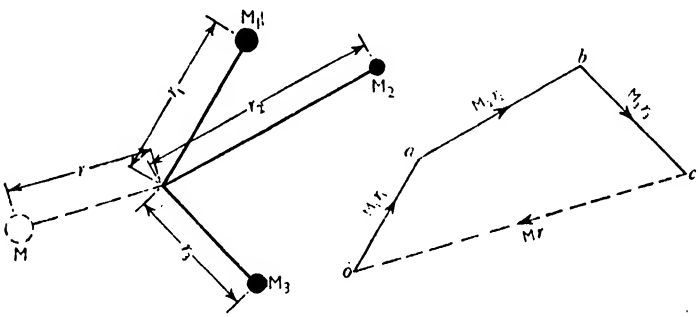


FIG. 3.

then draw through a the line ab parallel to r_2 and equal in length to $M_2 r_2$; finally through b draw the line bc parallel to r_3 and equal in length to $M_3 r_3$. The closing line cO therefore defines in magnitude and direction the mass-arm product of the required balance-weight.

(*b*) *In Different Planes.* For a number of reasons it is sometimes impracticable to counteract the unbalanced effect of prescribed masses by means of a weight rotating in the same plane.

Consider the system indicated in Fig. 4, where the mass of weight M_1 placed at radius r_1 in relation to the shaft AB is to be balanced by the weight M_2 placed at radius r_2 with reference to the shaft. It will be noticed that the planes of rotation of M_1 and its balance-weight M_2 are separated by the distance d.

When the shaft is stationary, the system is said to possess *static* balance if the condition $M_1 r_1 = M_2 r_2$ is satisfied. Once the shaft begins to revolve, however, it is subject to an unbalanced couple $\mathfrak{T}$, amounting to $\frac{M_1 r_1 \omega^2 d}{g}$ when $M_1 r_1 = M_2 r_2$, in consequence of which the system is said to lack *dynamic* balance. It is evident that com-

plete balance is unattainable in systems of this type, for the couple cannot be eliminated.

It follows that dynamic balance is in general of much greater importance than static balance, since in the present case want of the former results in the shaft 'wobbling' between the positions indicated by the dotted lines in the figure, due to the action of $\mathfrak{C}$. Dynamic balance accordingly implies static balance, but the converse statement is not true.

An interesting example of the unbalanced couple denoted above by $\mathfrak{C}$ is to be found in the case of an airscrew 'out of track', that is when the blades are unsymmetrically mounted with respect to the

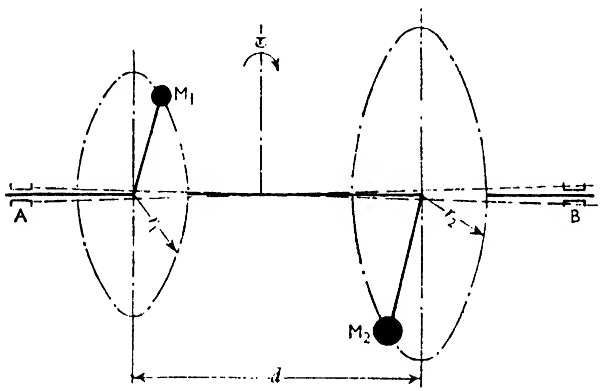


FIG. 4.

shaft. The magnitude of the disturbance may be indicated by reference to the fact that, for a speed of 1,300 r.p.m. the couple $\mathfrak{C}$ is of the order of 5,000 lb.-in. units for a wooden propeller of 12 ft. diameter and 80 lb. weight, having in this connection a manufacturing tolerance of 0·13 in. It is thus seen, when account is taken of the factors of safety employed in the construction of aircraft, that slight departures from the specified tolerances may lead to appreciable increases in stress.

Ex. 2. Determine the unbalanced couple produced by inaccurate mounting of a uniform disc on the shaft shown in Fig. 5, when the system is rotating at 3,000 r.p.m.; the disc measures 10 in. in diameter, weighs 100 lb., and is incorrectly mounted to such an extent that $\beta = 5$ minutes of arc.

If M be the weight of the whole disc, and R its radius, the weight m of the element contained between the limits $d\theta$ and dr shown in the figure is given by the relation

$$m = \frac{Mr}{\pi R^2}.dr.d\theta.$$

This produces in the plane of the disc a centrifugal force C defined by

$$C = \frac{mr\omega^2}{g},$$

where ω is the angular velocity of the shaft about its axis.

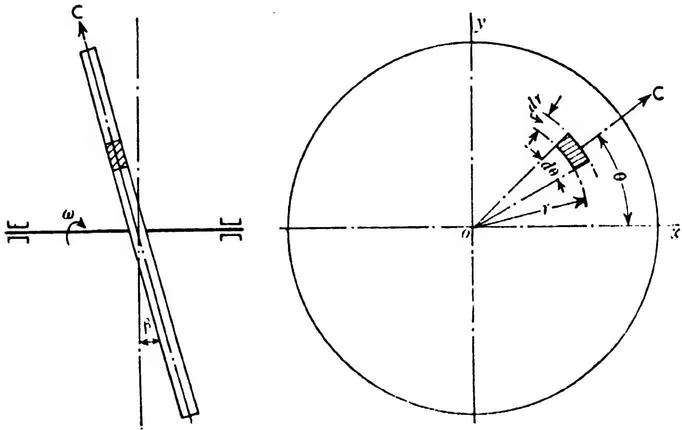


FIG. 5.

Since the y-component of this force is $C \sin \theta$, the specified element is acted on by an inertia couple amounting to $(C \sin \theta)r. \sin \theta. \sin \beta$. But as β is small, we may take $\sin \beta$ to be practically equal to the angle β measured in radians, when the last expression reduces to $Cr\beta. \sin^2 \theta$, and the total couple becomes

$$\int_0^R \int_0^{2\pi} Cr\beta \sin^2 \theta,$$

or

$$\frac{M\omega^2\beta}{\pi R^2 g} \int_0^R \int_0^{2\pi} r^3 \sin^2 \theta . dr . d\theta,$$

or

$$\frac{M\omega^2\beta R^2}{4g}.$$

Therefore the magnitude of the couple is approximately 2,000 lb.-ft., for the given values $M = 100$ lb., $R = \frac{5}{12}$ ft., $\beta = 0{\cdot}0015$ rad., $\omega = 314{\cdot}16$ rad. per sec., with $g = 32{\cdot}2$ ft. per sec. per sec. The axis of this couple revolves with the disc and acts in a plane almost perpendicular to that of rotation.

We may now examine without difficulty the system shown in Fig. 6, representing the two masses M_1 and M_2 concentrated at the radii r_1 and r_2, respectively, with respect to the shaft AB rotating with angular velocity ω. Here the inertia forces amount to $\frac{M_1 r_1 \omega^2}{g}$ and $\frac{M_2 r_2 \omega^2}{g}$; the bending-moment diagram has the form indicated by the hatched figure, and it also rotates with angular

velocity ω about AB. The lack of dynamic balance, as already remarked, results in reversals of load on the shaft and its bearings.

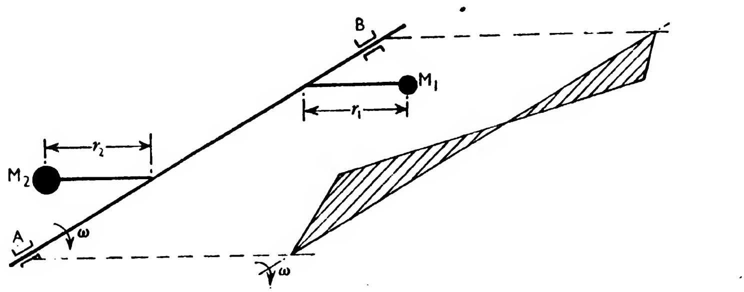


FIG. 6.

Ex. 3. To elucidate the point with a numerical example, take the three masses shown in Fig. 7, each of which weighs 300 lb. and rotates at the end of a crank having a common throw of 15 in.

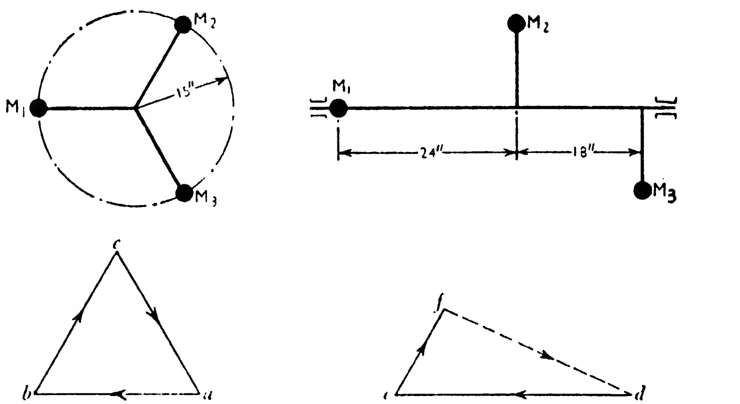


FIG. 7.

The planes of rotation are separated by the distances given in the figure, and the system revolves at 200 r.p.m., with the cranks 120 deg. apart.

Each of the prescribed masses causes a centrifugal force of

$$\frac{300 \times 15}{12 \times 32 \cdot 2}\left(\frac{200 \times 2\pi}{60}\right)^2 \text{ lb., or } 5{,}140 \text{ lb.,}$$

approximately, to act along its crank. The force polygon is constructed by first drawing the line *ab* parallel to the crank of M_1 and 5,140 units in length; from *b* draw *bc* parallel to the crank of M_2 and 5,140 units in length; finally, from *c* draw a line parallel to the crank of M_3 and 5,140 units in length. This results in a closed polygon, and the system is, therefore, statically balanced.

To evaluate the 'skewing' couple acting on the shaft when in rotation, take moments about any convenient (transverse) plane of reference. To fix ideas, let the plane be that in which M_3 rotates. Then the couple-polygon is constructed by drawing the line *de* parallel to the crank of M_1 and proportional to $(5{,}140 \times 3{\cdot}5)$ lb.-ft.; draw also *ef* parallel to the crank of M_2 and proportional to $(5{,}140 \times 1{\cdot}5)$ lb.-ft. The closing line *fd* accordingly defines the resultant couple, and it will be found to equal 15,600 lb.-ft. This quantity therefore indicates the degree of dynamic unbalance present in the system.

(*c*) *Any Number of Masses on a Shaft.* It is possible to pass in a straightforward manner to the system shown in Fig. 8, consisting

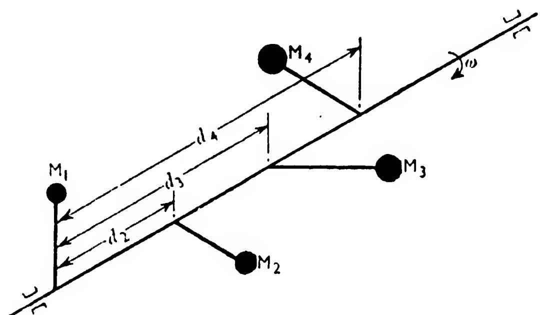


FIG. 8.

of a shaft rotating with angular velocity ω, and carrying the masses M_1, M_2, M_3, M_4, at radii r_1, r_2, r_3, r_4, respectively.

The corresponding forces C_1, C_2, C_3, C_4 are defined by the expressions

$$C_1 = \frac{M_1 r_1 \omega^2}{g}, \quad C_2 = \frac{M_2 r_2 \omega^2}{g}, \quad C_3 = \frac{M_3 r_3 \omega^2}{g}, \quad C_4 = \frac{M_4 r_4 \omega^2}{g}.$$

It has been shown that the resultant of these forces must be zero to secure static balance, hence

$$(M_1 r_1 + M_2 r_2 + M_3 r_3 + M_4 r_4)\omega^2 = 0$$

is the condition to be satisfied in this respect. Further, on taking the plane in which M_1 rotates as the origin, and writing $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$ for the couples associated in turn with the above mentioned forces, we have

$$\mathfrak{C}_2 = \frac{M_2 r_2 \omega^2 d_2}{g}, \quad \mathfrak{C}_3 = \frac{M_3 r_3 \omega^2 d_3}{g}, \quad \mathfrak{C}_4 = \frac{M_4 r_4 \omega^2 d_4}{g}.$$

Hence dynamic balance is obtained if the vectorial sum of these couples is zero, that is if the condition

$$(M_2 r_2 d_2 + M_3 r_3 d_3 + M_4 r_4 d_4)\frac{\omega^2}{g} = 0$$

is fulfilled.

In the general case of n masses arranged in this manner, we may therefore let

$$\sum_{s=1}^{n} M_s r_s = 0, \quad \ldots \ldots \ldots \quad (3.1)$$

and

$$\sum_{s=1}^{n} M_s r_s d_s = 0. \quad \ldots \ldots \ldots \quad (3.2)$$

represent, symbolically, the vectorial relations required for complete balance.

Further examples are appended, with a view to demonstrating that the most convenient way of approach to a solution depends on the nature of the problem to be solved.

Ex. 4. The three masses M_1, M_2, M_3 indicated in Fig. 9 weigh respectively 4 tons, 6 tons, 8 tons, and are placed at the ends of equal cranks which rotate in planes distant 13 ft., 9 ft., 4 ft. when reckoned from the left-hand end of the shaft shown in the figure.

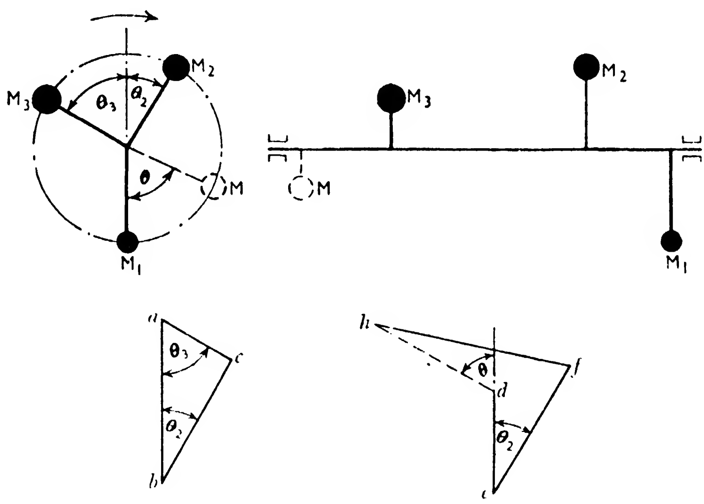


FIG. 9.

It is required to determine the values of the angles θ_2 and θ_3, as well as the magnitude and relative position of the balance-weight that must be placed at the end of an equal crank fixed at the left-hand end of the shaft, to ensure complete balance.

An examination of the given data shows that it is here most convenient to draw first the couple-polygon. If M denote the mass of the necessary balance-weight, the effect of this as yet unknown quantity on the couple polygon can be eliminated by taking moments of M_1, M_2, M_3 about the plane in which M rotates.

In terms of ton-foot units the moments of the masses M_1, M_2, M_3 about the specified plane are (4×13), (6×9), (8×4), respectively. Now for convenience suppose the crank of M_1 to be vertical, then the polygon, which must be closed in order to give the required balance, may be constructed by first drawing the line *ab* vertical and 52 units in length; the figure *abc* is completed by making *bc* 54 units in length, and *ac* 32 units in length. On measuring the angles thus formed, it will be found that $\theta_2 = 35$ deg., and $\theta_3 = 75$ deg., approximately.

These additional data enable us to draw the corresponding force-polygon *defh*, and thus find the magnitude and relative position for the balance-weight M.

We proceed in the usual manner, by drawing the lines *de*, *ef*, *fh* in turn parallel to *ab*, *bc*, *ca*, and proportional to 4 tons, 6 tons, 8 tons, respectively. The magnitude and relative position of the balance-weight M is defined by the closing line *hd*, the measurement of which yields the reqnired values of $M = 5$ tons, and $\theta = 55$ deg.

Ex. 5. In Fig. 10 M_1, M_2, M_3, M_4 represent four masses attached to the ends of equal cranks which are disposed symmetrically along a shaft in the manner indicated. It is proposed to secure inherent balance by appropriate arrangement of the masses,

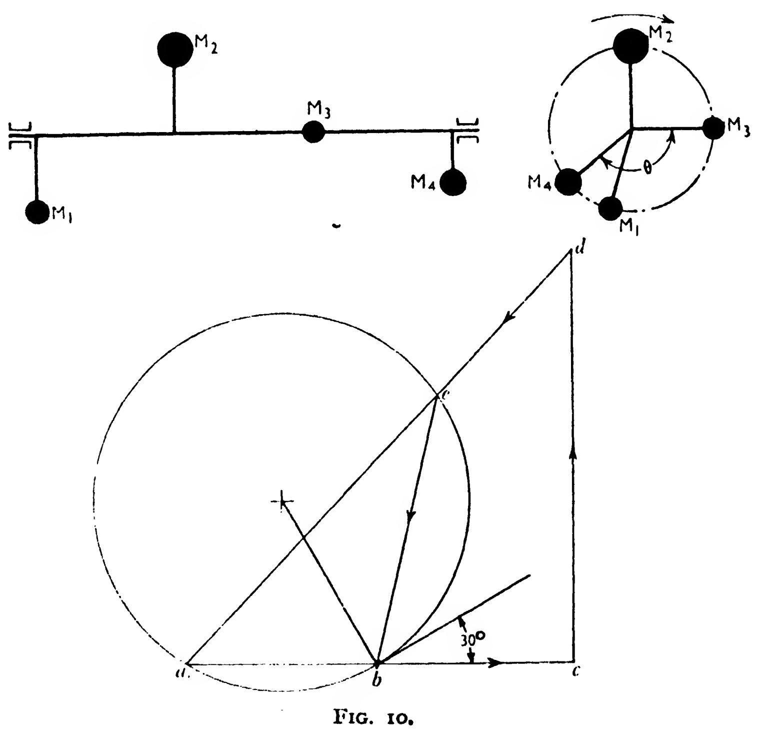


FIG. 10.

with $M_1 = 120$ lb. Find the necessary magnitudes for the other masses, and the angles between the inner and outer pairs of cranks.

Taking the crank of M_3 to be horizontal, first draw a line *ab* parallel to the direction of this crank and of length such as will subsequently represent M_3 to scale. If a circle of radius *ab* be drawn so as to pass through the points *a* and *b*, it follows from the geometrical properties of the figure that the line *ab* produced makes an angle of 30 deg. with the tangent at *b*, as shown in the figure. This is the required angle between the cranks of M_1 and M_4.

Now produce *ab* to *c*, making $bc = ab$, and draw a line *cd* at right angles to *ac*. Also find by trial a line *ad* that cuts the circle at *e* and divides *ad* so that $2ed = ae$. Complete the polygon by drawing the line *eb*.

It is readily seen that the figure *bcde* is the force-polygon, in which $bc = M_3$, $cd = M_2$, $de = M_4$ and $eb = M_1$. Measurement of these lines thus shows the necessary system of masses to be $M_1 = 120$ lb., $M_2 = 177$ lb., $M_3 = 83{\cdot}5$ lb., and $M_4 = 78{\cdot}3$ lb. Further, since the angle *cad* measures 48 deg., the relative positions of the inner and outer pairs of cranks are given by $\theta = 132$ deg. in the figure.

If in connection with the couple-polygon the plane of M_1 be taken as reference, that figure is represented by *acd*, which is closed, hence the system is dynamically balanced.

4. Disturbing Forces Acting on Engines. Even when the rotating masses on an engine are balanced, the mechanism is liable to disturbances that arise from other sources which may be referred to at this stage of the treatment.

(*a*) *Due to the Working Fluid.* On writing P for the total force exerted by the working fluid on the piston of an engine, we have on this account a disturbing force $P \tan \phi$ acting as indicated in Fig. 11. A horizontal component of this magnitude is in consequence transmitted direct to the crosshead, or to the cylinder-liner of internal combustion engines having no crossheads.

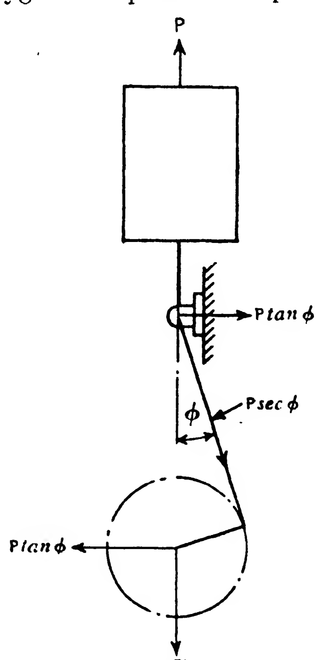


FIG. 11

(*b*) *Due to Inertia Effects.* These alone produce the system of forces denoted by S, V, H in Fig. 12, where H acts in the line of stroke, and V perpendicular to that direction. The relative importance of these forces, which will form the chief object of study in this chapter, depends on the type of engine under consideration. For example, both H and V may be of equal importance in the case of aero-engines, while H is commonly the most troublesome of the unbalanced components of force which act on the ordinary type of stationary engine.

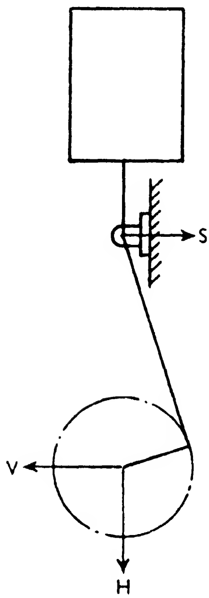


FIG. 12.

It is scarcely necessary to point out that the forces S and V are not identical, since they together account for the acceleration of the connecting rod.

5. Motion of a Connecting Rod. This part of an engine is the most difficult to examine from the present standpoint, due to the fact that one end of the rod executes reciprocating motion, while the other follows the circular path described by the crank-pin.

With a view to using the method of instantaneous centres in connection with the mechanism indicated by Fig. 13, let Ω be the instantaneous angular

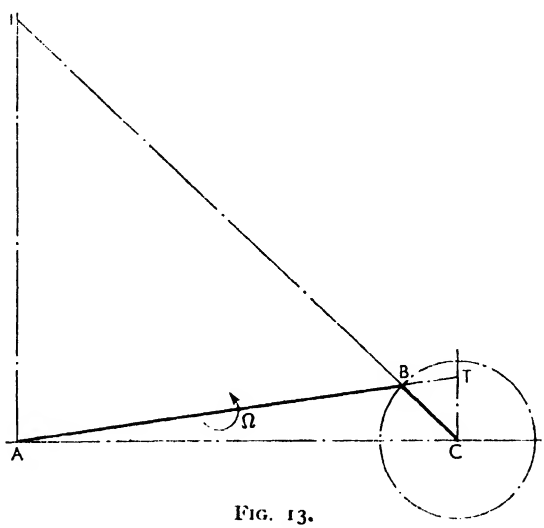


FIG. 13.

velocity of the rod AB, and ω the angular velocity of the crank CB. For analytical purposes draw through C a line at right

angles to AC, meeting AB produced in T; erect also a line through A and perpendicular to AC, to intersect the line CB produced in I.

Since the point I is the instantaneous centre of rotation for the connecting rod, we have

$$\begin{aligned}\text{velocity of } B \text{ about } I &= IB.\Omega \\ &= CB.\omega,\end{aligned}$$

whence
$$\Omega = \frac{CB}{IB}.\omega = \frac{BT}{BA}.\omega, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5.1)$$

by similar triangles. Inserting this relation in the known expression

$$\text{velocity of piston} = IA.\Omega,$$

we obtain

$$\begin{aligned}\text{velocity of piston} &= IA.\frac{BT}{BA}\omega = IA.\frac{CT}{IA}\omega \\ &= CT.\omega, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5.2)\end{aligned}$$

by similar triangles.

As accelerations, rather than velocities, are the principal quantities in dynamical problems, we proceed to determine the acceleration of the piston, by Klein's construction. In Fig. 14, let T correspond with the point found in the previous figure, and describe a circle on AB as diameter; also, with B as centre and BT as radius, draw a circle cutting the circle on AB in X and U. Finally join X and U with a line that cuts AB in Y, and the line AC in Z.

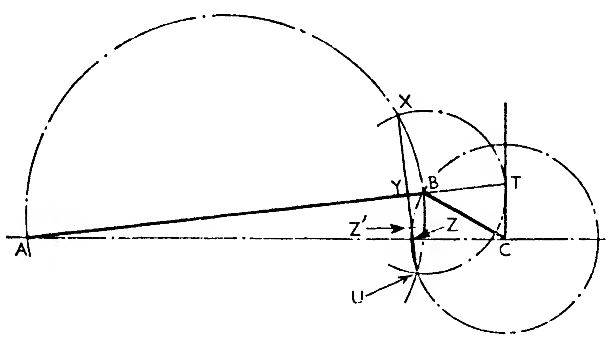


FIG. 14.

In this manner it is readily deduced that

acceleration of A = acceleration of A relative to B + acceleration of B,

and acceleration of A relative to $B = AB.\Omega^2$ along $AB + AB.\dot{\Omega}$ transverse to AB,

where $\dot{\Omega} = \frac{d\Omega}{dt}$. Hence, by equation (5.1),

$$AB.\Omega^2 = AB\left(\frac{BT}{AB}\right)^2.\omega^2$$

$$= \frac{(BT.\omega)^2}{AB},$$

which is given by Klein's method, in that it leads to $\frac{BT^2}{AB}$. The similar triangles in Fig. 14 enable us also to write the relations

$$BY = \frac{BX^2}{AB}$$

$$= \frac{BT^2}{AB}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

whence the acceleration of A relative to $B = YB.\omega^2$.

Now if, for the moment, we suppose that

transverse acceleration of A relative to $B = Z'Y.\omega^2$,

where Z' refers to a point on the line XZ, then the true acceleration of A is made up of the components $Z'Y.\omega^2$, $YB.\omega^2$ and $BC.\omega^2$; hence the true acceleration of A is denoted by $Z'C.\omega^2$. But this acceleration must be along the line of stroke AC, therefore the point Z' is identical with the known point Z. We may consequently write

$$\text{acceleration of } A \text{ (piston)} = ZC.\omega^2. \quad . \quad . \quad . \quad (5.4)$$

A simple construction enables us now to find the acceleration of any given point P on the connecting rod AB indicated in Fig. 15,

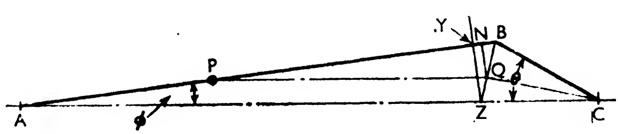


FIG. 15.

to which the points Y and Z have been transferred from the previous figure. The construction consists in joining the points Z and B, and drawing through P a line parallel to AC, to cut ZB in Q; continue by drawing through Q a line at right angles to AB, intersecting it at N. Also join QC.

In this figure the true acceleration of the point P consists of a

component along AB, and one at right angles to that direction, the values of which are given by the relations

$$\text{acceleration of } P \text{ relative to } B \text{ and along } AB = \frac{PB}{AB}.BY.\omega^2$$
$$= \frac{BN}{BY}.BY.\omega^2 = BN.\omega^2,$$

and transverse acceleration of P relative to $B = \dfrac{PB}{AB}.ZY.\omega^2$

$$= \frac{QN}{YZ}.YZ.\omega^2 = QN.\omega^2.$$

In view of these results the true acceleration of P may be expressed as the resultant of the components $QN.\omega^2$, $NB.\omega^2$ and $BC.\omega^2$. Since this resultant is $QC.\omega^2$ in the figure, it follows that

$$\text{acceleration of } P = QC.\omega^2. \quad . \quad . \quad . \quad . \quad (5\cdot5)$$

To obtain an analytical expression for the length ZC in equation (5·4), write ϕ and θ in turn for the angles BAC and BCA in Fig. 15; also let l be the length of the connecting-rod, and r the throw of the crank.

With this notation we have, from the relations (5·3) and angle $AYZ = 90$ deg.,

$$\begin{aligned} AZ &= AY \sec\phi \\ &= (l - BY)\sec\phi \\ &= \left(l - \frac{BT^2}{AB}\right)\sec\phi \\ &= \frac{l^2 - BT^2}{l\cos\phi}. \end{aligned}$$

If for conciseness the ratio $\dfrac{\text{throw of crank } (r)}{\text{length of connecting rod } (l)}$ be denoted by γ, our additional data may be expressed in the form

$$\frac{BT}{\gamma l} = \frac{\cos\theta}{\cos\phi}, \text{ and } \gamma\sin\theta = \sin\phi,$$

so that
$$\begin{aligned} AZ &= \frac{1}{l\cos\phi}\left(l^2 - \frac{\gamma^2 l^2\cos^2\theta}{\cos^2\phi}\right) \\ &= \frac{l}{\cos^3\phi}\{1 - \sin^2\phi - \gamma^2(1 - \sin^2\theta)\} \\ &= \frac{l(1 - \gamma^2)}{\cos^3\phi}. \end{aligned}$$

Inserting this value for AZ in the relation

$$ZC = AC - AZ,$$

now leads to

$$\begin{aligned} ZC &= l\cos\phi + \gamma l\cos\theta - \frac{l(1 - \gamma^2)}{\cos^3\phi} \\ &= l\{(1 - \gamma^2\sin^2\theta)^{\frac{1}{2}} + \gamma\cos\theta - (1 - \gamma^2)(1 - \gamma^2\sin^2\theta)^{-\frac{3}{2}}\}, \end{aligned}$$

since $\cos\phi = (1 - \gamma^2 \sin^2\theta)^{\frac{1}{2}}$. This relation for ZC may be written in cosine-terms alone, with the aid of the binomial theorem and known relations of the type

$$1 - 2\sin^2\theta = \cos 2\theta,$$
$$\tfrac{3}{2}\sin^2\theta - 2\sin^4\theta = \tfrac{1}{4}(\cos 2\theta - \cos 4\theta),$$
$$\tfrac{15}{8}\sin^4\theta - \tfrac{9}{4}\sin^6\theta = \tfrac{3}{128}(5\cos 2\theta - 8\cos 4\theta + 3\cos 6\theta),$$

for by this means we obtain the terms in the form

$$\sin^2\theta = -\tfrac{1}{2}(\cos 2\theta - 1),$$
$$\sin^4\theta = \tfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3),$$
$$\sin^6\theta = -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10),$$

to any particular order of approximation. Making these substitutions in the last expression for ZC yields, after effecting the expansions involved,

$$ZC = l\{\gamma\cos\theta + (\gamma^2 + \tfrac{1}{4}\gamma^4 + \tfrac{15}{128}\gamma^6 + \ldots)\cos 2\theta$$
$$- (\tfrac{1}{4}\gamma^4 + \tfrac{3}{16}\gamma^6 + \ldots)\cos 4\theta + (\tfrac{9}{128}\gamma^6 + \ldots)\cos 6\theta + \ldots\} \quad . \quad (5.6)$$

This equation determines, to the implied degree of approximation, the acceleration of the piston or point A in Fig. 15, since this quantity has been shown to equal $ZC.\omega^2$.

If, by way of illustration, the terms beyond γ^2 be neglected as of the second order, then

$$ZC.\omega^2 = r\omega^2(\cos\theta + \gamma\cos 2\theta),$$

which is an approximation that suffices for many practical purposes. We may easily verify this particular result, for by equation (5·2) we have

$$\text{velocity of piston} = CT.\omega,$$

whence, with ω constant,

$$\text{acceleration of piston} = \frac{d}{dt}(CT.\omega)$$
$$= \omega\frac{d}{dt}(r\sin\theta + r\cos\theta\tan\phi)$$
$$= \omega r\frac{d}{dt}(\sin\theta + \cos\theta\sin\phi),$$

approximately, as ϕ is comparatively small in actual engines. Thus

$$\text{acceleration of piston} = \omega r\frac{d}{dt}\left(\sin\theta + \frac{r\cos\theta\sin\theta}{l}\right)$$
$$= \omega^2 r(\cos\theta + \gamma\cos 2\theta),$$

since $\dfrac{d\theta}{dt} = \omega$.

6. Dynamical Equivalent of a Connecting Rod. Regard being had to the complicated motion of connecting-rods, it is manifest

that the analytical treatment of the matter will be greatly simplified if we reduce this part of the mechanism to a simple system of dynamically equivalent masses. In effecting the modification we must first decide on the number and arrangement of the equivalent masses to be included in the analysis, because these considerations depend partly on the approximation aimed at, and partly on the analytical way of approach to be followed in proceeding to the solution of a given problem.

Consider first a dynamically equivalent connecting-rod consisting of two masses situated on the longitudinal axis of a given rod, represented in Fig. 16. To specify the rod, write M_c for its weight,

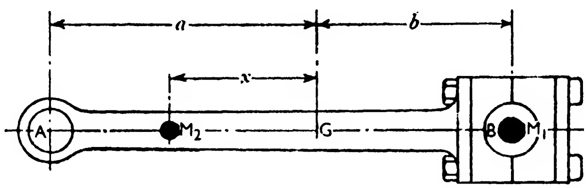


FIG. 16.

and k for its radius of gyration about the axis through its centre of gravity G and perpendicular to the plane of motion. Hence if G divide the rod into the lengths a and b shown in the figure, then $M_c k^2$ is the moment of inertia for the rod about the axis through G.

To define the relative positions of the two masses in question, suppose the actual rod to be approximately equivalent in this connection to the system formed by a mass of weight M_1 concentrated at the crank-pin end of the rod (B), and a mass of weight M_2 placed at a distance x measured from G in the direction of the gudgeon-pin (A).

In order that the specified masses shall be dynamically equivalent to the actual rod, the conditions

$$M_1 + M_2 = M_c, \quad M_2 x = M_1 b$$

must be satisfied, to secure equality between the mass-systems on the one hand, and coincidence of the centres of gravity on the other. The condition

$$M_2 x^2 + M_1 b^2 = (M_1 + M_2)k^2$$

must also be fulfilled, to ensure that the moments of inertia for the actual and equivalent systems are equal. It follows, on combining the second and third of these conditions, that

$$M_2 . \frac{M_1^2 b^2}{M_2^2} + M_1 b^2 = (M_1 + M_2)k^2,$$

or

$$\frac{M_1}{M_2}(M_1 + M_2)b^2 = (M_1 + M_2)k^2,$$

whence we have the relation $\dfrac{M_1}{M_2} = \dfrac{k^2}{b^2}$ for use in connection with the remaining condition. In this manner we obtain

$$\left.\begin{aligned} M_1 &= M_c\frac{k^2}{b^2 + k^2}, \\ M_2 &= M_c\frac{b^2}{b^2 + k^2}, \\ x &= \frac{k^2}{b}. \end{aligned}\right\} \qquad (6.1)$$

Therefore the inertia effect of the actual rod may be determined by considering only the forces associated with the mass $M_c\dfrac{k^2}{b^2 + k^2}$ situated at the crank-pin, and the mass $M_c\dfrac{b^2}{b^2 + k^2}$ placed at a point distant x from the centre of gravity G, as shown in Fig. 17.

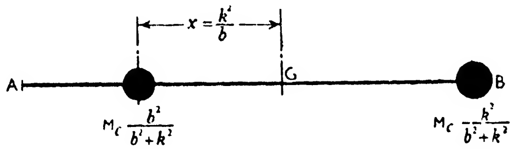


FIG. 17.

The general procedure will be understood if we now investigate an alternative system of equivalent masses for the given rod. Take this to consist of three masses denoted by M_A, M_B, M_G and placed respectively at the points A, B, G in Fig. 16. Here the conditions are

$$\begin{aligned} M_A + M_B + M_G &= M_c, \\ M_A a &= M_B . b, \\ M_A a^2 + M_B b^2 &= M_c k^2. \end{aligned}$$

A repetition of the foregoing analysis shows that these conditions will be satisfied if

$$\begin{aligned} M_A &= M_c\frac{k^2}{a(a + b)}, \\ M_B &= M_c\frac{k^2}{b(a + b)}, \\ M_G &= M_c\left(\frac{ab - k^2}{ab}\right) \end{aligned}$$

define the masses used for the purpose.

7. Dynamical Forces Acting on a Connecting-rod. For the present we shall suppose that the connecting-rod AB in Fig. 18 is dynamically equivalent to the two masses shown in Fig. 16, namely M_1 at the crank-pin B, and M_2 at the point P defined by the dimension x' in equations (6.1).

If the point Z be transferred from Fig. 14 to Fig. 18, then ZC represents the acceleration of the gudgeon-pin A to the same scale that BC represents the acceleration of the crank-pin B. To complete the construction shown in the figure, join ZB, and draw through the given point P a line parallel to AC, cutting ZB in

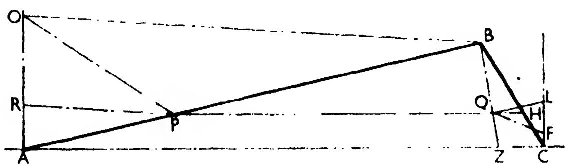


FIG. 18.

Q, and CB in H. Also draw through Q a line parallel to AB, cutting in L a line drawn through C and at right angles to AC. Now draw through A a line perpendicular to AC, and a line through P parallel to CQ; let these lines intersect at O. Join OB, and through P draw a line parallel to OB, to meet OA in R. Finally draw a line through Q and parallel to OB, cutting CL in F.

Since the acceleration of the point P is, by equation (5.5), equal to $QC.\omega^2$, on resolving this quantity into the components $QF.\omega^2$ and $FC.\omega^2$, it is seen that we must apply the forces

$$\frac{M_2}{g}\omega^2 QF \text{ at } B,$$

and

$$\frac{M_2}{g}\omega^2 FC \text{ at } A$$

in order to produce the necessary acceleration of the mass M_2 placed at P in the figure.

In proceeding to express these forces in terms of known quantities, we resolve the force $\frac{M_2}{g}\omega^2 QF$ into its components $\frac{M_2}{g}\omega^2 QH$, $\frac{M_2}{g}\omega^2 HC$, and $\frac{M_2}{g}\omega^2 CF$ acting through the point B.

The required relations for QH and HC in terms of the known lengths ZC and BC follow from the ratios

$$\frac{QH}{ZC} = \frac{QB}{ZB} = \frac{BP}{BA}$$

$$= \frac{b + \frac{k^2}{b}}{a + b},$$

whence $$QH = ZC.\frac{b + \frac{k^2}{b}}{a + b};$$

and $$\frac{HC}{BC} = \frac{AP}{AB}$$

$$= \frac{a - \frac{k^2}{b}}{a + b},$$

whence $$HC = BC.\frac{a - \frac{k^2}{b}}{a + b}.$$

These results, along with equations (6.1), enable us to write

$$\frac{M_2}{g}\omega^2 QH = \frac{M_c}{g}\omega^2\frac{b^2}{b^2 + k^2}.ZC\frac{b + \frac{k^2}{b}}{a + b}$$

$$= \frac{M_c}{g}\omega^2.ZC\frac{b}{a + b},$$

$$\frac{M_2}{g}\omega^2 HC = \frac{M_c}{g}\omega^2\frac{b^2}{b^2 + k^2}.BC\frac{a - \frac{k^2}{b}}{a + b}$$

$$= \frac{M_c}{g}\omega^2 BC\frac{b^2}{b^2 + k^2}\left\{\frac{ab - k^2}{b(a + b)}\right\}$$

for the above mentioned components of force in terms of known quantities.

An inspection of Fig. 18 now shows that the force denoted by the second of these expressions combines with the force $\frac{M_1}{g}\omega^2 BC$, to make up the total force acting along the crank; this inertia force is, in view of equation (6.1), accordingly equal to

$$\frac{M_c}{g}\omega^2 BC\left[\frac{b^2}{b^2 + k^2}\left\{\frac{ab - k^2}{b(a + b)}\right\} + \frac{k^2}{b^2 + k^2}\right],$$

or $$\frac{M_c}{g}\omega^2 BC\left\{\frac{ab^2 - bk^2 + (a + b)k^2}{(b^2 + k^2)(a + b)}\right\},$$

or $$\frac{M_c}{g}\omega^2 BC\frac{a}{a + b}.$$

Turning now to the preceding expressions that involve FC, these are most conveniently written in terms of the dimension CL, by means of the ratios

$$\frac{FC}{CL} = \frac{OR}{OA} = \frac{PB}{AB}$$
$$= \frac{k^2 + b^2}{b(a + b)},$$

so that
$$FC = CL\frac{k^2 + b^2}{b(a + b)}.$$

We thus obtain, by the aid of equations (6.1),

$$\frac{M_2}{g}\omega^2 FC = \frac{M_c}{g}\omega^2 CL\frac{b}{a + b}$$

for the value of the vertical force acting at the gudgeon-pin A, and

$$\frac{M_c}{g}\omega^2 CL\frac{b}{a + b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$

for the value of the vertical force acting at the crank-pin B.

The foregoing results define all the principal components of the force due to the motion of the connecting-rod, and these are for convenience of reference exhibited in Fig. 19; the corresponding

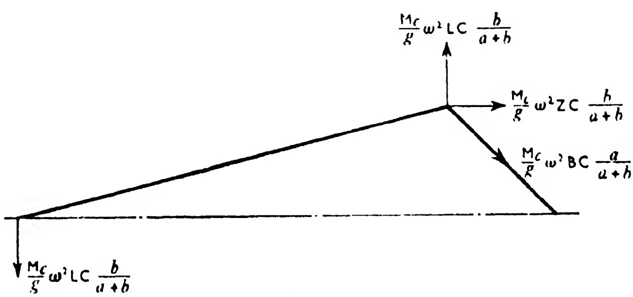


FIG. 19.

reactions, indicated in Fig. 20, appear as stresses in the structural system concerned.

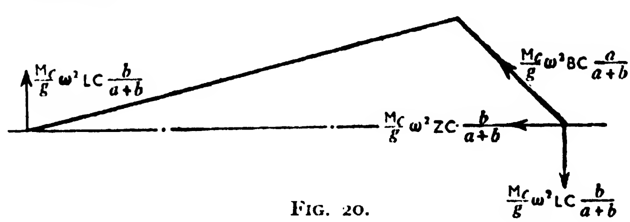


FIG. 20.

8. Forces due to Reciprocating Parts Other than the Connecting-rod. In finding the sum of the inertia effects

associated with engines, account must, of course, be taken of the parts that execute only reciprocating motion, such as, for example, the crosshead, and piston and its attachments. Let M_p be the total weight of these parts, which may be regarded in this respect as being concentrated at the point A in Fig. 21. The figure shown

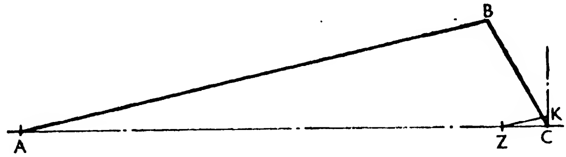


FIG. 21.

is obtained, with the point Z transferred from Fig. 14, by drawing through Z a line parallel to AB, to meet in K a line drawn through C and at right angles to AC.

It is readily deduced from the triangle ZCK that the inertia force due to the mass M_p gives rise to the reactions

$\frac{M_p}{g}\omega^2 CK$ vertically upwards through A,

$\frac{M_p}{g}\omega^2 KC$ vertically downwards through C,

$\frac{M_p}{g}\omega^2 CZ$ through C in the direction CA.

9. Inertia Force in the Line of Stroke. On summing the results obtained in Arts. 7 and 8 with reference to the components of force acting in the line of stroke, we have the unbalanced system indicated in Fig. 22, which therefore represents the inertia effects

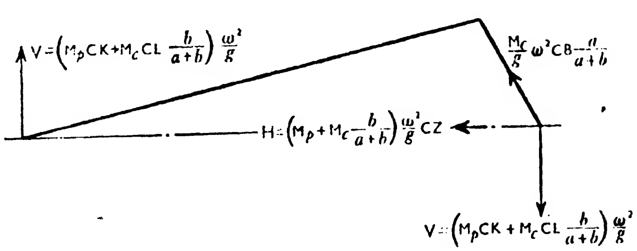


FIG. 22.

due to both the reciprocating and rotating parts of the specified engine. Here H and V are respectively the horizontal and vertical components of the total force arising from these sources of disturbance.

Hence we can write, in view of equation (5.6), for an engine

having a crank of throw r and a connecting-rod of length l,

$$H = \left(M_p + \frac{b}{l}M_c\right)\frac{\omega^2}{g}CZ$$

$$= \left(M_p + \frac{b}{l}M_c\right)\frac{\omega^2}{g}l\{\gamma\cos\theta + (\gamma^2 + \tfrac{1}{4}\gamma^4 + \ldots)\cos 2\theta$$
$$- (\tfrac{1}{4}\gamma^4 + \ldots)\cos 4\theta + \ldots\}$$

$$= \left(M_p + \frac{b}{l}M_c\right)\frac{r\omega^2}{g}\{\cos\theta + (\gamma + \tfrac{1}{4}\gamma^3 + \ldots)\cos 2\theta$$
$$- (\tfrac{1}{4}\gamma^3 + \ldots)\cos 4\theta + \ldots\}, \quad . \quad . \quad (9.1)$$

where $\gamma = \frac{r}{l}$. According to the present notation H is a minus quantity, but we have neglected this sign as irrelevant in matters pertaining to the numerical value of the periodic force under consideration.

10. Inertia Force Perpendicular to the Line of Stroke. It has been noticed that the vertical force V in Fig. 22 was formed by the components $\frac{M_c}{g}\omega^2 CL\frac{b}{l}$ associated with the mass M_c at P, and $\frac{M_p}{g}\omega^2 CK$ due to the parts of the mechanism which describe only reciprocating motion.

The method of Art. 7 may be applied to derive analytical expressions for CL and CK in the relation for V. To simplify the related diagram, transfer the points L, P, Q, Z from Fig. 18 to

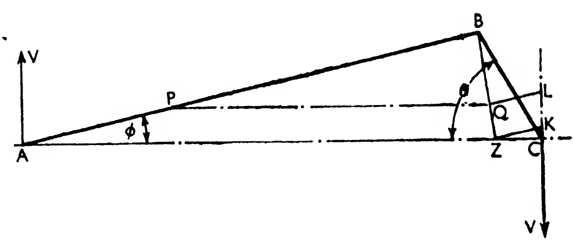


FIG. 23.

Fig. 23. Add to the resulting figure a line passing through Z and drawn parallel to AB, to meet in K the line CL.

In this manner we may readily deduce from the geometrical properties of the figure that

$$CK = CZ\tan\phi$$

$$= l\sin\phi + \frac{\gamma l\cos\theta\sin\phi}{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}} - \frac{l(1-\gamma^2)\sin\phi}{\cos^4\phi}$$

$$= \gamma l\sin\theta\{1 + \gamma\cos\theta(1-\gamma^2\sin^2\theta)^{-\frac{1}{2}}$$
$$- (1-\gamma^2)(1-\gamma^2\sin^2\theta)^{-2}\}, \quad . \quad . \quad . \quad . \quad (10.1)$$

since $\sin\phi = \gamma\sin\theta$, and $\cos\phi = (1-\gamma^2\sin^2\theta)^{\frac{1}{2}}$.

Further, from equations (6.1) we have

$$AP = a - x = a - \frac{k^2}{b} = \frac{ab - k^2}{b},$$

whence

$$\begin{aligned} CL &= AP \sin\phi + AP \sin\phi \cot\theta \tan\phi \\ &= AP\left\{\gamma \sin\theta + \frac{\gamma^2 \sin^2\theta \cot\theta}{(1 - \gamma^2 \sin^2\theta)^{\frac{1}{2}}}\right\} \\ &= \gamma\frac{ab - k^2}{b} \sin\theta\{1 + \gamma\cos\theta(1 - \gamma^2\sin^2\theta)^{-\frac{1}{2}}\} \qquad (10.2) \end{aligned}$$

Expansion of the terms in equations (10.1) and (10.2), with the help of the binomial theorem and trigonometrical relations of the type

$$\begin{aligned} \sin^3\theta &= \tfrac{1}{4}(-\sin 3\theta + 3\sin\theta), \\ \sin^5\theta &= \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta), \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

thus afford means of writing

$$\begin{aligned} V &= \left(M_p.CK + \frac{b}{l}M_c.CL\right)\frac{\omega^2}{g} \\ &= \left(M_p + \frac{b}{l}M_c\right)\frac{\omega^2 r}{g}\{-\sin\theta(\tfrac{1}{2}\gamma^2 + \tfrac{3}{8}\gamma^4 + \tfrac{5}{16}\gamma^6 + \ldots) \\ &\quad + \sin 2\theta(\tfrac{1}{2}\gamma + \tfrac{1}{8}\gamma^3 + \tfrac{15}{256}\gamma^5 + \ldots) \\ &\quad + \sin 3\theta(\tfrac{1}{2}\gamma^2 + \tfrac{7}{16}\gamma^4 + \tfrac{3}{8}\gamma^6 + \ldots) \\ &\quad - \sin 4\theta(\tfrac{1}{16}\gamma^3 + \tfrac{3}{64}\gamma^6 + \ldots) - \sin 5\theta(\tfrac{3}{16}\gamma^4 + \tfrac{1}{4}\gamma^6 + \ldots) \\ &\quad + \sin 6\theta(\tfrac{3}{256}\gamma^5 + \ldots) \ldots\} \\ &\quad + M_c\frac{\omega^2 r}{g}\left(\frac{ab - k^2}{l^2}\right)\{\sin\theta(1 + \tfrac{1}{2}\gamma^2 + \tfrac{3}{8}\gamma^4 + \tfrac{5}{16}\gamma^6 + \ldots) \\ &\quad - \sin 3\theta(\tfrac{1}{2}\gamma^2 + \tfrac{7}{16}\gamma^4 + \tfrac{3}{8}\gamma^6 + \ldots) \\ &\quad + \sin 5\theta(\tfrac{3}{16}\gamma^4 + \tfrac{1}{4}\gamma^6 + \ldots) + \ldots\} \quad \ldots \quad (10.3) \end{aligned}$$

for the inertia force V in Fig. 23. The number of terms to be included in this equation depends on the order of approximation aimed at in the result.

11. Graphical Representation of the Disturbing Forces. In this respect consider the horizontal component of force defined by equation (9.1), since, as already remarked, this is commonly the most important of the unbalanced effects in the stationary type of engine. If for brevity we write

$$M = M_p + \frac{b}{l}M_c, \quad \ldots\ldots \quad (11.1)$$

then M denotes the sum of the reciprocating masses identified with the gudgeon-pin, or crosshead.

On applying equation (9.1) to the case of a steam engine specified

by $\gamma = \frac{1}{3}$, and neglecting terms beyond γ^3 as of the second order of small quantities, we have

$$H = \frac{Mr\omega^2}{g}(\cos\theta + 0{\cdot}345\cos 2\theta - 0{\cdot}009\cos 4\theta).$$

We might construct, for one revolution of the crank, the graphs of $\cos\theta$, $0{\cdot}345\cos 2\theta$, and $0{\cdot}009\cos 4\theta$, as shown in the upper three

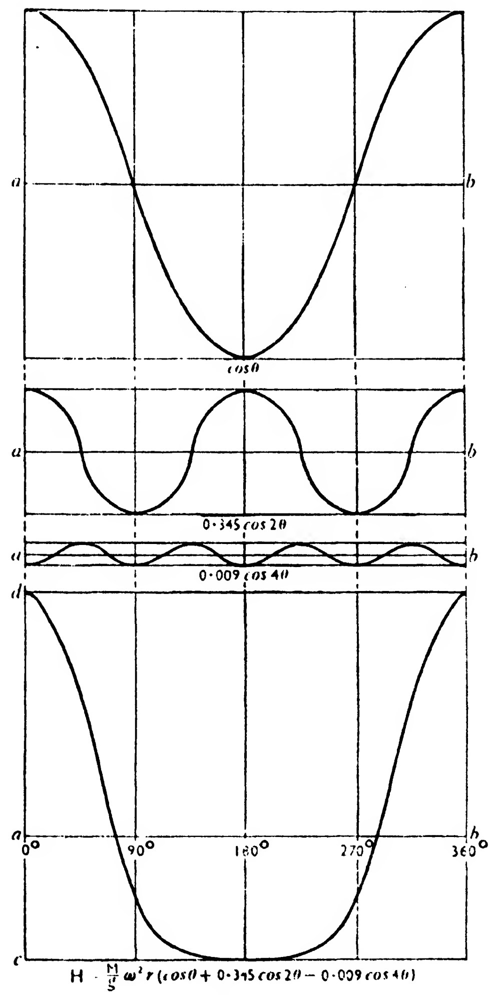


FIG. 24.

diagrams of Fig. 24, when the quantity H would be proportional to the algebraic sum of these graphs. This sum is indicated by the bottom graph, which accordingly consists of components that follow in turn a complete cosine-law of variation in one, two, and three revolutions of the crank. These are said to be the *harmonic*

components of the force in question, and are usually distinguished by the prefixes *primary*, *secondary*, *tertiary*, or by the corresponding ordinal numbers.

It will, however, be understood from equation (9.1) that there is in general a large number of such harmonic components associated with a given engine, but Fig. 24 suffices to demonstrate that in practical problems the ordinates of these graphs tend to assume negligibly small values beyond the secondary harmonic component of the inertia forces. In ordinary practice a sufficiently accurate result for many purposes is therefore given by the sum of the first and second harmonic components.

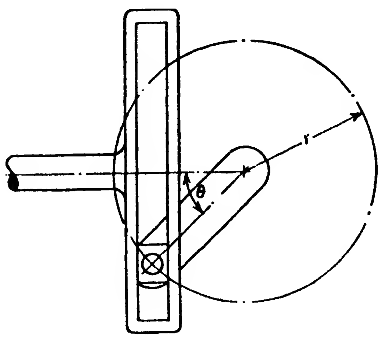


FIG. 25.

It is evident that the values of H are not equal for the positions corresponding to the inner and outer dead-centres when the obliquity of the connecting-rod is taken into account. If this obliquity be neglected, implying that the connecting-rod may be regarded as being infinitely long, then H assumes the value $\frac{M}{g}r\omega^2 \cos\theta$, and the mechanism corresponds with that shown in Fig. 25.

12. Balancing the Cranks. The fact that we have substituted for the actual connecting-rod an approximately equivalent system of masses indicates that the reciprocating parts of an engine cannot be balanced completely. This remark does not, however, apply to the rotating masses formed by the pin and webs of the crank concerned, for these can be balanced without difficulty by means of the pair of counter-weights shown dotted in Fig. 26. The same degree of balance can be obtained in other ways, such as, for example, by prolonging the webs to act as substitutes for the balance-weights indicated in the figure.

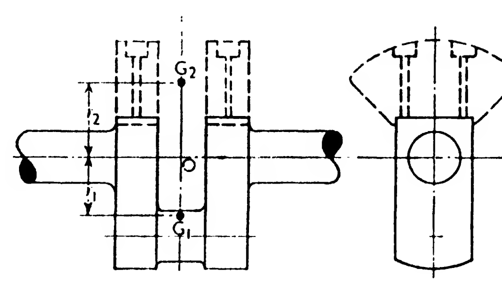


FIG. 26.

It will be assumed in what follows that the cranks are balanced to this extent, unless stated otherwise. The condition to be satisfied in this operation is that the centre of gravity of the combined

crank and counter-weights shall always lie at the point O on the axis of the crankshaft. In order to elucidate certain points, let M_1 be the total weight of the crank-pin and pair of webs, with the centre of gravity situated at the point G_1 distant r_1 from the axis of rotation. If balance is secured by means of two masses, each of weight M_2, and arranged so that their centre of gravity (G_2) acts at radius r_2 about the axis, then it follows from consideration of the moments involved that the crank is completely balanced if

$$M_1 r_1 = 2M_2 r_2,$$

hence each of the weights required for the purpose is given by

$$M_2 = \frac{M_1 r_1}{2r_2}.$$

Were the two weights not identical, the shaft would be subjected to the effect of a periodic couple of the type represented by $\mathfrak{C}$ in Fig. 4.

When the arrangement shown in Fig. 26 is used for this purpose, the counter-weights should be keyed, as well as bolted, to the crank-webs, since serious breakdowns sometimes occur due to failure of the bolt-connections.

Matters relating to the cost of production also must be considered, for obvious reasons, in connection with the methods used in balancing engines. On this account we are sometimes bound to employ arrangements in which couples of the type $\mathfrak{C}$ are present, such as is exemplified by the system shown in Fig. 27, where partial

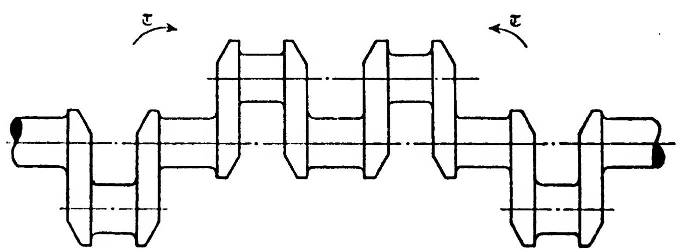


FIG. 27.

balance of the four cranks is obtained by arranging them in pairs set at 180 deg. apart. In this case the disturbance is only aggravated when, as is sometimes done, three, instead of five, symmetrically disposed bearings are used to support the crankshaft. Under these conditions, with no balance-weights for the cranks proper, the magnitude of the unbalanced couple $\mathfrak{C}$ tends to increase the range of variation in pressure on the bearings.

13. Horizontal Engines. In the foregoing analysis it has been supposed for brevity of treatment that the engine in question consisted of only one cylinder, but the following examples show

that we can proceed in a straightforward manner to any number and arrangement of cranks.

(*a*) *Single Cylinder.* Writing, as previously, M for $\left(M_p + \frac{b}{l}M_c\right)$ in equation (9.1), we have

$$H = \frac{M}{g}r\omega^2\{\cos\theta + (\gamma + \tfrac{1}{4}\gamma^3 + \ldots)\cos 2\theta$$
$$- (\tfrac{1}{4}\gamma^3 + \ldots)\cos 4\theta + \ldots\} \quad . \quad . \quad (13.1)$$

as the expression for the horizontal component of the inertia effects.

The primary harmonic component of this force, namely $\frac{M}{g}r\omega^2\cos\theta$, might be neutralized by the aid of the balance-weights shown in Fig. 26. This would, however, introduce into the system an unbalanced force acting in the vertical direction, which might well be as objectionable as the original disturbing force. It is therefore common practice to counteract only a fraction of the force designated by H. If, by way of illustration, only half of the above mentioned primary component be balanced, in addition to the effect of the crank proper, the balance-weights required for the purpose must produce a centrifugal force amounting to

$$\frac{a}{l}\frac{M_c}{g}r\omega^2 + \frac{M}{2g}r\omega^2,$$

or

$$\left\{\frac{a}{l}M_c + \tfrac{1}{2}\left(M_p + \frac{b}{l}M_c\right)\right\}\frac{r\omega^2}{g} \quad . \quad . \quad . \quad (13.2)$$

This compromise leads to the unbalanced system of forces indicated

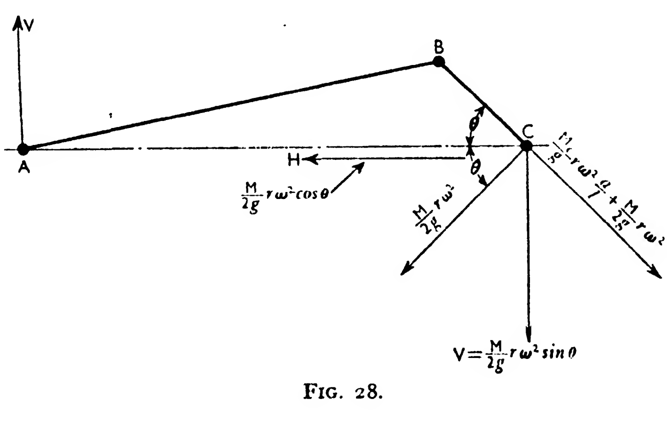


FIG. 28.

in Fig. 28, from which it is seen that an unbalanced force of magnitude $\frac{M}{2g}r\omega^2$ acts along a line that makes an angle θ with AC in the figure.

Partial balance of the connecting-rod might, alternatively, be

secured by extending the rod beyond the crank-pin B, when the dimension b in our analysis would assume a negative value. Then the factor $\left(M_p - \frac{b}{l}M_c\right)$, instead of $\left(M_p + \frac{b}{l}M_c\right)$, would appear in the expression for H; the former of these factors would vanish if the dimension b were such as to satisfy the condition $\frac{b}{l} = \frac{M_p}{M_c}$. Nevertheless, this cumbersome type of rod has little to be said in its favour, on either practical or dynamical grounds, for its use leads to increase in the value of the vertical component V defined in equation (10.3).

Ex. Apply the above results to the case of a 'uniflow' engine having a stroke of 30 in., and working at 144 r.p.m. In terms of our notation $M_1 = 1{,}960$ lb., $M_2 = 1{,}040$ lb., $x = 49{\cdot}1$ in., $k = 35{\cdot}7$ in., $M_p = 3{,}070$ lb., and $\gamma = \frac{1}{5}$.

Introducing the given values

$$M = M_p + \frac{b}{l}M_c = 4{,}110 \text{ lb.},$$

$$r = 1{\cdot}25 \text{ ft.},$$
$$\omega = 15{\cdot}1 \text{ rad. per sec.},$$
with
$$g = 32{\cdot}2 \text{ ft. per sec. per sec.},$$

into equation (9.1) yields

$$H = \frac{1{,}168{,}000}{32{\cdot}2}(\cos\theta + 0{\cdot}202\cos 2\theta - 0{\cdot}002\cos 4\theta \ldots) \text{ lb.};$$

this attains a maximum value of 19·45 tons, approximately, when $\theta = 0$.

Although a force of this magnitude operated for comparatively short intervals of time on the main bearings of the actual engine involved in this problem, the unbalanced effect proved to be a source of serious vibrations which were felt on neighbouring buildings. Subsequent investigation into the matter showed that the soil supporting the foundations was of a kind which transmitted the disturbance over long distances, in a manner that will be explained in Chap. IV.

(*b*) *Opposed Cylinders.* The forces H and V acting on the single-cylinder engine represented in Fig. 29 may be balanced, to

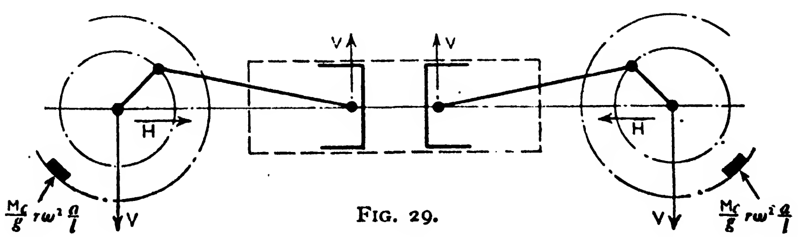


FIG. 29.

the implied approximation, by fitting to each crank a balance-weight of such dimensions as will neutralize the force specified by equation (13.2). It is here to be understood that identical masses are associated with each of the cranks, and that a suitable system of gearing ensures synchronous movement of the pistons.

Pass now to the consideration of the mechanism shown in Fig. 30, where the connecting-rods move in one plane, and similar masses are assigned to each cylinder. On writing H_1 and H_2 for

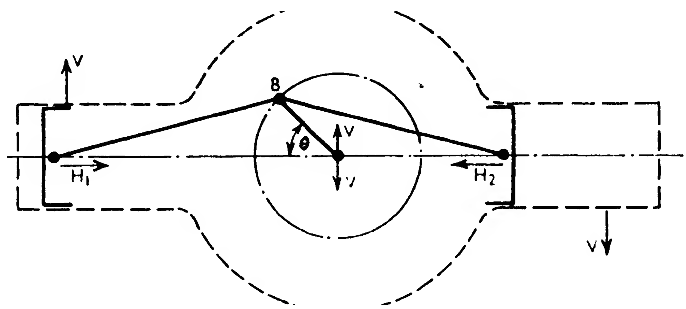


FIG. 30.

the horizontal components of the inertia forces indicated in the figure, we have, by equation (9.1),

$$H_1 = \frac{M_1}{g} r\omega^2 \{\cos\theta + (\gamma + \tfrac{1}{4}\gamma^3 + \ldots)\cos 2\theta - (\tfrac{1}{4}\gamma^3 + \ldots)\cos 4\theta + \ldots\},$$

$$H_2 = \frac{M_1}{g} r\omega^2 \{\cos(\pi - \theta) + (\gamma + \tfrac{1}{4}\gamma^3 + \ldots)\cos(2\pi - 2\theta)$$
$$- (\tfrac{1}{4}\gamma^3 + \ldots)\cos(4\pi - 4\theta) + \ldots\},$$

M_1 representing the reciprocating mass M_1 of each cylinder. Since the resultant of these forces is defined by

$$H_1 - H_2 = 2\frac{M_1}{g} r\omega^2 \cos\theta,$$

it is seen that primary harmonic components alone appear in the unbalanced effect. This is, however, the only advantage offered by the arrangement, because it leads to a transverse force V that gives rise to a disturbing couple when, as is commonly the case in these circumstances, the engine consists of a number of such pairs of cylinders.

Hence the unbalanced effect is equivalent to a moving mass situated at the centre of gravity of the prescribed system, and this remark applies also to the arrangement shown in Fig. 31, where the connecting-rods form at any instant a parallelogram. Here the disturbing forces acting in the line of stroke are those defined by the primary harmonic components of equation (9.1), which

might otherwise have been inferred from the fact that the centre of gravity (G) of the system executes simple harmonic motion. It is assumed that the cranks proper are balanced in the usual manner.

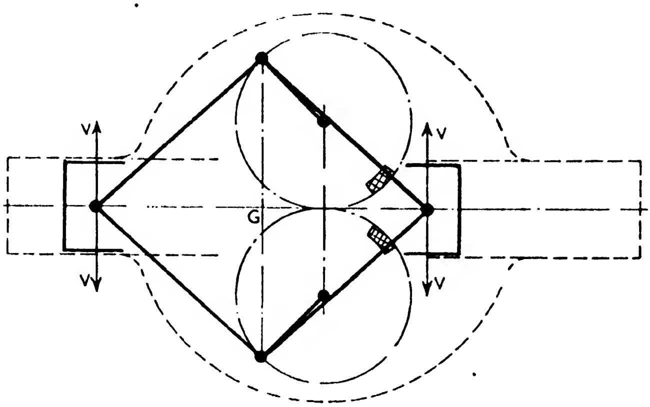


FIG. 31.

Although the opposed-piston engine shown in Fig. 32 is free from inertia forces acting in the line of stroke, the mechanism is nevertheless subjected to vertical forces denoted by V in the pre-

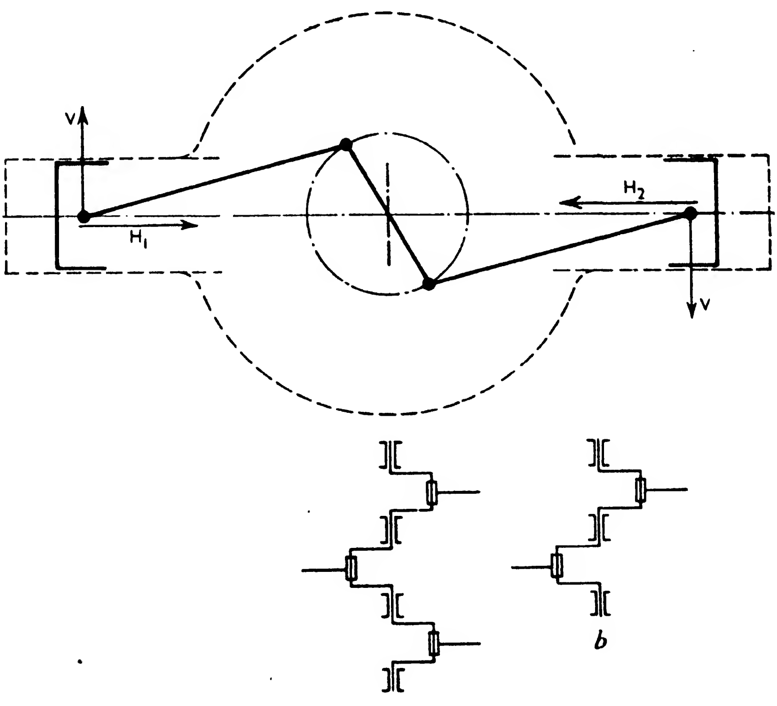


FIG. 32.

ceding analysis, but these are commonly of no practical importance. To avoid the cost involved in the manufacture of the three-throw crank necessary for this degree of balance, the two-throw crankshaft indicated in Fig. 32 (*b*) is sometimes used instead of the one represented by Fig. 32 (*a*), so that economy is obtained at the sacrifice of something in the way of balance.

14. Vertical Engines. The above procedure requires a slight modification when applied to the case of vertical engines, owing to the fact that the weight of the moving parts acts *against* the inertia force H during an up-stroke of the engine, and *with* that force during a down-stroke. Since this simple operation can be effected without difficulty, it is scarcely necessary to introduce the matter into the following applications of the theory.

(*a*) *Three-cylinder Engine.* In Fig. 33 let d be the axial distance between the centre lines of the cylinders, where it is supposed that the cranks are set at 120 deg. apart, and that $\left(M_p + \frac{b}{l}M_c\right)$ is the reciprocating mass attached to each of the three cranks.

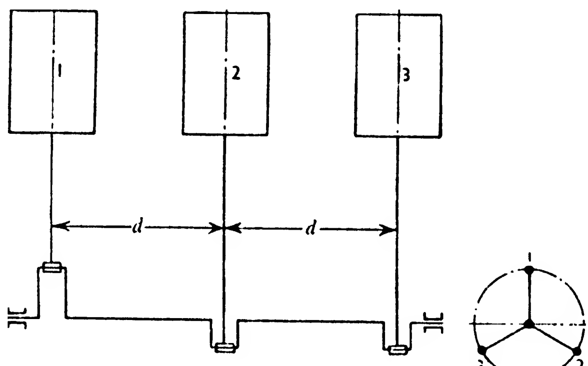


FIG. 33.

If θ define the position of the crank for cylinder No. 1 in the figure, the inertia forces acting in the line of stroke for the cylinders numbered 1, 2, 3 are, by equation (9.1),

$$\frac{Mr\omega^2}{g}\{A_1 \cos\theta + A_2 \cos 2\theta + A_4 \cos 4\theta + A_6 \cos 6\theta + \ldots\},$$

$$\frac{Mr\omega^2}{g}\left\{A_1 \cos\left(\theta + \frac{2\pi}{3}\right) + A_2 \cos\left(2\theta + \frac{4\pi}{3}\right) + A_4 \cos\left(4\theta + \frac{8\pi}{3}\right)\right.$$
$$\left. + A_6 \cos\left(6\theta + \frac{12\pi}{3}\right) + \ldots\right\},$$

$$\frac{Mr\omega^2}{g}\left\{A_1 \cos\left(\theta + \frac{4\pi}{3}\right) + A_2 \cos\left(2\theta + \frac{8\pi}{3}\right) + A_4 \cos\left(4\theta + \frac{16\pi}{3}\right)\right.$$
$$\left. + A_6 \cos(6\theta + 8\pi) + \ldots\right\},$$

where the A-coefficients represent constants for an engine having a crank-throw r, and rotating with angular velocity ω.

Since the sums

$$\frac{M}{g}r\omega^2 A_1\left\{\cos\theta + 2\cos(\theta+\pi)\cos\frac{\pi}{3}\right\},$$

$$\frac{M}{g}r\omega^2 A_2\left\{\cos 2\theta + 2\cos(2\theta+2\pi)\cos\frac{\pi}{2}\right\},$$

$$\frac{M}{g}r\omega^2 A_4\left\{\cos 4\theta + 2\cos(4\theta+4\pi)\cos\frac{4\pi}{3}\right\}$$

vanish separately, it is to be inferred that the primary, secondary and tertiary harmonic components of the force under examination are inherently balanced.

Proceeding in this manner to the higher harmonic components, it will be found that the 6th harmonic component is the first to appear in the system of unbalanced forces. This amounts to

$$\frac{3M}{g}r\omega^2 A_6\cos 6\theta,$$

where $A_6 = \frac{9}{128}\gamma^5 + \frac{45}{512}\gamma^7$, approximately. A continuation of the analysis would disclose the 12th and 18th harmonic components to be the succeeding unbalanced effects, since the intermediate orders are all zero in value. In so far as the force H contains only three unbalanced harmonic components up to the 18th order terms, the engine may be regarded as very satisfactory in this respect.

A correspondingly high degree of balance is not, however, attained as regards the 'fore and aft' couple acting in the plane of the paper. This may easily be demonstrated by taking moments about a line in the plane of rotation of crank No. 3, whence

primary component of the couple

$$= \frac{M}{g}r\omega^2 d.A_1\left\{2\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right)\right\}$$

which is not zero. Hence the unbalanced couple contains primary harmonic components; this is, in fact, a common source of trouble with this type of engine.

It is well to remember that in triple-expansion steam engines the moving masses assigned to each crank are in general not equal, and in these circumstances M has a different value for each crank.

(*b*) *Six-cylinder Engine.* The above-mentioned tilting couple vanishes if two similar engines are coupled together, with the pairs (1, 6), (2, 5), (3, 4) of cranks in phase, to form the six-throw crank-

shaft shown in Fig. 34. As in the previous example, the 6th, 12th, 18th,....... harmonic components represent the only unbalanced effects of the combined force H.

It is of practical interest to notice here that if ab in Fig. 34 be a mirror placed normal to the axis of the crank in Fig. 33, then its 'optical image' indicates the cranks numbered 4, 5, 6 in the figure. It follows, on generalizing, that any system of unbalanced couples can be neutralized by adding to the system the set of forces represented by the optical image of the original mechanism.

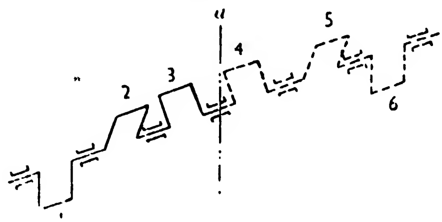


FIG. 34.

Appropriate combinations of the arrangements shown in Figs. 29 and 32 afford a means of examining the Junker heavy-oil engine for aircraft, and the Doxford oil-engine with opposed cylinders.

Readers interested in the related problem of four-cylinder engines may be referred to an instructive treatment by Professor C. E. Inglis,[1] who has examined the matter with particular reference to marine installations.

15. Vee and Radial Engines. The above expressions for the forces H and V may be applied to the case of vee engines, provided due consideration is given to the relative positions of the lines of stroke.

Taking, by way of example, the engine represented by Fig. 35, where the mass M as defined in equation (11.1) is to be associated with each of the cylinders inclined at 45 deg. to the vertical, we thus find that $\frac{M}{g} r\omega^2 \cos\theta$ and $\frac{M}{g} r\omega^2 \sin\theta$ are the primary harmonic components of H in equation (9.1). Since these components are equivalent to a force $\frac{M}{g} r\omega^2$ acting along the crank, this unbalanced effect may be combined with that due to the crank proper, and the total force neutralized by means of a single counter-weight placed opposite the crank in the usual manner. It may be noted that in this type of engine it is common practice to leave unbalanced the higher harmonic components of H.

When other pairs of cylinders are added to that shown in the

[1] *Trans. Inst. N.A.*, vol. 43, page 248 (1911).

figure, a tilting couple, acting in a direction perpendicular to the plane of the paper, is in general introduced into the unbalanced effects.

It is readily seen that a simple extension of the method enables us to examine radial engines, since these may be treated as a number of vee engines arranged symmetrically in one or more planes.

Mention has already been made of the fact that equal importance may well be attached to the unbalanced forces H and V.

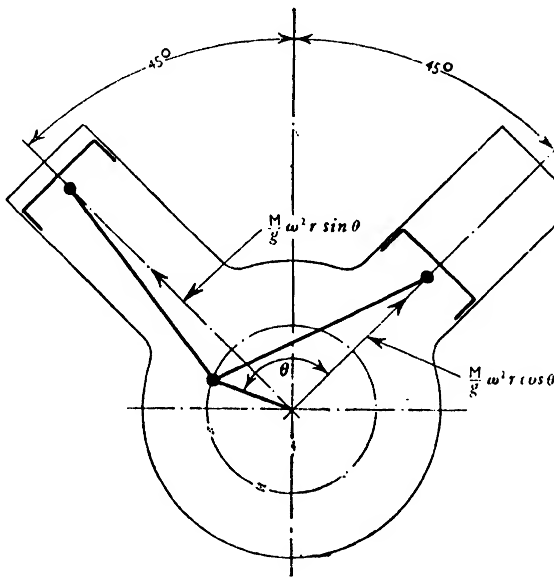


FIG. 35.

The relative order of magnitude of these components may be indicated by reference to a particular engine of this class, developing 450 b.h.p., for which it was found that the maximum values of H and V amounted to 696 lb. and 1,607 lb., respectively, when the engine was revolving at 2,000 r.p.m. The significance of these forces becomes manifest when regard is had to the slender form of construction used in aircraft, as this renders the structure liable to vibrations in a number of directions.

Reference should be made under this heading to the 'articulated' type of connecting-rod which is used for the purpose of coupling a number of rods to a single crank, in a manner similar to that illustrated in Fig. 36. Since the longitudinal axes of the articulated connecting-rods do not pass through the centre of the corresponding crank-pin, it is of some interest to consider the consequences of this method of construction.

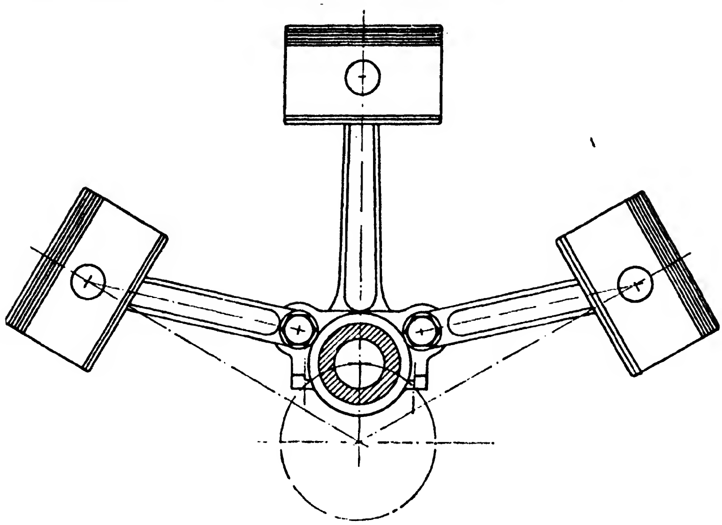


FIG. 36.

16. ' Offset ' Engines. With the object of reducing in value the thrust denoted by $P \tan \phi$ in Fig. 11, the cranks of engines are sometimes given a certain amount of ' offset ' in the direction of rotation, equal to the dimension f in Fig. 37.

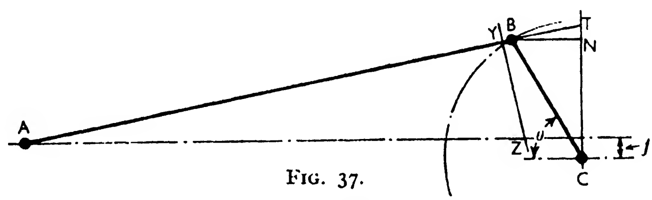


FIG. 37.

To investigate the dynamical effect of this modification on the component of force defined by

$$H = \left(M_p + \frac{b}{l}M_c\right)\frac{\omega^2}{g}ZC$$

in Art. 5, we have, making use of the geometrical relations of Fig. 37 for an engine rotating with uniform angular velocity ω,

$$ZC.\omega^2 = \frac{d}{dt}(\omega.CT)$$

$$= \omega r\frac{d}{dt}\{\sin\theta + (\tfrac{1}{2}\gamma + \tfrac{1}{8}\gamma^3 + \ldots)\sin 2\theta - (\tfrac{1}{16}\gamma^3 + \ldots)\sin 4\theta + \ldots$$

$$-\frac{f}{l}(1 + \tfrac{3}{8}\gamma^2 + \ldots)\cos\theta + \frac{f}{l}(\tfrac{3}{8}\gamma^2 + \ldots)\cos 3\theta + \ldots\}$$

$$= r\omega^2\left(\cos\theta + \frac{f}{l}\sin\theta + \gamma\cos 2\theta + \ldots\right), \quad . \quad . \quad (16.1)$$

where, as usual, $\gamma = \dfrac{\text{throw of crank } (r)}{\text{length of connecting rod } (l)}$.

Although the amount of offset used in practice does not greatly influence H compared with its value for the normal type of engine, the method of construction does not make for improvement in the way of balance. A comparison of equations (16.1) and (5.6) will suffice to elucidate the point, whence it follows that the effect of offset is to introduce sine-terms of all the odd multiples of θ into the expression for H. Consequently, in the case of three-cylinder engines having offset cranks the unbalanced harmonic components comprise the series 3rd, 6th, 9th, 12th, 15th, 18th, . . . orders, compared with the 6th, 12th, 18th, . . . orders for the normal type of engine. These additional components of unbalanced force, though small in magnitude, tend, for reasons that will be explained in Chapter III, to increase the probability of 'resonance' in various parts of the structural system concerned.

We may mitigate the consequences to the extent of neutralizing the primary harmonic component, equal to

$$\frac{M}{g}r\omega^2\left(\cos\theta + \frac{f}{l}\sin\theta\right),$$

by means of a balance-weight arranged in the rotating system so as to induce a centrifugal force amounting to $\dfrac{Mr\omega^2}{gl}(l^2 + f^2)^{\frac{1}{2}}$. It is well to observe here, as the matter is sometimes forgotten in practice, that the counter-weight for H should be set at an angle α with respect to the balance-weight for the crank proper, where $\tan\alpha = \dfrac{f}{l}$ in Fig. 38. These two counter-weights may conveniently

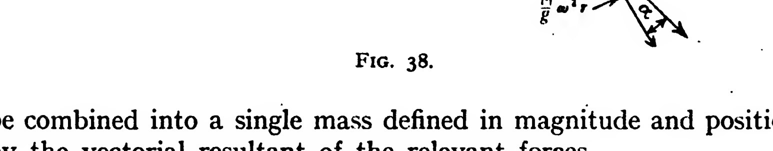


FIG. 38.

be combined into a single mass defined in magnitude and position by the vectorial resultant of the relevant forces.

17. Indicator Diagrams and Inertia Forces. The effect of the inertia forces on the indicator diagram for a given engine may be examined by taking a convenient number of points in the working cycle and dividing the corresponding force H by the effective area of the piston. When the resulting graph is superposed on the indicator diagram, we obtain a graphical representation of the

effective driving force on the system, and it may readily be shown that the related torque is influenced more and more by the inertia effects as the speed of the engine increases.

The point may be elucidated with the help of Fig. 39, where the above mentioned superposition has been effected to derive the

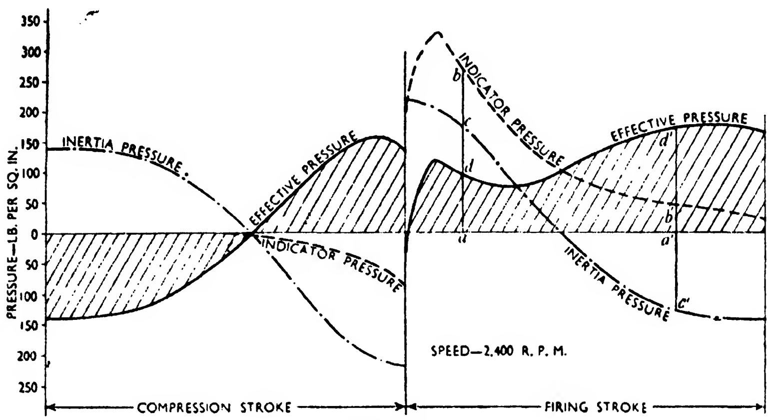


FIG. 39.

graph of the effective force denoted by the hatched diagram in the figure. The shape of this diagram, referring to an engine with a speed of 2,400 r.p.m., clearly differs in important particulars from the indicator diagram involved.

Consider first what happens on this account during the firing stroke. In the early stages, when the linear velocity of the mechanism is increasing, the moving parts 'absorb' energy in much the same manner as would a flywheel; energy is in the same sense 'restored' to the mechanism in the later stages of the firing stroke, when the linear velocity is decreasing. Hence, on taking a and a' in the figure as representative points in the stroke, the effective pressure for the point a is the difference between the ordinates ab and ac, that is ad; similarly, the effective pressure for the point a' is given by the sum of the ordinates $a'b'$ and $a'c'$, that is $a'd'$. A corresponding transformation of energy takes place in the succeeding compression stroke, as is illustrated in the figure.

Since the driving torque on the crank is proportional to the ordinates of the hatched diagram, this, rather than the indicator diagram, should be used in designing the crankshaft and other parts of the mechanism in accordance with specified working stresses. When a prescribed engine includes a number of cranks, their combined effect in this connection is to be evaluated by the method exemplified in Art. 14 (a). Increase in the number of cranks on an engine, when examined as a whole, tends to remove certain

characteristics which appear in Fig. 39, such as, for example, the sudden reversal of load on the gudgeon-pin when it reaches the end of the compression stroke. The nature of this particular disturbance resembles a hammer-blow on the bearings of engines rotating at high speeds, and instances of troublesome lubrication have been traced to this source.

Notwithstanding the undesirable consequences of inertia effects on internal combustion engines generally, Fig. 39 demonstrates that these forces make for uniformity in the effective pressure during the firing stroke, when the pressure of the working fluid attains its maximum value. In the general subject of design, therefore, inertia forces become of considerable importance when high-speed engines are under examination. It is also useful to notice that the reciprocating parts of engines contribute to the effect exerted by a flywheel which may be attached to the mechanism.

18. Dynamics of Engine Mechanisms. We shall now approach the main problem by way of the various reactions on the principal parts of engines, and thus present another aspect of the matter.

Taking AB in Fig. 40 to represent the actual connecting-rod of the engine in question, it is most convenient for our present

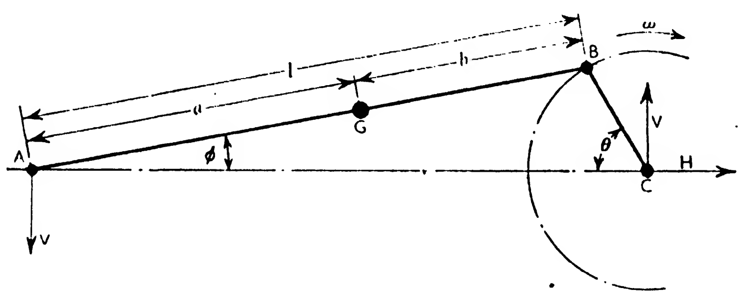


FIG. 40.

purpose to suppose that the rod is dynamically equivalent to two masses, one concentrated at the crank-pin B, and the other at the gudgeon-pin A. It follows from the treatment of Art. 6 that in these circumstances the weight of each mass is inversely proportional to its distance from the centre of gravity (G) of the connecting-rod. Moreover, on writing M_c for the weight of the rod, and M_A and M_B for the masses to be assigned in this manner to the ends A and B of the rod, it is readily deduced that the dynamically equivalent system includes also the couple

$$\frac{M_c}{g}(k^2_A - al)\ddot{\phi}$$

for a rod having a radius of gyration k_A about the end A. Here

$\ddot{\phi}$ denotes the angular acceleration of the rod about the gudgeon-pin, and a the distance of G measured from that pin.

Now for simplicity let the connecting-rod in Fig. 40 be disconnected in the manner indicated by Fig. 41, then, with C as the

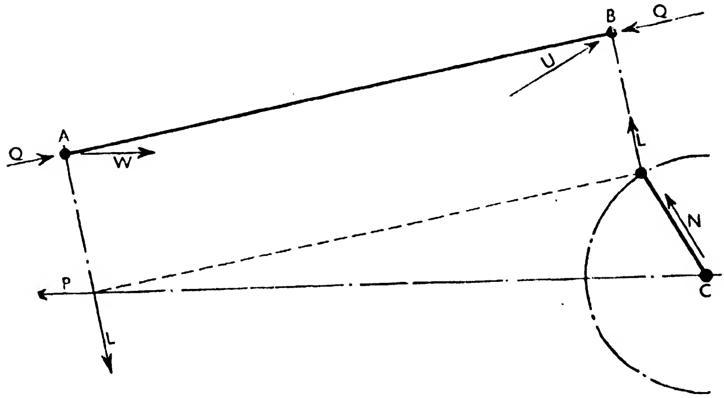


FIG. 41.

origin for the x- and y-axes, it is easily seen that the system comprises, in the notation of Fig. 42:

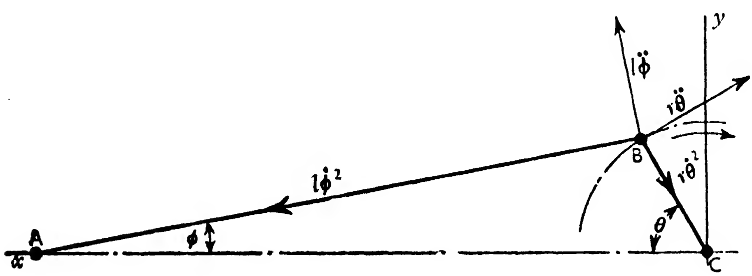


FIG. 42.

(i) A thrust Q acting along the connecting-rod, with equal and opposite reactions at the points A and B;

(ii) A couple $\frac{M_c}{g}(k^2_A - al)\ddot{\phi}$, with equal and opposite components L acting normal to the rod at the points A and B;

(iii) A force N acting through the point B in a direction parallel to the crank, and equal to $\frac{M_c ra}{gl}\dot{\theta}^2$, or $\frac{M_c a\gamma}{g}\dot{\theta}^2$, where

$$\gamma = \frac{\text{throw of crank } (r)}{\text{length of connecting-rod } (l)};$$

(iv) A force U acting through the point B and normal to the crank, amounting to $\frac{M_c ra}{gl}\ddot{\theta}$, or $\frac{M_c a\gamma}{g}\ddot{\theta}$;

(v) A reaction W equal to $\frac{M_c(l-a)}{gl}\ddot{x}$, or $\frac{M_A}{g}\ddot{x}$, where $\ddot{x}$ is the acceleration of the gudgeon-pin referred to the line of stroke ; this is due to that part of the rod which is taken as being concentrated at the point A.

Hence if $\mathfrak{T}$ be the torque resisting motion of the crank, and I_c the combined moments of inertia for the crankshaft, flywheel and the machinery connected to it, we have, on equating the moments of force to the appropriate product of mass and acceleration,

$$Qr\sin(\theta+\phi) - Ur - Lr\cos(\theta+\phi) - \mathfrak{T} = I_c\ddot{\theta}\ . \quad (18.1)$$

as the equation of motion for the mechanism.

Further, on writing P for the force exerted by the working fluid on the piston, and M_q for the mass of the reciprocating parts concentrated at A and *not* included in M_A, we obtain

$$P - \left(Q\cos\phi + L\sin\phi + \frac{M_A}{g}\ddot{x}\right) = \frac{M_q}{g}\ddot{x} \quad . \quad (18.2)$$

as a result of equating all the forces which act through the point A.

Taking the crank-angle θ to be the most convenient co-ordinate of reference in connection with these equations, we proceed to express all the variables in terms of θ and its derivatives.

For example, to derive the required expression for ϕ, we use the known geometrical relations

$$\cos\phi = (1-\gamma^2\sin^2\theta)^{\frac{1}{2}},$$

$$\sin\phi = \frac{r}{l}\sin\theta = \gamma\sin\theta.$$

Differentiation with respect to time yields

$$\dot{\phi}\cos\phi = \gamma\dot{\theta}\cos\theta,$$

or

$$\dot{\phi} = \gamma\frac{\cos\theta}{\cos\phi}\dot{\theta}$$

$$= \gamma\cos\theta(1-\gamma^2\sin^2\theta)^{-\frac{1}{2}}\dot{\theta} \quad . \quad . \quad (18.3)$$

For brevity write α for $\frac{\dot{\phi}}{\dot{\theta}}$, then

$$\frac{d\alpha}{d\theta} = \frac{d}{d\theta}\left(\frac{\dot{\phi}}{\dot{\theta}}\right) = \frac{\dot{\theta}\frac{d\dot{\phi}}{d\theta} - \dot{\phi}\frac{d\dot{\theta}}{d\theta}}{\dot{\theta}^2}$$

$$= \frac{\frac{\dot{\theta}\ddot{\phi}}{\dot{\theta}} - \frac{\dot{\phi}\ddot{\theta}}{\dot{\theta}}}{\dot{\theta}^2} = \frac{\ddot{\phi} - \alpha\ddot{\theta}}{\dot{\theta}^2},$$

whence, on rearranging the terms,

$$\ddot{\phi} = \dot{\theta}^2\frac{d\alpha}{d\theta} + \alpha\ddot{\theta}.$$

But we also have the relation

$$\alpha = \frac{\gamma \cos \theta}{\cos \phi}$$
$$= \gamma \cos \theta(1 + \tfrac{1}{2}\gamma^2 \sin^2 \theta + \ldots)$$

when $\dfrac{1}{\cos \phi}$ is expanded by the aid of the binomial theorem. Inserting this relation for α in the expression for $\ddot{\phi}$ leads finally to

$$\ddot{\phi} = \gamma(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta), \quad . \quad . \quad . \quad . \quad (18.4)$$

if we neglect terms beyond the second order in γ.

Turning to the corresponding relations for $\ddot{x}$, the first step consists in resolving along the x- and y-axes the components of acceleration referred to the point B. These resolutions give

$$\ddot{x} + l\ddot{\phi} \sin \phi + l\dot{\phi}^2 \cos \phi = -r\ddot{\theta} \sin \theta - r\dot{\theta}^2 \cos \theta,$$

and
$$l\ddot{\phi} \cos \phi - l\dot{\phi}^2 \sin \phi = r\ddot{\theta} \cos \theta - r\dot{\theta}^2 \sin \theta.$$

To eliminate the $\ddot{\phi}$-terms from these equations, multiply the first by $\cos \phi$ and the second by $\sin \phi$, then subtraction of the former from the latter results in

$$\ddot{x} \cos \phi - l\dot{\phi}^2 = r\ddot{\theta}(\cos \theta \sin \phi + \sin \theta \cos \phi)$$
$$+ r\dot{\theta}^2(\cos \theta \cos \phi - \sin \theta \sin \phi),$$

or
$$\ddot{x} = \frac{r\ddot{\theta} \sin (\theta + \phi)}{\cos \phi} + \frac{r\dot{\theta}^2 \cos (\theta + \phi)}{\cos \phi} + \frac{l\dot{\phi}^2}{\cos \phi}.$$

Next substitute the values of ϕ and $\dot{\phi}$ from equation (18.3), when it will be found, on expanding the resulting expression, that

$$\ddot{x} = r\ddot{\theta}(\sin \theta + \tfrac{1}{2}\gamma \sin 2\theta) + r\dot{\theta}^2(\cos \theta - \gamma \sin^2 \theta) + \gamma^2 l\dot{\theta}^2 \cos^2 \theta$$
$$= r\ddot{\theta}(\sin \theta + \tfrac{1}{2}\gamma \sin 2\theta) + r\dot{\theta}^2(\cos \theta + \gamma \cos 2\theta), \quad . \quad . \quad . \quad (18.5)$$

if terms beyond γ^2 be neglected.

Now introduce the relation for Q given by equation (18.2) into equation (18.1), and thus obtain

$$\frac{r \sin (\theta + \phi)}{\cos \phi}\left(P - \frac{M_q}{g}\ddot{x} - L \sin \phi - \frac{M_A}{g}\ddot{x}\right)$$
$$- Ur - Lr \cos (\theta + \phi) - \mathfrak{T} = I_c\ddot{\theta}.$$

Finally, insert in this equation the above relations for $\ddot{x}$, ϕ, L and

U expressed in terms of θ and its derivatives. The procedure, with M written for $(M_A + M_q)$, leads to

$$Pr(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \mathfrak{T} = \left(I_c + \frac{M_c}{g}ra\gamma\right)\ddot{\theta} + Lr\cos(\theta + \phi)$$
$$+ r(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)\left(L\sin\phi + \frac{M}{g}\ddot{x}\right)$$
$$= \left\{\left(I_c + \frac{M_c}{g}ra\gamma\right) + \frac{M_c}{g}\gamma^2(k^2_A - al)\cos^2\theta + \frac{Mr^2}{g}(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)^2\right\}\ddot{\theta}$$
$$+ \left\{\frac{Mr^2}{g}(\cos\theta + \gamma\cos 2\theta)(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \frac{M_c\gamma^2}{2g}(k^2_A - al)\sin 2\theta\right\}\dot{\theta}^2 \quad \text{......(18.6).}$$

This determines, to the stated approximation, the motion of the specified engine. The law of variation for the force P exerted by the working fluid on the piston is to be obtained from the indicator diagram, and the corresponding law for $\mathfrak{T}$ from a record taken with the aid of a torsiometer. With the help of these diagrams we may therefore express the quantity

$$\{Pr(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \mathfrak{T}\},$$

denoting the difference between the turning moment due to the working fluid and the resisting torque, in the form of a Fourier series involving θ and its multiples. In applying this series to equation (18.6) in particular, we require only the terms in θ and 2θ.

The forces H and V obtained in this manner will in general differ in value from those given by equations (9.1) and (10.3), owing to the different assumptions involved in the dynamically equivalent systems for the connecting rods. This difference in the values depends partly on the shape or configuration of the rod in question, but the matter is commonly of minor importance from the practical point of view. It is scarcely necessary to say that the results leading up to the last equation might have been used independently for the purpose of examining the inertia effect of particular parts of the specified engine.

Suppose, for instance, that we wish to find the inertia force H_1 caused in the line of stroke by the mass M situated at the gudgeon-pin. The above results enable us to write

$$H_1 = \frac{M}{g}\ddot{x}$$
$$= \frac{Mr}{g}\{(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)\ddot{\theta} + (\cos\theta + \gamma\cos 2\theta)\dot{\theta}^2\}$$

as the expression for the force; if the crank rotates with uniform angular velocity defined by $\dot{\theta} = \omega$, then $\ddot{\theta} = 0$, and the expression reduces to

$$H_1 \doteq \frac{Mr\omega^2}{g}(\cos\theta + \gamma\cos 2\theta).$$

If, on the other hand, it be required to evaluate the transverse force V_1 produced by the connecting-rod alone, we have shown that

$$V_1 = \frac{M_c(k^2{}_A - al)}{gl}\ddot{\phi},$$

whence, by equation (18.4),

$$V_1 = \frac{M_c(k^2{}_A - al)\gamma}{gl}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad . \quad . \quad (18.7)$$

Hence the magnitude of this force becomes

$$V_1 = \frac{M_c(k^2{}_A - al)\gamma\omega^2}{gl}\sin\theta$$

when the crank is rotating with uniform angular velocity ω, the minus sign being irrelevant in this respect. Moreover, since the distance AC in Fig. 40 is equal to

$$(l\cos\phi + r\cos\theta),$$

or $$\{l(1 - \tfrac{1}{2}\gamma^2\sin^2\theta + \ldots) + r\cos\theta\},$$

it follows that the disturbing couple associated with V_1 in equation (18.7) is given by

$$V_1\{l(1 - \tfrac{1}{2}\gamma^2\sin^2\theta + \ldots) + r\cos\theta\},$$

or

$$\frac{M_c(k^2{}_A - al)\gamma}{gl}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\{l(1 - \tfrac{1}{2}\gamma^2\sin^2\theta + \ldots) + r\cos\theta\},$$

or

$$\frac{M_c(k^2{}_A - al)\gamma}{g}(1 + \gamma\cos\theta)(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

if terms beyond γ^2 be rejected. The magnitude of this couple therefore amounts to

$$\frac{M_c\gamma\omega^2}{g}(k^2{}_A - al)(1 + \gamma\cos\theta)\sin\theta$$

when the crank is revolving at uniform angular velocity ω.

In systems containing several cranks and sets of reciprocating parts, we proceed as in Art. 14(*a*), by taking into consideration the different phase-angles between the cranks.

It will be manifest that the procedure leading to equation (18.6) has the practical disadvantage of having involved a considerable amount of labour in the process of eliminating the reactions that

do not appear in the equation. We shall therefore devote what remains of this chapter mainly to the development and exemplification of a method in which a generalized equation for prescribed mechanisms is obtained without recourse to irrelevant factors. The practical value of this analytical instrument of investigation will be shown to lie in the ease with which it enables us to pass from one co-ordinate of reference to another, by means of a particular formula.

It may make the several applications of the method intelligible if we now examine in outline the fundamental idea of the theory, though Professor E. T. Whittaker's comprehensive treatise [1] on the subject must be referred to for a full discussion on the matter.

19. Generalized Co-ordinates and Lagrange's Equations. While we commonly specify the position of a moving particle at any instant by its co-ordinates (x, y, z) referred to a set of rectangular axes fixed in space, the position may equally well be defined by the values of any three prescribed functions of x, y, z, if from the values in question the corresponding values of x, y, z can be obtained uniquely. These functions may be used as the co-ordinates of the point, in which circumstances the values of x, y, z when expressed explicitly in terms of the functions serve as formulæ in the process of transforming the rectangular co-ordinates into the new system. Illustrations of the operation have been given in the preceding Article, where it was found convenient to state the co-ordinates x and ϕ in terms of the angle or co-ordinate θ. By the aid of the well-known formulæ

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\},\quad \left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\},\quad \left.\begin{aligned} x &= r\cos\phi \\ y &= r\cos\theta\sin\phi \\ z &= r\sin\theta\sin\phi \end{aligned}\right\},$$

it is possible in a like manner to pass from rectangular co-ordinates to polar, cylindrical or spherical co-ordinates, taken in turn.

In what follows we shall confine ourselves to cases where the number of independent co-ordinates or variables necessary to determine the configuration of a system is equal to its number of *degrees of freedom*. Hence the number of variables used in the solution of a problem must be just sufficient to fix the position of the system under examination, since otherwise one co-ordinate could not be varied without affecting the others, which would violate the conditions of motion for a system having a specified number of degrees of freedom.

The 'absolute' system of units will be used in the following discussion, since our present aim is that of examining the analytical aspect of the matter.

[1] *Analytical Dynamics.*

(*a*) *Dynamics of a Particle.* According to Newton's second law the equations of free motion for a particle of mass M, referred to its rectangular co-ordinates (x, y, z), are given by the relations

$$M\ddot{x} = X,\ M\ddot{y} = Y,\ M\ddot{z} = Z,$$

where the components X, Y, Z of the actual forces resolved parallel to the fixed rectangular axes represent the *effective forces* on the particle.

It is frequently desirable at this stage of a problem to be able to express such equations of motion in terms of any other system of co-ordinates, and thus generalize the method used in Art. 18, where $\ddot{\phi}$ and $\ddot{x}$ were written in terms of $\ddot{\theta}$.

Suppose we wish to state the above equations of motion in terms of the three co-ordinates q_1, q_2, q_3, then the first step consists in writing the original co-ordinates (x, y, z) as functions of the new co-ordinates (q_1, q_2, q_3), which may be indicated analytically by

$$x = f_1(q_1, q_2, q_3),\ y = f_2(q_1, q_2, q_3),\ z = f_3(q_1, q_2, q_3).$$

With this notation, on differentiating with respect to time, the expressions for the component velocities $\dot{x}, \dot{y}, \dot{z}$ become

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3,$$

$$\dot{y} = \frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2 + \frac{\partial y}{\partial q_3}\dot{q}_3,$$

$$\dot{z} = \frac{\partial z}{\partial q_1}\dot{q}_1 + \frac{\partial z}{\partial q_2}\dot{q}_2 + \frac{\partial z}{\partial q_3}\dot{q}_3,$$

where $\dot{x}, \dot{y}, \dot{z}$ are explicit functions of $q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3$, linear and homogeneous in $\dot{q}_1, \dot{q}_2, \dot{q}_3$. It may be noted in passing that the quantities $\dot{x}^2, \dot{y}^2, \dot{z}^2$ are consequently homogeneous quadratic functions of $\dot{q}_1, \dot{q}_2, \dot{q}_3$.

Since

$$\frac{d}{dt}\frac{\partial x}{\partial q_1} = \frac{\partial^2 x}{\partial q_1{}^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_1}\dot{q}_2 + \frac{\partial^2 x}{\partial q_3 \partial q_1}\dot{q}_3,$$

and

$$\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial^2 x}{\partial q_1{}^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2 + \frac{\partial^2 x}{\partial q_1 \partial q_3}\dot{q}_3,$$

it is seen that

$$\frac{d}{dt}\frac{\partial x}{\partial q_1} = \frac{\partial \dot{x}}{\partial q_1}, \quad \ldots \ldots \quad (19.1)$$

with similar relations for y and z. It is also evident that

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}, \quad \ldots \ldots \quad (19.2)$$

with similar relations for y and z.

Consider now the work δW_1 done by the effective forces when

the co-ordinate q_1 undergoes an infinitesimal change δq_1 without affecting either q_2 or q_3. If δx, δy, δz are the changes thus produced in x, y, z, respectively, we have

$$\delta W_1 = M(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z);$$

to transform this equation into terms of the new co-ordinates (q_1, q_2, q_3), we can at once write down

$$\delta W_1 = M\left(\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right)\delta q_1.$$

Now the rule for differentiating a product yields

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{d}{dt}\frac{\partial x}{\partial q_1},$$

which leads, on substituting the values given by equations (19.1) and (19.2), to

$$\begin{aligned}\ddot{x}\frac{\partial x}{\partial q_1} &= \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1}\\ &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2}\right) - \frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2}\right),\end{aligned}$$

and similar relations for y and z. Hence, inserting these results in the last expression for δW_1,

$$\delta W_1 = \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1}\right)\delta q_1, \quad . \quad . \quad . \quad . \quad (19.3)$$

where $2T = M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The symbol T evidently denotes the *kinetic energy* of the particle.

Moreover, if $Q_1\delta q_1$ be the work done in the specified displacement of the particle, so that Q_1 is the force involved, it follows that

$$\delta W_1 = Q_1\delta q_1,$$

or, by equation (19.3),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1 \quad . \quad . \quad . \quad . \quad . \quad (19.4)$$

Expressions of the same form would have been derived had the particle described the infinitesimal displacements of δq_2 in q_2, or δq_3 in q_3. The last equation will therefore in general contain q_1, q_2, q_3 and their derivatives with respect to the time, so that it cannot be solved without the help of the other equations in the set. This indicates that T must be expressed in terms of q_1, q_2, q_3 and their time-derivatives before we can form the relation for the work done by the effective forces. The work done by the actual forces, namely $Q_1\delta q_1$, $Q_2\delta q_2$ and $Q_3\delta q_3$, must be obtained direct from an examination of a given problem.

The above results can, in certain circumstances, be applied to problems involving constrained motion of a particle, and these

conditions may be mentioned here. If, on the one hand, the particle is constrained to move on a prescribed *surface*, any two independent specified functions of the rectangular co-ordinates x, y, z may be used as its co-ordinates q_1 and q_2, provided that the rectangular co-ordinates can be uniquely expressed as explicit functions of q_1, q_2 by the aid of the equations for the given surface in rectangular co-ordinates and the equations formed by writing q_1, q_2 equal to their values in terms of x, y, z. If, on the other hand, the particle is constrained to move in a prescribed *path*, any specified function of x, y, z may be used as its co-ordinate q_1, provided that the rectangular co-ordinates can be uniquely expressed as functions of q_1 by means of the two rectangular equations for the path and the equations formed by writing q_1 equal to its value in x, y, z.

(*b*) *Dynamics of Rigid Bodies.* When a system of particles form a rigid body which is capable of describing either free or constrained motion, the sum of the effective forces acting on all the particles is dynamically equivalent to the set of actual forces operating on the system. This may be indicated, in view of the fact that the kinetic energy T of the system is equal to the sum of the kinetic energies of all the particles, by writing

$$2T = \Sigma M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

where the summation sign extends over all the particles.

Suppose now that n degrees of freedom are associated with the rigid body or system when its motion is influenced by constraints which may or may not vary with the time. If the constraints vary with the time, then a set of any n independent variables q_1, q_2, . . ., q_n may be taken as the co-ordinates of the system, provided that the position of every particle in the system is uniquely determined when $q_1, q_2, \ldots, q_n$ and the time are known, and also that $q_1, q_2, \ldots, q_n$ follow explicitly when the positions of all the particles in the system and the time are given. If, on the contrary, the constraints do not vary with the time, a set of any n independent variables $q_1, q_2, \ldots, q_n$ may be taken as the co-ordinates of the system, provided that the position of every particle in the system is uniquely determined when $q_1, q_2, \ldots, q_n$ and the constraints are known, and also that $q_1, q_2, \ldots, q_n$ follow explicitly when the positions of all the particles are given.

The expressions defining the constraints in terms of the rectangular co-ordinates for the particles and of the co-ordinates $q_1, q_2, \ldots, q_n$ represent the *geometrical relations* of the system, which may or may not contain the time explicitly. The geometrical relations must not, however, contain explicitly either the time-derivatives of the rectangular co-ordinates for the particles or those of the co-ordinates $q_1, q_2, \ldots, q_n$ unless they can by integration be

freed from these derivatives. If the geometrical relations do not contain the time explicitly they enable us to express the co-ordinates x, y, z explicitly as functions of $q_1, q_2, \ldots, q_n$. On the other hand, if these relations contain the time explicitly they afford a means of writing the co-ordinates x, y, z of every particle in the system as functions of $t, q_1, q_2, \ldots, q_n$.

It has already been shown that the formula (19.4) is applicable in circumstances where the geometrical relations do not contain the time t explicitly, so that there remains to be examined the case where the constraints vary with t. For this purpose consider the work δW_1 done by the effective forces when the co-ordinate q_1 undergoes the infinitesimal change δq_1 without affecting the other co-ordinates. Here we must write

$$x = f(t, q_1, q_2, \ldots, q_n),$$

whence, on differentiating with respect to the time,

$$\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial x}{\partial q_n}\dot{q}_n,$$

which is an explicit function of $t, q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. Further differentiation yields

$$\frac{d}{dt}\frac{\partial x}{\partial q_1} = \frac{\partial^2 x}{\partial t \partial q_1} + \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_1}\dot{q}_2 + \ldots + \frac{\partial^2 x}{\partial q_n \partial q_1}\dot{q}_n,$$

$$\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial^2 x}{\partial q_1 \partial t} + \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2 + \ldots + \frac{\partial^2 x}{\partial q_1 \partial q_n}\dot{q}_n,$$

$$\frac{\partial x}{\partial q_1} = \frac{\partial \dot{x}}{\partial \dot{q}_1},$$

so that, on repeating the foregoing procedure, we obtain

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial \dot{x}}{\partial q_1}.$$

Proceeding in this manner, and summing the effect of all the particles in the system, it will be found that

$$\begin{aligned}\delta W_1 &= \Sigma M\left(\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right)\delta q_1 \\ &= \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1}\right\}\delta q_1,\end{aligned}$$

since

$$\begin{aligned}\ddot{x}\frac{\partial x}{\partial q_1} &= \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) \\ &= \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1} \\ &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2}\right) - \frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2}\right).\end{aligned}$$

Finally, if $Q_1\delta q_1$ denote the work done by the forces in changing q_1 by the infinitesimal amount δq_1, we can equate $Q_1\delta q_1$ to the last expression for δW_1, and thus derive

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1.$$

This result demonstrates that the formula (19.4) holds even when the constraints vary with the time.

The n independent variables $q_1, q_2, \ldots, q_n$ are called the *generalized co-ordinates* of the system, and their time-rates of variation $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ are the corresponding *generalized components of velocity*.

In systems having n degrees of freedom the expression for the kinetic energy T may accordingly be written in the form

$$\begin{aligned} 2T &= \Sigma M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= a_{11}\dot{q}_1{}^2 + a_{22}\dot{q}_2{}^2 + \ldots + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots, \end{aligned} \qquad (19.5)$$

where

$$\left.\begin{aligned} a_{rr} &= \Sigma M\left\{\left(\frac{\partial x}{\partial q_r}\right)^2 + \left(\frac{\partial y}{\partial q_r}\right)^2 + \left(\frac{\partial z}{\partial q_r}\right)^2\right\}, \\ a_{rs} &= \Sigma M\left(\frac{\partial x}{\partial q_r}\cdot\frac{\partial x}{\partial q_s} + \frac{\partial y}{\partial q_r}\cdot\frac{\partial y}{\partial q_s} + \frac{\partial z}{\partial q_r}\cdot\frac{\partial z}{\partial q_s}\right) = a_{sr}, \end{aligned}\right\} \qquad (19.6)$$

the process of summation being effected by assigning the values 1, 2, 3, ..., n in turn to the suffixes r and s. It is to be noticed that T is a homogeneous quadratic function of the generalized components of velocity. Moreover, the symbols a_{rr}, a_{rs}, representing the *coefficients of inertia* for the system, are seen to be in general functions of the co-ordinates $q_1, q_2, \ldots, q_n$, so that they usually vary with the configuration of the system. The coefficients a_{rr}, a_{rs} are subject to certain algebraic limitations, for an examination of which the reader may be referred to another source[1] of information, as the matter does not commonly enter into practical problems.

To define another quantity, we have, by equation (19.5),

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_r} &= \Sigma M\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_r} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_r} + \dot{z}\frac{d\dot{z}}{\partial \dot{q}_r}\right) \\ &= \Sigma M\left(\dot{x}\frac{\partial x}{\partial q_r} + \dot{y}\frac{\partial y}{\partial q_r} + \dot{z}\frac{\partial z}{\partial q_r}\right), \end{aligned}$$

since it has been shown that the relation $\dfrac{\partial \dot{x}}{\partial \dot{q}_r} = \dfrac{\partial x}{\partial q_r}$ holds for all the n co-ordinates. Consequently, if we write

$$\frac{\partial T}{\partial \dot{q}_r} = p_r, \qquad (19.7)$$

then the symbol p_r denotes the *generalized component of momentum*.

[1] H. Lamb, *Higher Mechanics*, page 182, second edition.

This may easily be verified in the case of a single particle, when

$$p_x = \frac{\partial T}{\partial \dot{x}}$$
$$= M\dot{x},$$

with corresponding relations for the y- and z-components of the momentum.

Our procedure has led from the case of the single particle implied in equation (19.4) to the complex system involved in equation (19.5), for which the motion is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.8)$$

This formula yields n independent equations of motion for a system having n degrees of freedom when the values 1, 2, 3, ..., n are in turn assigned to the suffix r. We thus obtain n ordinary differential equations of the second order, in which $q_1, q_2, \ldots, q_n$ are the dependent variables and t is the independent variable. It is evident that these equations are sufficient to determine the motion when the initial circumstances are known, because the number of differential equations is equal to the number of dependent variables.

The symbol Q_r in the last expression represents the *generalized component of force*. In calculating the value of any one of these components we may, of course, neglect all the forces that on the whole do no work, such as, for example, the internal forces of a rigid body, or the reactions between smooth surfaces in contact. If we wish to determine the latter forces, it is only necessary to suppose the constraints removed and the number of degrees of freedom correspondingly increased, when substitution of the forces associated with the constraints gives a complete set of equations based on the new assumptions.

J. L. Lagrange[1] first formulated the expression (19.8), and it occupies an outstanding position in the general subject of dynamics on account of the fact that the formula is applicable to all systems of finite freedom.

In order to point out the general character of the last equation, let T denote the kinetic energy of the moving parts of the engine examined in Art. 18 and indicated by Fig. 42. By Lagrange's formula we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_x$$

for the inertia force in the line of stroke,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = Q_y$$

[1] *Méchanique Analytique* (1788).

for the inertia force perpendicular to the line of stroke, and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

for the related torque on the crankshaft. These results show that the generalized component of velocity $\dot{q}_r$ in equation (19.8) may denote either linear or angular velocities, and that the generalized component of momentum may refer to momentum or to moment of momentum. An application of the theory of dimensions to a given equation of motion readily removes any ambiguity in this respect, but the operation is usually unnecessary, for a slight acquaintance with mechanics suffices to show that the Q_r-terms refer to forces when linear velocities are involved, and to torques or couples when angular velocities are present in the equation.

(*c*) *Conservative Forces.* If X, Y, Z are the components of the forces acting on a moving particle referred to its rectangular co-ordinates (x, y, z), the work W done by the forces in displacing the particle from the point P_1 having co-ordinates (x_1, y_1, z_1) to the point P_2 having co-ordinates (x_2, y_2, z_2) is given by the relation

$$W = \int_{P_1}^{P_2} (X\,dx + Y\,dy + Z\,dz).$$

As each of the quantities X, Y, Z is in general a function of the three variables x, y, z, the path described by the particle must be known before W can be evaluated. We may suppose this to be so by writing

$$f_1(x, y, z) = 0,\ f_2(x, y, z) = 0$$

as the equations of the path in question, when the method of successive elimination enables us to express y explicitly in terms of x, and z explicitly in terms of x. Substituting the resulting values for y and z in the expression for X, we obtain X as a function of the single variable x, when $\int_{x_1}^{x_2} X\,dx$ may be found by a simple quadrature; a repetition of the process with respect to Y and Z leads further to the values of $\int_{y_1}^{y_2} Y\,dy$ and $\int_{z_1}^{z_2} Z\,dz$. The work done in the prescribed displacement is equal to the sum of these three integrals.

If, however, $(X\,dx + Y\,dy + Z\,dz)$ is an 'exact differential', it follows that there is a function $U = F(x, y, z)$ such that

$$\frac{\partial U}{\partial x} = X,\ \frac{\partial U}{\partial y} = Y,\ \frac{\partial U}{\partial z} = Z.$$

Hence, since

$$\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz,$$

or $X\,dx + Y\,dy + Z\,dz$ is the complete differential of the function, we have

$$\int (X\,dx + Y\,dy + Z\,dz) = F(x, y, z)$$
$$= U,$$
$$W = \int_{P_1}^{P_2} (X\,dx + Y\,dy + Z\,dz) = F(x_2, y_2, z_2) - F(x_1, y_1, z_1)$$
$$= U_2 - U_1.$$

When the forces are such that the function $U = F(x, y, z)$ exists, they are known as *conservative forces*, and U is called the *force function*. Our result thus shows that the work done by conservative forces on the particle is independent of the path, being equal to the difference between the values of the force function for the final and initial positions.

To demonstrate that this applies also to a system of n particles, consider the rth particle of the system; let (x_r, y_r, z_r) specify its position, and X_r, Y_r, Z_r the corresponding components of force acting on the particle. Then the work W done in displacing the system from the point P_1 to P_2 is given by

$$W = \Sigma \int_{P_1}^{P_2} (X_r\,dx_r + Y_r\,dy_r + Z_r\,dz_r),$$

where the summation sign Σ extends over all the particles. If there is a function $U = F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n)$ such that

$$\frac{\partial U}{\partial x_r} = X_r, \quad \frac{\partial U}{\partial y_r} = Y_r, \quad \frac{\partial U}{\partial z_r} = Z_r,$$

it is known that $\Sigma(X_r\,dx_r + Y_r\,dy_r + Z_r\,dz_r)$ is an 'exact differential', with $U = F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n)$ as its indefinite integral. Here U is the force function involved, and the work done by the forces on the system in moving it between the initial and final configurations is equal to the value of the force function for the final position *minus* its value for the initial position; that is, the work done is independent of the path described by the system.

In any system subjected to conservative forces the negative of the force function U represents the *potential energy* of the system, and if it is acted on by frictional agencies that may be neglected, the potential energy is equivalent to the related *strain energy* V.

When the geometrical relations for a system do not involve the time t, in either the force function or the potential energy, we can substitute for the rectangular co-ordinates of a specified set of particles their values in terms of the generalized co-ordinates $q_1, q_2, \ldots, q_n$, and in this manner write U and V in terms of the generalized co-ordinates. This leads to an expression for U by

means of which we can write δU_r for the work done by the force Q_r in changing q_r by the infinitesimal amount δq_r, whence, on equating,

$$Q_r\delta q_r = \frac{\partial U}{\partial q_r}\delta q_r$$
$$= -\frac{\partial V}{\partial q_r}\delta q_r,$$

because δU_r is approximately equal to $\frac{\partial U}{\partial q_r}\delta q_r$ or $-\frac{\partial V}{\partial q_r}\delta q_r$. Consequently

$$Q_r = -\frac{\partial V}{\partial q_r},$$

so that we may substitute this value for Q_r in equation (19.8) and obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r} \quad . \quad . \quad . \quad . \quad (19.9)$$

as the equation for motion under these conditions. This is the fundamental equation in the general theory of vibrations discussed in Chap. III.

(*d*) *Rotating Systems.* This type of system deserves mention on account of the fact that engineering practice presents us with problems pertaining to structures that rotate about fixed axes, such as, for example, the wheels of turbines, and the blades of airscrews.

To examine the subject with reference to rigid bodies, consider

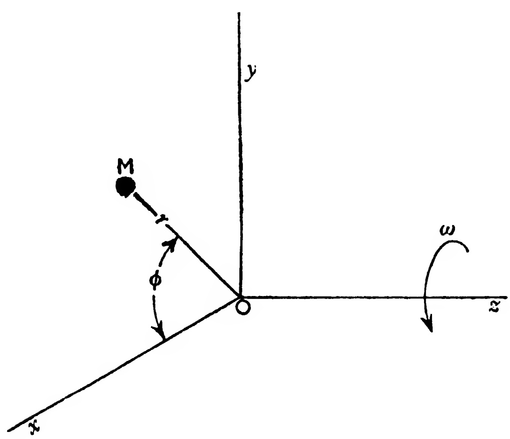


FIG. 43.

a system rotating with uniform angular velocity ω about the z-axis, where n degrees of freedom are supposed to be involved independent of the velocity ω. Take Fig. 43 to represent a particle of mass M in the system, where the position of M at a given

instant is defined by (x, y, z) in rectangular co-ordinates, and by (r, ϕ) in cylindrical co-ordinates. The preceding analysis shows that the latter co-ordinates can be stated in terms of n co-ordinates $q_1, q_2, \ldots, q_n$, these expressions being independent of the time.

Making the substitutions

$$x = r\cos\phi,\ y = r\sin\phi,\ z = z$$

in equation (19.5), and summing for all the particles of the system, on the assumption that the effective forces depend only on the co-ordinates $q_1, q_2, \ldots, q_n$, we find

$$2T = \Sigma M\{\dot{r}^2 + r^2(\dot{\phi} + \omega)^2 + \dot{z}^2\} \quad . \quad . \quad . \quad (19.10)$$

for the kinetic energy T of the system when rotating with uniform angular velocity ω about the fixed axis, and

$$2T_0 = \Sigma M(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)$$

for the kinetic energy T_0 of the system when $\omega = 0$.

Now the quantity ΣMr^2 will be a known function of $q_1, q_2, \ldots, q_n$. Also, the quantity $\Sigma Mr^2\dot{\phi}$ will be a known function of $q_1, q_2, \ldots, q_n$, $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$, being linear in the velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$; it will be zero in the expression for T_0 if the motion of every particle has no component in the direction of ϕ increasing; it will, on the other hand, be a perfect differential with respect to t of a function of the single co-ordinate q involved when n is equal to unity. We may combine these two practical and important cases by assuming that $\Sigma Mr^2\dot{\phi}$ is of the form $\frac{dF}{dt}$, where F is a known function of the co-ordinates $q_1, q_2, \ldots, q_n$.

In this notation, with $2W$ written for ΣMr^2, equation (19.10) becomes

$$T = T_0 + \omega\frac{dF}{dt} + \omega^2 W.$$

Moreover, if $(Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q_n\delta q_n)$ be the work done in an arbitrary infinitesimal displacement by the external forces $Q_1, Q_2, \ldots, Q_n$, from equation (19.8) we have

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{q}_r}\right) + \frac{d}{dt}\left(\omega\frac{\partial F}{\partial q_r}\right) - \frac{\partial T_0}{\partial q_r} - \omega\frac{d}{dt}\left(\frac{\partial F}{\partial q_r}\right) - \omega^2\frac{\partial W}{\partial q_r} = Q_r,$$

or

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{q}_r}\right) - \frac{\partial T_0}{\partial q_r} = -\frac{\partial}{\partial q_r}(-\omega^2 W) + Q_r$$

$$= -\frac{\partial}{\partial q_r}(-\omega^2 W) - \frac{\partial V}{\partial q_r}, \quad . \quad (19.11)$$

by equation (19·9), where $r = 1, 2, \ldots, n$.

It is thus seen that we can in the given circumstances investigate the motion of the rotating system by assuming the angular velocity

ω to be zero and adding the term $-\frac{1}{2}\Sigma Mr^2\omega^2$ to the potential energy V in equation (19.9). This modification of the potential energy offers a means of treating systems constrained to rotate about a fixed axis as if the rotation were absent.

(*e*) *Dynamical Similitude.* The process of deriving information with the help of models of given mechanisms and structures implies a knowledge of the relations between the ratios of the masses and forces referred to the model on the one hand, and to the actual structure on the other.

In terms of the ratio model : actual structure, let us write

$l : 1$ for the ratio of the linear dimensions,

$m : 1$ for the ratio of the masses,

$w : 1$ for the ratio of the rates of working,

$f : 1$ for the ratio of the forces.

Now if X be the force acting on each particle of mass M, its equation of motion is of the form

$$M\ddot{x} = X,$$

hence if M is altered in the ratio $m : 1$, $\ddot{x}$ is accordingly altered in the ratio $lw^2 : 1$, and X in the ratio $f : 1$. It is thus seen that

$$f = lmw^2$$

is the relation which must be satisfied to ensure dynamical similarity between the actual structure and its model.

For example, if the forces in question are due to gravity, then $f = m$, and $lw^2 = 1$. In this case, therefore, the rates of working vary inversely as the square root of the linear dimensions.

If $f = -1$, to take another particular case, we have the model acted on by reversed forces measured in the direction of the actual forces; this procedure is sometimes followed in tests with models, and notably in the case of bridges. It is always possible, at least as a first approximation, to suppose that a structure is subjected to constraints which are independent of the time, and to forces that depend only on the configuration of the structural system. The motion of such a system having n degrees of freedom can then be determined by Lagrange's formula (19.8), namely

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r,$$

with $r = 1, 2, 3, \ldots, n$; as previously, the kinetic energy T for the system is a homogeneous quadratic function of the generalized velocities, involving the co-ordinates in any way.

For analytical purposes now write

$$t' = it$$

in Lagrange's equation, where $i = \sqrt{-1}$, and denote differentiation

with respect to t' by accents. Since $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)$ and $\frac{\partial T}{\partial q_r}$ are homogeneous of degree -2 in dt, in the new notation the equation becomes

$$\frac{d}{dt'}\left(\frac{\partial T_1}{\partial q_r'}\right) - \frac{\partial T_1}{\partial q_r} = -Q_r,$$

where $r = 1, 2, \ldots, n$, and T_1 is the same function of $q_1', q_2', \ldots q_n', q_1, q_2, \ldots, q_n$ that T is of $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, q_1, q_2, \ldots, q_n$.

The n expressions given by the last equation may clearly be taken as the equations of motion for the model with reversed forces, provided that t' be regarded as denoting the time. Also, if $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2 \ldots, \beta_n$ represent in turn the initial values of $q_1, q_2, \ldots, q_n, \dot{q}_1\, \dot{q}_2, \ldots, \dot{q}_n$ in a specified motion of the actual system, it follows that $\alpha_1, \alpha_2, \ldots, \alpha_n, -i\beta_1, -i\beta_2, \ldots, -i\beta_n$ will be the corresponding quantities for the model with reversed forces. Hence we have, on generalizing, the theorem: *In any dynamical system subjected to constraints independent of the time and to forces that depend only on the configuration, 'real' integrals of the equations of motion are obtained even when* $\sqrt{-1}t$ *is substituted for* t *and* $-\sqrt{-1}\beta_1, -\sqrt{-1}\beta_2, \ldots, -\sqrt{-1}\beta_n$ *for the initial velocities* $\beta_1, \beta_2, \ldots, \beta_n$, *respectively*. Hence in the case of models with reversed forces the procedure yields equations which represent the motion of the actual structures concerned.

(*f*) *Impulsive Forces*. Simple algebraic methods usually suffice for the solution of problems which involve the action of impulsive forces on rigid bodies, because this type of motion does not depend on the integration of differential equations. We can, in fact, treat many questions which come under this heading as simple problems in *maxima* and *minima*, by means of two theorems which will now be stated.

It is to be noted, first, that if a particle initially at rest in the position defined by the co-ordinates (x, y, z) be displaced to the position $(x + \delta x, y + \delta y, z + \delta z)$, we may write $\delta x = u_1\delta t$, $\delta y = v_1\delta t$, $\delta z = w_1\delta t$ and regard the displacement as being effected in the time δt by imposing on the particle a velocity having components u_1, v_1, w_1 parallel respectively to the axes of x, y, z. Secondly, if the particle is initially in motion with a velocity having the components u, v, w parallel to the same axes, the given displacement could be brought about by imposing on the particle an additional velocity defined by the components $u_1 - u$, $v_1 - v$, $w_1 - w$ parallel to the respective axes.

Pass now to a system of particles subjected to a set of impulsive forces, and let Q_x, Q_y, Q_z be the components of the given forces which act on a representative particle of mass M in such a manner

that its components of velocity before and after the impact are u, v, w and u_1, v_1, w_1, respectively.

In any infinitesimal change of the system which results in the co-ordinates of the representative particle being altered by amounts equal to δx, δy, δz, summation over the complete system of particles shows that

$$\Sigma M\{(u_1-u)\delta x+(v_1-v)\delta y+(w_1-w)\delta z\}=\Sigma(Q_x\delta x+Q_y\delta y+Q_z\delta z),$$

since the virtual work done by the effective forces is equal to that done by the actual forces.

If the particle were at rest and u_2, v_2, w_2 were the components of velocity which would have to be imposed on it to bring about the assumed displacement in the interval of time δt, we have in the above equation $\delta x = u_2\delta t$, $\delta y = v_2\delta t$, $\delta z = w_2\delta t$, whence

$$\Sigma M\{(u_1 - u)u_2 + (v_1 - v)v_2 + (w_1 - w)w_2\}$$
$$=\Sigma(u_2Q_x + v_2Q_y + w_2Q_z) \quad . \quad . \quad (19.12)$$

Two cases deserve special notice here. (i) When the impulse is such that the initial motion is continued after the impact, then $u_2 = u$, $v_2 = v$, $w_2 = w$, therefore

$$\Sigma M\{(u_1 - u)u + (v_1 - v)v + (w_1 - w)w\}$$
$$= \Sigma(uQ_x + vQ_y + wQ_z) \quad . \quad . \quad (19.13)$$

(ii) In the actual motion brought about by the impulsive forces, we have $u_2 = u_1$, $v_2 = v_1$, $w_2 = w_1$ in equation (19.12), so that

$$\Sigma M\{(u_1 - u)u_1 + (v_1 - v)v_1 + (w_1 - w)w_1\}$$
$$= \Sigma(u_1Q_x + v_1Q_y + w_1Q_z) \quad . \quad . \quad (19.14)$$

To deduce the first of the theorems in question, suppose the system to start from rest, in which circumstance $u = v = w = 0$ in equations (19.12) and (19.14). The expressions then reduce to

$$\Sigma M(u_1u_2 + v_1v_2 + w_1w_2) = \Sigma(u_2Q_x + v_2Q_y + w_2Q_z), \quad . \quad (19.15)$$
$$\Sigma M(u_1^2 + v_1^2 + w_1^2) \quad . \quad = \Sigma(u_1Q_x + v_1Q_y + w_1Q_z). \quad . \quad (19.16)$$

But the first of these expressions is identically

$$\tfrac{1}{2}\Sigma M[(u_1^2 + v_1^2 + w_1^2) + (u_2^2 + v_2^2 + w_2^2)$$
$$-\{(u_2 - u_1)^2 + (v_2 - v_1)^2 - (w_2 - w_1)^2\}];$$

also, on subtracting equation (19.16) from (19.15),

$$\tfrac{1}{2}\Sigma M[(u_2^2 + v_2^2 + w_2^2) - (u_1^2 + v_1^2 + w_1^2)$$
$$- \{(u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2\}]$$
$$= \Sigma\{(u_2 - u_1)Q_x + (v_2 - v_1)Q_y + (w_2 - w_1)Q_z\} \quad . \quad . \quad (19.17)$$

Therefore if u_2, v_2, w_2 are the components of velocity of the representative particle M in any possible motion of the system which could result in the points of application of the impulsive forces having the same velocities as in the actual motion, the second

member of equation (19.17) is zero, and we have *Thomson's theorem* [1]: *If any number of points on a system are suddenly set in motion with given velocities, the kinetic energy of the resulting motion is less than that of any other kinematically possible motion which the system can have with the given velocities, the difference being the energy of the motion which must be compounded with either to produce the other.*

With reference to the second theorem, let P_x, P_y, P_z be the components of the impulsive force which must act on the representative particle M in order to change its component velocities from u, v, w to u_2, v_2, w_2. Under these conditions equation (19.14) gives

$$\Sigma M\{(u_2 - u)u_2 + (v_2 - v)v_2 + (w_2 - w)w_2\} \\ = \Sigma(u_2P_x + v_2P_y + w_2P_z)\,; \quad . \quad . \quad (19.18)$$

subtracting this from equation (19.12) we find

$$\Sigma M\{(u_1u_2 + v_1v_2 + w_1w_2) - (u_2^2 + v_2^2 + w_2^2)\} \\ = \Sigma\{u_2(Q_x - P_x) + v_2(Q_y - P_y) + w_2(Q_z - P_z)\}, \quad . \quad . \quad (19.19)$$

the left-hand side of which is seen to be identically

$$\tfrac{1}{2}\Sigma M[(u_1^2 + v_1^2 + w_1^2) - (u_2^2 + v_2^2 + w_2^2) \\ - \{(u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2\}].$$

If the Q-forces differ from the P-forces only by the impulsive actions and reactions due to the introduction into the original system of additional constraints that do no work in the assumed motion, then the second member in equation (19.19) is zero, and we have *Bertrand's theorem* [2]: *If a given set of impulses is applied to different points on any system in motion, the kinetic energy of the resulting motion is greater than the kinetic energy of the motion which the system would have under the action of the same impulses and constraints and of any additional constraints due to the reactions of perfectly smooth or perfectly rough fixed surfaces, or rigid connections between particles of the system.*

When Bertrand's theorem is applied to cases where the forces are continuous, but not impulsive, it will be found that the increase of kinetic energy in unit time is diminished by the introduction of new constraints that do not influence the potential energy.

It may be noticed in connection with the principle of 'least constraint' enunciated by C. F. Gauss [3] that the measure of the 'constraint' is practically equivalent to the kinetic energy of the motion which, combined with the motion that the system would have if all the constraints were removed, would result in the actual motion. It follows from Bertrand's theorem that the constraint

[1] Lord Kelvin and P. G. Tait, *Treatise on Natural Philosophy*, Part I, page 294.

[2] Due to J. C. F. Sturm, *Comptes Rendus*, vol. 13, page 1046 (1841).

[3] *J. für Math.* (*Crelle*), vol. IV, page 232 (1829); or *Werke*, vol. 5, page 23.

thus defined is less than in any assumed motion acquired by the introduction of additional constraining forces.[1]

Lord Rayleigh has pointed out [2] that the theorems of Thomson and Bertrand are both included in the statement that the introduction of new constraints makes for increase in the inertia, or moment of intertia, of a system. Another connection between these theorems has been discovered by Professor G. I. Taylor,[3] who found the reduction in energy brought about by constraints in the Bertrand case to be *less* than the increase brought about by the same constraints in the Thomson case.

The following applications of Lagrange's method to the problem of balancing engines in particular, and mechanisms in general, will serve to elucidate various points in the preceding analysis.

20. Dynamical Equivalent of the Reciprocating Parts of an Engine. Consider a steam engine of the type examined in Art. 18, and let:

M = weight of the reciprocating parts concentrated at the gudgeon- or crosshead-pin;

M_c = weight of the connecting-rod, shown in Fig. 40;

k_A = radius of gyration of the connecting-rod about the gudgeon-pin;

$$\gamma = \frac{\text{throw of crank } (r)}{\text{length of connecting-rod } (l)}.$$

Since the kinetic energy T of the mechanism under consideration is equal to half the sum of the products of mass and square of the velocity for each reciprocating part, we can at once write down, from an inspection of Fig. 40 and the known data,

$$2T = \frac{1}{g}\left\{M\dot{x}^2 + \left(I_c + M_c\frac{a}{l}r^2\right)\dot{\theta}^2 + M_c(k^2{}_A - al)\dot{\phi}^2\right\} \quad . \quad (20.1)$$

in terms of the previous notation.

Taking the crank-angle θ to be the most convenient co-ordinate of reference for the present purpose, we substitute θ for q_r in Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r$$

and thus obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta', \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.2)$$

where Q_θ' represents the value of the torque-reaction due to the reciprocating parts. It is next necessary to introduce the geometrical

[1] See also E. J. Routh, *Elementary Rigid Dynamics*, Art. 391.

[2] *Theory of Sound*, vol. I, page 100.

[3] *Proc. Lond. Math. Soc.*, vol. 21, page 413 (1923).

relations evaluated in Art. 18 by way of expressing the variables x and ϕ in terms of θ and its derivatives, with a view to deriving an equation of the form

$$2T = a_{11}\dot{\theta}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.3)$$

where a_{11} denotes the appropriate coefficient of inertia for the mechanism. Since

$$\frac{\partial T}{\partial \dot{\theta}} = a_{11}\dot{\theta},\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = a_{11}\ddot{\theta} + \frac{da_{11}}{d\theta}\dot{\theta}^2,\ \frac{\partial T}{\partial \theta} = \tfrac{1}{2}\frac{da_{11}}{d\theta}\dot{\theta}^2$$

we have, on inserting these values in the formula (20.2),

$$a_{11}\ddot{\theta} + \tfrac{1}{2}\frac{da_{11}}{d\theta}\dot{\theta}^2 = Q_\theta', \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.4)$$

the ordinary sign of differentiation being used because the relation for T is now supposed to be written in terms of θ.

To obtain the value of a_{11}, the geometrical relations give

$$\dot{x} = r\dot{\theta}(\sin\theta + \tfrac{1}{2}\gamma \sin 2\theta) \text{ and } \dot{\phi} = \gamma\dot{\theta}\cos\theta,$$

hence, on making these substitutions in equation (20.1),

$$a_{11} = \frac{1}{g}\Big\{(I_c + M_c\gamma ra) + M_c\gamma^2(k^2{}_A - al)\cos^2\theta + Mr^2(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)^2\Big\} \quad . \quad . \quad (20.5)$$

Consequently

$$\frac{da_{11}}{d\theta} = \frac{1}{g}\{-M_c\gamma^2(k^2{}_A - al)\sin 2\theta + 2Mr^2(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)(\cos\theta + \gamma\cos 2\theta)\}, \quad . \quad . \quad (20.6)$$

to the implied degree of approximation.

Turning now to the corresponding relation for the torque-reaction, let $\mathfrak{T}_m$ be its mean value taken over a working cycle, then

$$\mathfrak{T}_m = \int_0^{2\pi} Q_\theta' d\theta.$$

On integrating this expression between the given limits, it will be found that the terms containing $\sin\theta\cos\theta$ disappear, and that the relation reduces to

$$\mathfrak{T}_m = \int_0^{2\pi} a_{11}\ddot{\theta} d\theta.$$

By equation (20.4)

$$\mathfrak{T}_m = \frac{1}{2g}\Big\{I_c + M_c\gamma ra + M_c\gamma^2(k^2{}_A - al) + Mr^2(1 + \tfrac{1}{4}\gamma^2)\Big\}\ddot{\theta},$$

hence the inertia- or flywheel-effect of the reciprocating parts is equivalent to that produced by a weight of

$$\frac{1}{2gr^2}\Big\{M_c\gamma ra + M_c\gamma^2(k^2{}_A - al) + Mr^2(1 + \tfrac{1}{4}\gamma^2)\Big\}$$

concentrated at the crank-pin. The result indicates the extent to

which the mass of the reciprocating parts augments that of the flywheel, to the present order of approximation.

21. Motion of Reciprocating Engines. We shall investigate this problem with reference to the engine implied in the previous Article, on the assumption that the frictional agencies may be neglected for the present, and that the crank-webs are initially balanced.

As all the principal forces are to be included in the solution, the foregoing specification must be extended accordingly, and this will be effected by writing:

I_c = moment of inertia about the axis of rotation for the crank-shaft, flywheel and the machinery connected with it;
P = force exerted by the working fluid on the piston;
$\mathfrak{T}$ = resisting torque on the crankshaft corresponding to the crank-angle θ.

It will easily be seen in view of the foregoing results that the expression for the kinetic energy T of the prescribed mechanism is

$$2T = \frac{1}{g}\left\{M\dot{x}^2 + \left(I_c + M_c\frac{a}{l}r^2\right)\dot{\theta}^2 + M_c(k^2{}_A - al)\dot{\phi}^2\right\} \quad (21.1)$$

Our aim is, as before, that of arranging this relation in the form

$$2T = a_{11}\dot{\theta}^2,$$

and by this means expressing Lagrange's formula in the form

$$a_{11}\ddot{\theta} + \frac{1}{2}\frac{da_{11}}{d\theta}\dot{\theta}^2 = Q_\theta \quad . \quad . \quad . \quad . \quad . \quad (21.2)$$

for an engine having a_{11} as its coefficient of inertia, where Q_θ accordingly indicates the sum of the external forces acting on the system.

On inserting the above mentioned geometrical relations for $\dot{x}$ and $\dot{\phi}$ in equation (21.1), it will be found that the coefficient of inertia

$$a_{11} = \frac{1}{g}\{(I_c + M_c\gamma ra) + M_c\gamma^2(k^2{}_A - al)\cos^2\theta$$
$$+ Mr^2(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)^2\} \quad . \quad . \quad (21.3)$$

The quantity $\frac{da_{11}}{d\theta}$ is therefore given by equation (20.6).

Regard being had to the preceding remarks concerning the external forces associated with engines, it follows that here

$$Q_\theta = Pr(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \mathfrak{T}.$$

This quantity can be expressed in the form of a Fourier series, with the help of the indicator diagram and a corresponding record taken with a torsiometer.

Hence, on substituting the above values of a_{11}, $\dfrac{da_{11}}{d\theta}$ and Q_θ in equation (21.2), we have

$$\frac{1}{g}\{(I_c+M_c\gamma ra)+M_c\gamma^2(k^2_A-al)\cos^2\theta+Mr^2(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta)^2\}\ddot{\theta}$$
$$+\frac{1}{g}\left\{Mr^2(\cos\theta+\gamma\cos 2\theta)(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta)-\frac{M_c\gamma^2}{2}(k^2_A-al)\sin 2\theta\right\}\dot{\theta}^2$$
$$= Pr(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta)-\mathfrak{T} \quad . \quad . \quad (21.4)$$

as the required equation of motion. The present method has thus led to an expression which is the same as equation (18.6), though different procedures have been followed in the two cases.

Now the last equation is obviously of the form

$$f(\theta)\ddot{\theta}+\tfrac{1}{2}f'(\theta)\dot{\theta}^2+F(\theta)=0, \quad . \quad . \quad . \quad . \quad (21.5)$$

where $f'(\theta)$ represents $\dfrac{d}{d\theta}f(\theta)$, and $f(\theta)$ and $F(\theta)$ are known functions of θ for a given engine. This class of differential equation is examined in treatises devoted to the subject,[1] which the reader must consult for a complete discussion on the matter. We may, however, observe that while it is not in general possible to obtain a solution giving θ explicitly in terms of the time t, it is nevertheless practicable to derive a solution in the form

$$t=\pm\int\sqrt{\frac{f(\theta)}{c_1-2\int F(\theta)d\theta}}d\theta+c_2, \quad . \quad . \quad (21.6)$$

where c_1 and c_2 represent constants which depend on the initial conditions of motion. A first approximation to the solution may be obtained by neglecting all but the first of the harmonic terms in $\int F(\theta)d\theta$ which enter by way of the Fourier series that relates to the terms P and $\mathfrak{T}$. The method of successive approximation then affords a means of including a number of the harmonic terms in the solution, but the first approximation suffices for many practical purposes.

It is scarcely necessary to add that due account must be taken of the working cycle for the engine in question, since for a two-stroke cycle the integral $\int F(\theta)d\theta$ in the last equation becomes

$\displaystyle\int_0^{2\pi}\{Pr(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta)-\mathfrak{T}\}d\theta$, compared with

$$\int_0^{4\pi}\{Pr(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta)-\mathfrak{T}\}d\theta$$

for a four-stroke cycle.

In the special case where the crank of the engine rotates with

[1] See A. R. Forsyth, *Differential Equations*, Chap. IV, fourth edition.

uniform angular velocity ω, we have $\ddot{\theta} = 0$ in equation (21.4), and therefore

$$\frac{\omega^2}{g}\left\{Mr^2(\cos\theta+\gamma\cos 2\theta)(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta)-\frac{M_c\gamma^2}{2}(k^2_A-al)\sin 2\theta\right\}$$
$$= Pr\,(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \mathfrak{T}.$$

It is of practical interest to notice here that the inertia effect of the rotating parts implied in the term I_c has vanished, as showing that the flywheel and other rotating parts do not affect the motion of an engine so long as its speed remains constant.

Moreover, if we neglect the effect on the motion of the couple produced by the connecting-rod, the last equation becomes

$$\frac{Mr^2\omega^2}{g}(\cos\theta+\gamma\cos 2\theta)(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta) = Pr(\sin\theta+\tfrac{1}{2}\gamma\sin 2\theta)-\mathfrak{T},$$

so that the corresponding torque-reaction

$$\mathfrak{T}_1 = r(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)\left\{P \pm \frac{Mr\omega^2}{g}(\cos\theta + \gamma\cos 2\theta)\right\},$$

where, for reasons already stated, the *plus* and *minus* signs apply to vertical engines.

In cases where the engine under consideration contains a number of cylinders and sets of reciprocating parts, on applying our method and taking account of the phase angles between the cranks, it will be found that certain of the cos θ and sin θ terms cancel. For example, if the engine be a six-cylinder type having similar masses associated with each of the cylinders, we obtain, by equation (21.4),

$$\frac{1}{g}\{I_c + 6M_c\gamma ra + 12M_c\gamma^2(k^2_A - al) + 9Mr^2(1 - \gamma + \tfrac{1}{4}\gamma^2)\}\ddot{\theta}$$
$$= Pr(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \mathfrak{T}.$$

Hence if $B\sin(pt + \alpha)$ denote the first term of the Fourier series for the right-hand quantities in this equation, to the first order of approximation we have the motion defined by

$$\ddot{\theta} = C\sin(pt + \alpha), \quad . \quad . \quad . \quad . \quad . \quad (21.7)$$

where

$$C = \frac{gB}{I_c + 6M_c\gamma ra + 12M_c\gamma^2(k^2_A - al) + 9Mr^2(1 - \gamma + \tfrac{1}{4}\gamma^2)}.$$

Thus, to our order of approximation, the specified engine is inherently balanced when revolving with uniform speed, as has already been shown.

Although equation (21.4) contains terms for all the principal masses associated with engines, it is incomplete to the extent of not including terms for such parts as the valve-gear and governor. In circumstances where the inertia effect of the valve-gear must be taken into consideration, it is only necessary to add terms representing the kinetic energy of the gear to the above expression for T,

and proceed in the usual manner to a solution of the resulting equation. Further discussion on the point is for convenience deferred to Art. 37, where Lagrange's formula is used for the purpose of evaluating the kinetic energy of specified types of valve-gear. From a practical point of view the inertia effect of the governor on the motion of its engine is, of course, of minor importance, but it may be remarked that the value of T for the combined system is given in equation (22.1). Therefore that expression should be substituted for T in equation (21.1), if the mass of the governor is to be included in a system of this type.

Had the geometrical relations been used to express T in equation (21.1) in terms of x and its derivatives, as will be exemplified in Art. 31, the equation corresponding with (21.4) would then give the component of the inertia force in the line of stroke. Similarly, the equation of motion derived from the relation for T expressed in terms of y and its derivatives would determine the component of the inertia force in a direction perpendicular to the line of stroke. We can in this way readily refer the unbalanced forces to any co-ordinate, and investigate various problems in the design of such mechanisms, some of which may now be mentioned.

(*a*) *Effect of Friction on the Motion of Cranks.* For given initial conditions the motion of the crank implied in equation (21.4) is naturally affected by the friction acting on the main bearings. If we suppose, by way of illustration, that the friction is proportional to the angular velocity $\dot{\theta}$, then the equation for the damped motion is of the type

$$\ddot{\theta} + F\dot{\theta} = C \sin (pt + \alpha),$$

provided we can neglect all but the first harmonic component of the applied forces. Then, on integrating,

$$\dot{\theta} = - c_1 e^{-Ft} + c_2 \cos (pt + \alpha + \beta)$$

defines the angular velocity of the crank under these conditions, where c_1, c_2, β denote constants which depend on the initial circumstances of motion. Thus it is seen, on taking Fig. 44 to represent the graphs of the quantities in this relation, that the effect of the frictional agencies gradually dies away after an accidental disturbance, leaving only the motion imposed by the periodic forces. It may be noted in passing that the figure also illustrates the manner in which damping devices, working according to the specified law of friction, would cause a momentary disturbance in the motion to die away.

(*b*) *Flywheels.* To determine the *minimum* dimensions of the flywheel required for the engine when operating under a load that varies in a known manner, let ω denote the mean of the maximum and minimum values of the angular velocity. The permissible

range of variations in the speed is usually given in the specification of an engine. On confining ourselves to the first term in the Fourier

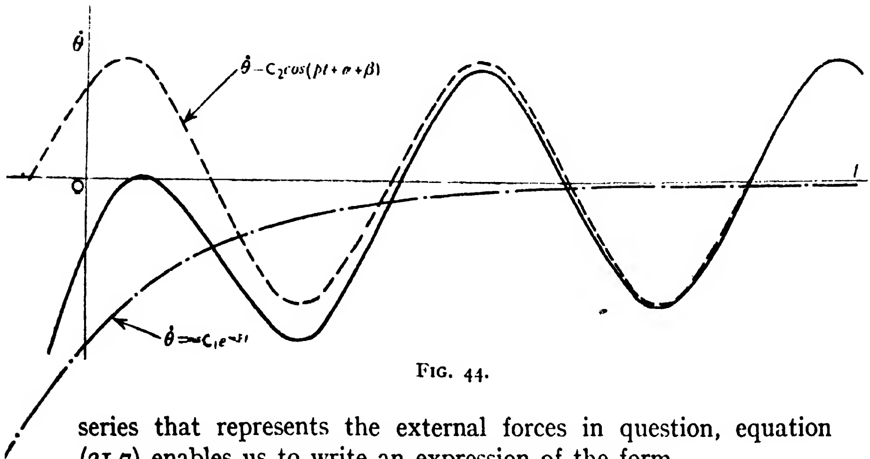


FIG. 44.

series that represents the external forces in question, equation (21.7) enables us to write an expression of the form

$$\dot{\theta} = C \cos (pt + \alpha) + D,$$

where C, p, α, D are known constants for a prescribed engine. Here the coefficient of inertia for the engine is included in C.

The permissible variation in speed, Δ, is therefore defined by

$$\Delta = \frac{\dot{\theta}_{\text{max.}} - \dot{\theta}_{\text{min.}}}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21.9)$$

The value of Δ depends on the nature of the load; for instance, $\Delta = 0{\cdot}005$ may be taken to represent an average value for spinning mills, and $\Delta = 0{\cdot}03$ for general purposes.

Consequently if all the moving masses are known with the exception of the dimensions for the flywheel, the last equation may be solved for the given value of Δ. That is, if Fig. 45 be

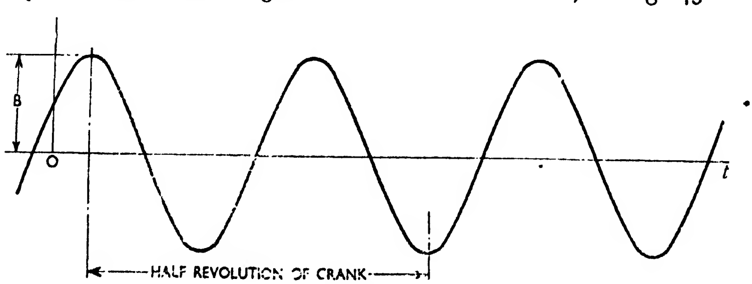


FIG. 45.

regarded as the graph of the first term in the Fourier series for a six-cylinder engine, which may be indicated by $B \cos (pt + \alpha)$, the dimensions of the flywheel are involved in the term I_c of equation

(21.7). For a fixed value of Δ it is apparent that an increase in the 'amplitude' B necessitates a corresponding increase in I_c, and *vice versa*, provided the other masses remain constant throughout the change. If, on the contrary, the magnitudes of the reciprocating masses are increased, the moment of inertia of the flywheel must be accordingly decreased, otherwise Δ will be affected by the change.

Taking a general view of the problem of flywheels, it is seen that the symbol I_c should include the effective moment of inertia of the rotating masses. Hence that symbol should contain a term for the mass of the fluid that rotates with airscrews on the one hand, and with marine propellers on the other. To effect this modification in the case of marine installations, for example, we usually add 25 per cent. to the actual moment of inertia of the propeller.

(*c*) *Firing Order of Internal Combustion Engines.* It is to be inferred from the foregoing results that the chief factors of a mechanical nature which enter into the design of internal combustion engines include the inertia forces and the permissible cyclic variation in speed for a load which varies according to a specified law. These forces influence, in a way which will be explained in Chapter VI, the torsional vibrations of the crankshaft concerned, so that it is advisable to distribute as evenly as possible the stresses produced on this account. In order to minimize the localized stresses in the crankshafts of engines having an even number of cylinders greater than three and operating on the four-stroke cycle, it is usual practice to place consecutively working cylinders as far apart as possible along the axis of the shaft. Changes in this respect do not affect the P-term in equation (21.4), other quantities remaining constant.

(*d*) *Supercharging of Internal Combustion Engines.* This method of working does, however, affect the P-term in the general equation of motion compared with a non-supercharged cycle, since the effect of supercharging is to increase the mean effective pressure on the piston. Supercharging accordingly makes for lighter flywheels on engines operating under prescribed types of load, to an extent which may be determined by equation (21.4).

A special significance is attached to the matter in the case of non-supercharged aero-engines working under service conditions. This is so because such engines are liable to disturbed carburation when the aeroplane is executing manœuvres, and equation (21.4) shows that this kind of disturbance makes for increase in the unbalanced forces and couples. If the point is examined with the aid of the indicator diagrams, it will be found that the harmonic components of the lower orders, in particular, are sensitive to disturbances arising from this source. This may be illustrated by

reference to a certain aero-engine of the geared type having nine cylinders. In the case of the harmonic component of a frequency equal to the angular velocity of the airscrew, it was found that a change of 4 per cent. in the value for each cylinder resulted in a total change of nearly 50 per cent. for the complete engine. In this connection due consideration should be given also to the first of the ' half order ' harmonic components of the unbalanced effects, since theory shows, and experience confirms, that these form a common source of troublesome vibration on aircraft.

(e) *Exhaust-turbine and Marine Engine Installations.* Although the chief aim in adding an exhaust-turbine to a marine engine is that of improving the thermal efficiency of the latter, the modification in general affects the quantities denoted by P and I_c in equation (21.4) when referred to the original engine. This arises from the fact that any change in the valve-setting, steam-pressure, or vacuum on the condenser which may be brought about by the modification leads to a corresponding change in the cyclic variation of P. Similarly, the value of I_c for the engine alone is increased by an amount equal to the effective moment of inertia of the turbine and its gearing; in evaluating this increase account must be taken of the gear-ratio involved, in accordance with the method of Art. 116. Thus if I represent the moment of inertia for an exhaust-turbine revolving n times the angular velocity $\dot{\theta}$ of the engine or shaft, then $\frac{1}{2}n^2I\dot{\theta}^2$ is the kinetic energy of the turbine referred to the speed of the engine. Adding this quantity to T in equation (21.1), we proceed as before to a solution of the resulting equation of motion.

The mechanical effect of fitting an exhaust-turbine to a marine engine is therefore that of increasing the ' flywheel effect ' of the system as a whole, and this has been shown to make for uniform speed under given conditions of load. It has another advantage which deserves mention, in that the modification lowers the natural or ' critical ' frequency of vibration for a propeller shaft of prescribed dimensions, as will be explained in Chapter VI. It is well to notice that a change in the critical speed of a given shaft accordingly leads to a change in the positions of the ' nodes ' or points of zero displacement on the shaft when it is executing vibratory motion, for the new positions of the nodes may or may not prove to be advantageous. For instance, if the position of an ' anti-node ' or point of maximum displacement in torsional vibration coincides with that if the gearing on the shaft, the gear stresses are likely to increase owing to oscillations at the anti-node. Various types of couplings offer a measure of control over the position of the nodes in troublesome cases, as is demonstrated in Ex. 7 of Art. 38, where a spring type of coupling is examined in this respect. It

is therefore essential to determine, before introducing an exhaust-turbine, the relative positions of the nodes for the original and modified conditions of operation.

22. Stability of Governors. In instances where an engine is required to rotate within a limited range of speed when the load varies in a known manner, we must investigate the motion of the system formed by the engine and its governor.

To fix ideas, suppose the engine investigated in Art. 21 to be regulated by the governor indicated in Fig. 46, the spindle of which rotates n times the angular velocity $\dot{\theta}$ of the crankshaft. Writing ψ for the inclination of the links to the vertical axis at the instant when the position of the crank is defined by θ, it is readily seen that the system has two degrees of freedom, since the co-ordinates θ and ψ can undergo independent variations about their positions of equilibrium. On this account we must first express the inertia of the governor in terms of coefficients referred to θ on the one hand, and to ψ on the other.

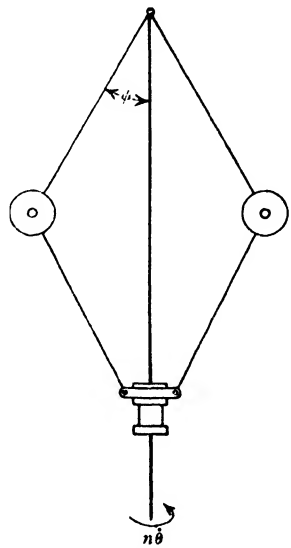


FIG. 46.

For this purpose let:

a_{11} = coefficient of inertia for the engine alone, in relation to θ;
b = coefficient of inertia for the links, balls and other parts of the governor, in relation to θ;
c = coefficient of inertia for the links, balls and other parts of the governor, in relation to ψ.

The expression for the kinetic energy T of the engine and governor is here given by equation (19.5) in the form

$$2T = (a_{11} + n^2 b)\dot{\theta}^2 + c\dot{\psi}^2, \quad . \quad . \quad . \quad . \quad (22.1)$$

where a_{11} is defined by equation (21.3). It will make for brevity of working if we write I for $(a_{11} + n^2 b)$, then

$$2T = I\dot{\theta}^2 + c\dot{\psi}^2. \quad . \quad . \quad . \quad . \quad (22.2)$$

Further, if V is the potential energy of the governor-system alone when the excess of driving power over the resistance opposing motion of the engine amounts to $\mathfrak{T}_1$, with the help of equation (19.9) we can write

$$\frac{d}{dt}(c\dot{\psi}) - \tfrac{1}{2}\frac{\partial c}{\partial \psi}\dot{\psi}^2 - \tfrac{1}{2}\frac{\partial I}{\partial \psi}\dot{\theta}^2 = -\frac{\partial V}{\partial \psi}, \quad . \quad . \quad . \quad (22.3)$$

$$\frac{d}{dt}(I\dot{\theta}) = \mathfrak{T}_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.4)$$

where $\dfrac{\partial I}{\partial \psi} = n^2 \dfrac{\partial b}{\partial \psi}$. The second of these equations might equally well have been deduced direct from the fact that the excess torque is equal to the time-rate of variation of the angular momentum with respect to θ.

At this stage we may introduce into the analysis an overall coefficient of friction, by assuming the frictional forces to be proportional to the velocity $\dot{\psi}$. If K denote this coefficient, the equation for the damped motion is given by adding the term $-K\dot{\psi}$ to the right-hand side of equation (22.3), whence

$$\frac{d}{dt}(c\dot{\psi}) - \tfrac{1}{2}\frac{\partial c}{\partial \psi}\dot{\psi}^2 - \tfrac{1}{2}\frac{\partial I}{\partial \psi}\dot{\theta}^2 = -\frac{\partial V}{\partial \psi} - K\dot{\psi} \quad . \quad . \quad (22.5)$$

The value of K is to be derived from tests with the system in question.

To specify the equilibrium-position of the system, it is feasible to assume that the displacement of the governor from its 'no load' configuration is proportional the magnitude of the torque $\mathfrak{T}_1$. This will result in $\mathfrak{T}_1$ becoming zero when the angular velocity of the crank agrees with the condition of equilibrium, so that if $\psi = \alpha$ define this position, we have

$$\mathfrak{T}_1 = -\beta(\psi - \alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.6)$$

where β represents a known constant for the specified system. Under these circumstances it follows, on writing ω for the angular velocity of the crank in the state of equilibrium, that the conditions $\dot{\theta} = \omega$ and $\psi = \alpha$ must be simultaneously satisfied by equation (22.5). Making these substitutions thus leads to

$$\tfrac{1}{2}\frac{\partial I}{\partial \psi}\omega^2 = \frac{\partial V}{\partial \psi},$$

or
$$\frac{n^2\omega^2}{2}\frac{\partial b}{\partial \psi} = \frac{\partial V}{\partial \psi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.7)$$

To investigate the motion of the governor which may be caused by a slight variation in the mean speed of the engine, suppose the disturbance to be such that as a result of ω changing by an amount equal to x the angle ψ undergoes a correspondingly slight change equal to y. Then the disturbed motion is defined by the relations

$$\dot{\theta} = \omega + x, \; \psi = \alpha + y.$$

Substituting these values in equations (22.4) and (22.5), and cancelling the terms that refer to the steady motion, we have

$$I\dot{x} + \omega\frac{\partial I}{\partial \psi}\dot{y} + \beta y = 0, \quad . \quad . \quad . \quad . \quad (22.8)$$

$$c\ddot{y} - \frac{\omega^2}{2}\frac{\partial^2 I}{\partial \psi^2}y - \omega\frac{\partial I}{\partial \psi}x + \frac{\partial^2 V}{\partial \psi^2}y + K\dot{y} = 0, \quad . \quad . \quad (22.9)$$

as the required equations for the disturbed motion. As the values of these coefficients are those corresponding to the equilibrium-position of the governor, it follows that the coefficients denote constant quantities for the prescribed small variations about that position.

We now combine the last two expressions, by differentiating equation (22.9) with respect to the time t and substituting in the result the value of $\dot{x}$ given by equation (22.8). Thus it appears that

$$cI\dddot{y}+KI\ddot{y}+\left\{I\left(\frac{\partial^2 V}{\partial\psi^2}-\frac{\omega^2}{2}\frac{\partial^2 I}{\partial\psi^2}\right)+\omega^2\left(\frac{\partial I}{\partial\psi}\right)^2\right\}\dot{y}+\beta\omega\frac{\partial I}{\partial\psi}=0. \quad . \quad . \quad (22.10)$$

In order to complete the solution we make the practically true assumption that the small quantities denoted by x and y vary as $e^{\lambda t}$, where e is the base of hyperbolic functions. Hence, making these substitutions,

$$cI\lambda^3+KI\lambda^2+\left\{I\left(\frac{\partial^2 V}{\partial\psi^2}-\frac{\omega^2}{2}\frac{\partial^2 I}{\partial\psi^2}\right)+\omega^2\left(\frac{\partial I}{\partial\psi}\right)^2\right\}\lambda+\beta\omega\frac{\partial I}{\partial\psi}=0, \quad . \quad (22.11)$$

or

$$\lambda^3+p_1\lambda^2+p_2\lambda+p_3=0, \quad . \quad . \quad . \quad (22.12)$$

where

$$p_1=\frac{K}{c},\ p_2=\frac{1}{cI}\left\{I\left(\frac{\partial^2 V}{\partial\psi^2}-\frac{\omega^2}{2}\frac{\partial^2 I}{\partial\psi^2}\right)+\omega^2\left(\frac{\partial I}{\partial\psi}\right)^2\right\},\ p_3=\frac{\beta\omega}{cI}\frac{\partial I}{\partial\psi}.$$

For stability it is necessary that none of the roots of this cubic in λ has a positive real part. On the basis of this information it is not difficult to establish the fact [1] that this condition will in any case be fulfilled if the coefficients p_1, p_2, p_3 are essentially positive, and also $p_1p_2 > p_3$. Provided the coefficients are always positive, then, by equation (22.11), the relation

$$K\left\{I\left(\frac{\partial^2 V}{\partial\psi^2}-\frac{\omega^2}{2}\frac{\partial^2 I}{\partial\psi^2}\right)+\omega^2\left(\frac{\partial I}{\partial\psi}\right)^2\right\}>\beta c\omega\frac{\partial I}{\partial\psi} \quad . \quad (22.13)$$

will secure stability of the governor throughout slight disturbances about the equilibrium-position.

The last relation is of general application, so that we may proceed to explain the method of evaluating the relevant coefficients for given types of governors.

(*a*) *Watt's Type.* Suppose the above-mentioned engine to be fitted with the governor indicated in Fig. 47, which rotates n times the angular velocity $\dot{\theta}$ of the crank. The mechanism consists of four similar links of length l and weight w, and two balls each of weight W; the weight of each link will be taken to act through its mid-length. It will also be assumed that the upper and lower joints of the links lie on the axis of rotation; this does not in general

[1] W. S. Burnside and A. W. Panton, *Theory of Equations.*

apply to both of the joints on actual mechanisms of this type, but the modification necessary on this account may readily be effected by the aid of the geometrical relations for the governor.

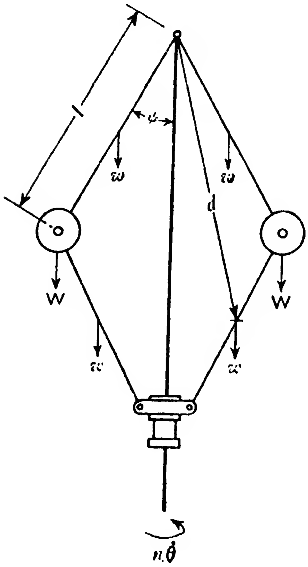


FIG. 47.

By the principle of virtual work we have, for use in connection with equation (22.1),

$$b = \frac{1}{g}(2Wl^2 \sin^2 \psi + wl^2 \sin^2 \psi)$$

$$= \frac{l^2 \sin^2 \psi}{g}(2W + w)$$

as the coefficient of inertia b referred to the crank-angle θ, and

$$c = \frac{1}{g}\{2Wl^2 + wl^2(1 + 4 \cos^2 \psi)\}$$

$$= \frac{l^2}{g}\{2W + w(1 + 4 \cos^2 \psi)\}$$

as the coefficient of inertia c referred to the angle ψ. In evaluating these quantities we take account of the fact that d in the figure is given by $d^2 = l^2(\frac{1}{4} + 2 \cos^2\psi)$. The same procedure may be used to find the expression for the potential energy V of the governor, which leads to

$$\delta V = -\ 2Wl\delta(\cos \psi) - 4wl\delta(\cos \psi),$$

whence $$V = -\ 2l(W + 2w) \cos \psi.$$

The relation (22.13) contains certain partial differentials of the above expressions, which are thus seen to be

$$\frac{\partial^2 V}{\partial \psi^2} = 2l(W + 2w) \cos \psi,$$

$$\frac{\partial^2 I}{\partial \psi^2} = n^2\frac{\partial^2 b}{\partial \psi^2} = \frac{2n^2l^2}{g}(2W + w) \cos 2\psi,$$

$$\left(\frac{\partial I}{\partial \psi}\right)^2 = \frac{n^4l^4}{g^2}(2W + w)^2 \sin^2 2\psi,$$

$$c\frac{\partial I}{\partial \psi} = \frac{n^2l^4}{g^2}(2W + w) \sin 2\psi\{2W + w(1 + 4 \cos^2 \psi)\}.$$

Inserting these values in (22.13), we ascertain the relation to be satisfied as regards stability of the system under any prescribed conditions of load. Consequently if one of the quantities in the expression is unknown, we may use the relation for the purpose of determining that quantity. For example, if the moment of

inertia of the flywheel is the only unknown quantity, its *minimum* value can be derived from the resulting relation, for the quantity in question is included in the term I_c of the coefficient a_{11}. In certain circumstances the overall coefficient of friction K is the unknown term in the inequality, and this may be evaluated in a like manner.

It is well, again, to observe that in all numerical applications of the foregoing analysis the angle ψ should be given its value corresponding to the 'steady running' conditions for the engine, as was done in connection with equation (22.9). This remark applies also to the following results for the Porter and spring-loaded types of governors.

(*b*) *Interaction between an Engine and its Governor.* Before proceeding, we may touch on this aspect of the general problem, though an investigation into the matter implies a knowledge of the way in which the stresses associated with the vibratory motion are propagated through the materials concerned. These stresses are transmitted in a manner which will be described in Chapters III and IV. Our chief object of study here is, however, the possible relations between the p-coefficients in equation (22.12).

If, on the one hand, $p_1 p_2 < p_3$, the treatment of Art. 42 shows that a disturbance in the motion of the engine would cause the governor to execute vibrations having an amplitude or displacement which increases indefinitely with the time t, as represented in

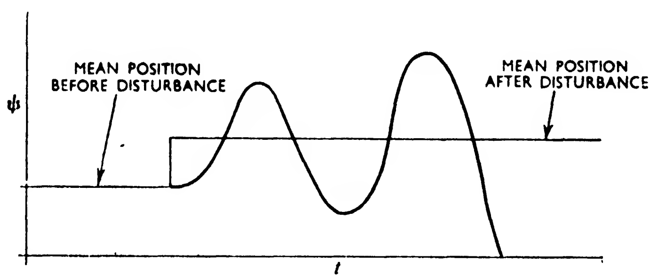


FIG. 48.

Fig. 48. It is obvious that the governor would then be useless—if not dangerous—in operation.

On the other hand, if $p_1 p_2 = p_3$, it follows from the same theory that the disturbed mechanism would describe oscillations with a constant displacement about the equilibrium-position of the governor. The graph of this motion, persisting indefinitely with the time t, is indicated by the full line in Fig. 49. It thus appears that the governor is, again, inefficient for practical purposes, due to its sluggish action making for 'hunting'.

These objectionable characteristics are avoided if the relation

(22.13) is satisfied, that is if $p_1p_2 > p_3$. This is so because the governor then attains a position of stability shortly after a variation

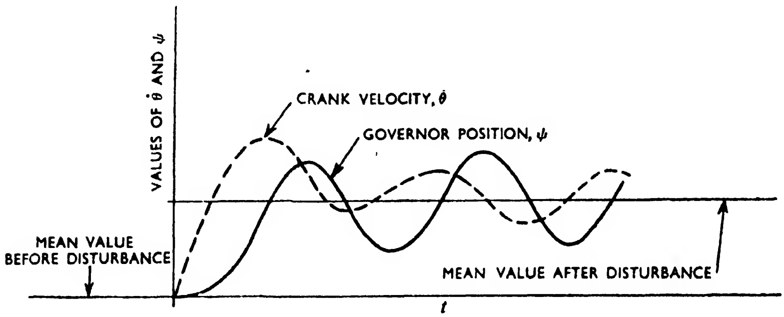


FIG. 49.

in load on the engine has occurred, in an interval of time which depends mainly on the magnitude of the resistance offered to motion by the dashpot. Then the full line in Fig. 50 may be taken to indicate the motion which would follow a slight disturbance, where the rate of damping is influenced mainly by the value of K in equation (22.12).

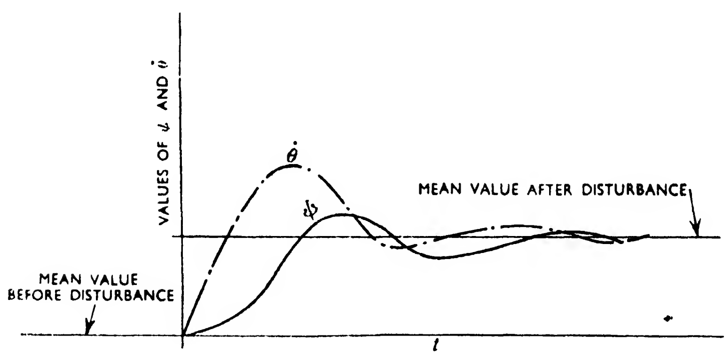


FIG. 50.

The configuration of the governor at any instant is given by the solution of equation (22.10). This is, as may be readily verified, of the form

$$y = G_1e^{-K_1t} + G_2e^{K_2t}\cos(pt + \varepsilon), \quad . \quad . \quad (22.14)$$

where G_1, G_2, ε are constants which depend on the initial circumstances of motion, and K_1, K_2 represent the frictional resistances involved in the problem.

Had we eliminated the y-terms, instead of the x-terms, in the work of deriving equation (22.14), however, the resulting expression would have been of the same form, hence the relation for x is also

of the same type as the last equation. The initial conditions of motion with respect to x and y are not, however, in general the same, and this accounts for the difference in the shape of the corresponding graphs in Figs. 49 and 50. This may be explained in the following manner with reference to the subsequent treatment. Suppose that at time $t = 0$ a change in load on the engine leads to the slight variation x about the mean value of the angular velocity $\dot{\theta}$ of the crank, in consequence of which a finite angle of inclination appears simultaneously on the graphs shown dotted in the figures. But before the governor can react to this change of load, the disturbance must be transmitted in the form of a *stress-wave* through the elastic material connecting the engine and governor, in accordance with the theory of Chapter IV. The time occupied in propagating this wave is comparatively short, but finite in value, so that a corresponding interval of time separates the instants at which the related variations x and y appear on the graphs in Figs. 49 and 50. Owing to the effect of this phenomenon on the motion, combined with that of the frictional agencies, at the instance $t = 0$ the graph of ψ starts tangentially to the horizontal axes in the figures.

It is thus seen that recourse must be had to the analysis given in Chapters III and IV in arriving at a graphical solution, exhibiting the relative variations in the speed of the crank and the configuration of the governor which would be produced by a specified change in the load on a given engine.

The following theory shows also that if the frictional resistances to the motion of a specified system of this type are negligibly small, a sudden change in load will cause the system to oscillate, about a mean position, with a period that is sensibly constant and independent of the magnitude of the change.

A further remark may be made here. This arises from the fact that a certain amount of energy is beyond the control of a governor, since a quantity of the working fluid is present in the passages connecting the governor-valve and piston at the instant when the mechanism starts to move in consequence of a change in the load on the prime-mover concerned. Hence if the load is suddenly reduced from full to zero value on a double-acting two-stroke engine, for instance, the uncontrollable energy associated with each cylinder is equal to twice the maximum work done in a stroke; the steam present in the cylinders of a triple-expansion engine, measured from the governor-valve to the exhaust port, similarly continues to do work after the governor has reacted to a variation in the load on the crank-shaft. The mechanical equivalent of this energy should, strictly speaking, be added to the kinetic energy T for the engine under examination, after due account has been taken of the thermodynamical efficiency involved in the process. The resulting equation

would, to the implied order of approximation, then include terms for all the forces acting on the system, whether arising from mechanical or thermodynamical sources.

(c) *Porter's Type.* Take Fig. 51 to represent the mechanism, where W_1 is the combined weight of the central mass and sleeve, W that of each ball, and w that for each of the four links; the centre of gravity for each link, of length l, will be assumed to act through the mid-length, as indicated in the figure. We shall further suppose the governor to rotate at n times the angular velocity $\dot{\theta}$ of the crank-shaft of its engine, the latter being specified in this regard by the coefficient a_{11} in equation (21.3).

Making the previous assumption that the upper and lower joints of the links lie on the axis of rotation, the coefficient of inertia b for the governor referred to the crank-angle θ is easily seen to be the same as that found for the mechanism indicated in Fig. 47, namely

$$b = \frac{l^2 \sin^2 \psi}{g}(2W + w).$$

Our procedure leads also to

$$c = \frac{l^2}{g}\{2W + w(1 + 4\cos^2 \psi) + 2W_1 \sin^2 \psi\}$$

for the related coefficient of inertia c in terms of ψ; it is to be noted here that the central mass and sleeve move vertically twice as fast as the balls. There remains to be derived an expression for the potential energy V of the governor, for which purpose we apply the principle of virtual velocities, and thus obtain

$$\delta V = -2Wl\delta(\cos \psi) - 4wl\delta(\cos\psi) - 2W_1 l\delta(\cos\psi),$$

whence $V = -2l(W + 2w + W_1)\cos\psi$.

Hence, on differentiating these equations, we have

$$\frac{\partial^2 V}{\partial \psi^2} = 2l(W + 2w + W_1)\cos\psi,$$

$$\frac{\partial^2 I}{\partial \psi^2} = n^2\frac{\partial^2 b}{\partial\psi^2} = \frac{2n^2 l^2}{g}(2W + w)\cos 2\psi,$$

$$\left(\frac{\partial I}{\partial \psi}\right)^2 = \frac{n^4 l^4}{g^2}(2W + w)^2 \sin^2 2\psi,$$

for use in connection with the relation (22.13).

With this information we can, after effecting the appropriate substitutions in the relation for stability, determine any one of the quantities when the values of the remaining terms in the expression are known.

(d) *Spring-loaded Type.* Here the system is formed by four links each of length l and weight w, two balls each of weight W, a

sleeve weighing W_s, and the spring indicated in Fig. 52. In what follows we shall suppose, as before, that the upper and lower joints for the links lie on the axis of rotation, and that the governor rotates at n times the angular velocity $\dot{\theta}$ of the crank of the engine, which is specified for our present purpose by the coefficient a_{11} in equation (21.3). The weight of the spring will be neglected in comparison with the other masses concerned.

As regards the spring, let its length be such that the spring is

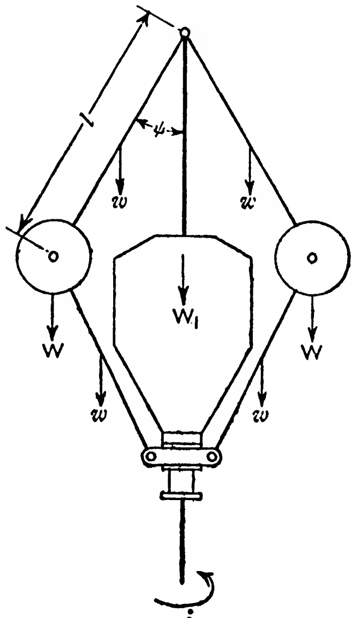


FIG. 51.

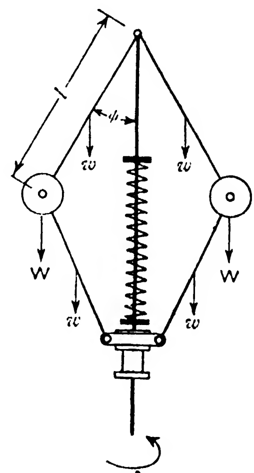


FIG. 52.

free from load when $\psi = 0$, so that if Rz is the force required to compress it through unit distance, then $z = 2l(1 - \cos\psi)$.

An application of the foregoing method now gives

$$V = -2l(W + 2w + W_s)\cos\psi + \frac{2Rl^2}{g}(1 - \cos\psi)^2$$

for the potential energy V of the governor,

$$b = \frac{l^2}{g}(2W + w)\sin^2\psi$$

for its coefficient of inertia referred to the crank-angle θ, and

$$c = \frac{l^2}{g}\{2W + w(1 + 4\cos^2\psi) + 4W_s\sin^2\psi\}$$

for its coefficient of inertia referred to the angle ψ.

Hence, on differentiating these expressions, we have

$$\frac{\partial^2 V}{\partial \psi^2} = 2l(W + 2w + W_s)\cos\psi + \frac{4Rl^2}{g}(\cos\psi - \cos 2\psi),$$

$$\frac{\partial^2 I}{\partial \psi^2} = n^2\frac{\partial^2 b}{\partial \psi^2} = \frac{2n^2l^2}{g}(2W + w)\cos 2\psi,$$

$$\left(\frac{\partial I}{\partial \psi}\right)^2 = \frac{n^4l^4}{g^2}(2W + w)^2 \sin^2 2\psi$$

for the terms which appear in the relation (22.13). These expressions suffice to determine the conditions for stability of the specified type of governor.

It is to be noticed that this mechanism is liable to sluggish action, due to the force exerted by the spring increasing with the ' rise ' of the sleeve. Moreover, the results show that $V = 0$ is a possible condition in this case, for which reason the governor may cease to regulate the speed of its engine if the counteracting effect of the dashpot and other frictional agencies is inadequate for the purpose.

23. Testing of Governors. Work of this nature may be carried out either with the mechanism disconnected from its engine, or with the engine and governor examined together.

In the former case, with a mechanism of the dimensions implied in equation (22.1), the kinetic energy T_1 for the governor *alone* is defined by the relation

$$2T_1 = n^2 b\dot{\theta}^2 + c\dot{\psi}^2.$$

Moreover, if $\mathfrak{T}_2$ is the driving torque on the governor-spindle, by equation (22.4), we have,

$$\begin{aligned}\mathfrak{T}_2 &= \frac{d}{dt}(n^2 b\dot{\theta}) \\ &= n^2 b\ddot{\theta} + n^2\frac{db}{dt}\dot{\theta} \\ &= n^2 b\ddot{\theta} + n^2\dot{\theta}\dot{\psi}\frac{\partial b}{\partial \psi}. \qquad (23.1)\end{aligned}$$

Since the work done in unit time by this torque is equal to $n\mathfrak{T}_2\dot{\theta}$, the power supplied to the governor can therefore be expressed in the form

$$n\mathfrak{T}_2\dot{\theta} = n^3 b\ddot{\theta}\dot{\theta} + n^3\dot{\theta}^2\dot{\psi}\frac{\partial b}{\partial \psi} \qquad (23.2)$$

But, on writing $2F$ for the rate at which the mechanical energy of the system is dissipated by the inherent frictional agencies implied in the symbol K of equation (22.5), we have also

$$n\mathfrak{T}_2\dot{\theta} = n\frac{d}{dt}(T_1 + V) + n\dot{\psi}\frac{\partial F}{\partial \dot{\psi}}$$

$$= nc\dot{\psi}\ddot{\psi} + \frac{n}{2}\dot{\psi}^3\frac{\partial c}{\partial \psi} + n^3 b\ddot{\theta}\dot{\theta} + \frac{n^3}{2}\dot{\theta}^2\dot{\psi}\frac{\partial b}{\partial \psi} + n\dot{\psi}\frac{\partial V}{\partial \psi} + n\dot{\psi}\frac{\partial F}{\partial \dot{\psi}},$$

from consideration of the energy of the system. Equating this expression to the right-hand side of equation (23.2) and cancelling the common factors, it is seen that

$$c\ddot{\psi} + \tfrac{1}{2}\dot{\psi}^2\frac{\partial c}{\partial \psi} = \frac{n^2}{2}\dot{\theta}^2\frac{\partial b}{\partial \psi} - \frac{\partial V}{\partial \psi} - \frac{\partial F}{\partial \psi} \quad . \quad . \quad . \quad (23.3)$$

is the equation for the disturbed motion of the mechanism. All the quantities involved here can be found by experimental means, so the damped oscillations of the governor can be determined by this equation.

A remark is called for as regards the friction on the governor-sleeve which is included in K of equation (22.9). This may be found experimentally in terms of the vertical movement of the sleeve. To state this resisting force in terms of the angle ψ, let K' denote the coefficient of friction thus obtained, and z the vertical movement of the sleeve corresponding to a slight change y in the inclination of the links. Then the experimentally determined frictional resistance on the sleeve is to be expressed in the form $K'\dot{z}$ before it can be incorporated into the foregoing analysis. The required relation is easily found with the help of the principle of virtual work, which shows that the sleeve contributes the quantity $2K'l^2 \sin^2 \psi$ to the overall coefficient K. With this information we can express the experimental data in terms of the velocity $\dot{y}$ in equation (22.9), as it ought to be.

When an engine and its governor are tested together, first importance is to be attached to the ratio $x : y$ which is involved in equations (22.8) and (22.9), for it has been shown that the efficiency of the governor as a regulator is conditioned by the value of that ratio within the limits defined by the upper and lower stops on the governor. If, as sometimes occurs in practice, the value of this ratio is too small to satisfy the relation (22.13), the efficiency of the mechanism may be improved by effecting suitable changes in either the mass associated with the sleeve or the stiffness of the spring. It is, of course, always essential to ensure that the 'power' of the governor is sufficient to overcome the resistances opposing any change in the configuration, whence it follows that the value of the quantity given by equation (23.3) should be greater than the work expended in overcoming the resistances which act in a specified change of ψ.

Our analysis has revealed a number of possible sources of trouble, the chief of which may now be mentioned.

(i) A governor may oscillate continuously due to the 'flywheel effect' of the engine as a whole being too small in value, owing to the value of the coefficient a_{11} in equation (21.3) being too small for the load concerned. In this case the coefficient a_{11} should be taken as the unknown quantity when investigating the stability of the system with the aid of the relation (22.13).

(ii) A governor may be sluggish in action due to the lack of 'power'. The matter may be examined analytically by using the relation (22.13) for the purpose of finding the time taken by the governor to move between the configurations corresponding to prescribed speeds of the engine. If the interval of time is greater than that required, it is in general necessary to increase the power of the governor, or to substitute one of the relay type.

(iii) The dashpot may be incorrectly adjusted for the conditions of load on the engine. Trouble on this account is easily investigated by making an alteration in either the load on the spring of the governor or the viscous resistance offered by the dashpot, and noting the speeds of the engine before and after the change is carried out under a constant load. If the adjustment of the dashpot is at fault, the frictional coefficient K should be treated as the unknown quantity in the relation (22.13). Due care should be exercised in effecting the suggested changes, since it has been shown that a variation in the resistance initiated by the dashpot makes for the condition $p_1p_2 = p_3$ in equation (22.12), so that an excessive reduction in the resistance may lead to the 'hunting' phenomenon indicated in Fig. 49. If, on the contrary, the change results in the condition $p_1p_2 < p_3$, the engine is then liable to the objectionable type of motion illustrated by Fig. 48.

(iv) Faulty operation of governors may also be due to an excessive amount of energy stored in the working fluid that is present in the passages connecting the governor-valve and inlet port on the cylinder at the instant when the mechanism reacts to a variation in load on the engine. The relative magnitude of this energy is usually small for modern installations, but instances are sometimes met with where the design leaves something further to be desired in this respect.

24. Fourier's Expansion. The point to be discussed here arises from the fact that our data commonly include graphical records pertaining to various quantities involved in the equations of motion for mechanisms and structures, and this information must be introduced into the related expressions. For example, if the indicator diagram and torsiograph-record for a specified engine give Fig. 53 as the graph of the quantity

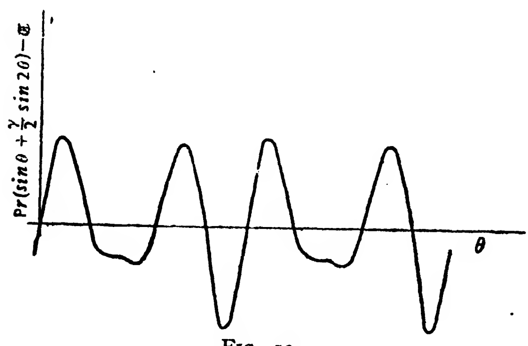


FIG. 53.

$\{Pr(\sin\theta + \frac{1}{2}\gamma\sin 2\theta) - \mathfrak{T}\}$ in equation (21.4), this information must be expressed in the form of a series. The same remark applies also to the deflection-time graphs for structures executing vibrations under the influence of external forces.

A number of graphical methods are available for effecting the operation in question, all of which are based on a theorem due to J. B. Fourier,[1] who discovered that under certain conditions a function $f(\theta)$ can be expressed in the form

$$\begin{aligned} f(\theta) &= A_0 + A_1\cos\theta + A_2\cos 2\theta + \ldots + A_n\cos n\theta \\ &\qquad + B_1\sin\theta + B_2\sin 2\theta + \ldots + B_n\sin n\theta \\ &= A_0 + \sum_{n=1}^{\infty}(A_n\cos n\theta + B_n\sin n\theta), \end{aligned} \qquad (24.1)$$

where $A_0, A_1, \ldots, A_n, B_1, B_2, \ldots, B_n$ denote constants which can be evaluated for a given graph.

The validity of this expansion is here assumed,[2] since we confine ourselves to systems for which it is always possible to determine, at least for practical purposes, the coefficients on the assumption that the above series may be integrated term by term between the limits $-\pi$ and π.

To define the coefficient A_0, we integrate both sides of the last expression over the range $(-\pi, \pi)$, and so obtain

$$\int_{-\pi}^{\pi} f(\theta)d\theta = A_0\int_{-\pi}^{\pi} d\theta + \sum_{n=1}^{\infty}\{A_n\int_{-\pi}^{\pi}\cos n\theta d\theta + B_n\int_{-\pi}^{\pi}\sin n\theta d\theta\}$$

But since

$$\int_{-\pi}^{\pi}\cos n\theta d\theta = 0, \quad \int_{-\pi}^{\pi}\sin n\theta d\theta = 0,$$

it follows that

$$\int_{-\pi}^{\pi} f(\theta)d\theta = 2\pi A_0,$$

hence

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta \qquad (24.2)$$

determines the coefficient A_0.

The remaining coefficients can be evaluated by multiplying both sides of equation (24.1) in turn by $\cos m\theta$ and $\sin m\theta$, and integrating between the limits $-\pi$ and π. The right-hand side of the equation then contains terms of the types

$$A_n\int_{-\pi}^{\pi}\cos m\theta\cos n\theta d\theta = \tfrac{1}{2}A_n\int_{-\pi}^{\pi}\{\cos(m-n)\theta + \cos(m+n)\theta\}d\theta,$$

$$B_n\int_{-\pi}^{\pi}\sin m\theta\sin n\theta d\theta = \tfrac{1}{2}B_n\int_{-\pi}^{\pi}\{\cos(m-n)\theta - \cos(m+n)\theta\}d\theta.$$

[1] *Théorie analytique de la Chaleur* (1822).

[2] See H. S. Carslaw, *Theory of Fourier's Series and Integrals* for a discussion on the point.

It is readily proved that the former expression is zero when n is any integer other than m, and πA_m when $n = m$; also that the latter expression is zero when n is any integer other than m, and πB_m when $n = m$. Since

$$A_n\int_{-\pi}^{\pi} \sin m\theta \cos n\theta d\theta = \tfrac{1}{2}A_n\int_{-\pi}^{\pi} \{\sin(m+n)\theta + \sin(m-n)\theta\}d\theta,$$

$$B_n\int_{-\pi}^{\pi} \cos m\theta \sin n\theta d\theta = \tfrac{1}{2}B_n\int_{-\pi}^{\pi} \{\sin(m+n)\theta - \sin(m-n)\theta\}d\theta,$$

both of which vanish whatever integral values m and n may have, it follows that

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\theta d\theta, \quad . \quad . \quad . \quad . \quad . \quad (24.3)$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin m\theta d\theta. \quad . \quad . \quad . \quad . \quad . \quad (24.4)$$

Equation (24.1) may consequently be written in the form

$$f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big\{\cos n\theta\int_{-\pi}^{\pi} f(\phi)\cos n\phi d\phi$$
$$+ \sin n\theta\int_{-\pi}^{\pi} f(\phi)\sin n\phi d\phi\Big\} \quad . \quad . \quad (24.5)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(\phi)\cos n(\phi-\theta)d\phi, \quad . \quad . \quad . \quad (24.6)$$

where the expression on the right of equation (24.5) represents a Fourier series of period 2π, provided the expansion is valid.

If the series determines $f(\theta)$ for values of θ between the limits $-\pi$ and π, it will not in general apply in cases where the values of θ lie outside those limits, for that could only occur if $f(\theta)$ itself were periodic and of period 2π.

Mention will now be made of the two principal types of functions, in so far as they enter into this matter.

All the B_m-coefficients will vanish when the function $f(\theta)$ is *even*, that is when $f(-\theta) = f(\theta)$ holds between the limits $-\pi \leq \theta \leq \pi$, for we then have from equation (24.4)

$$\begin{aligned}
\pi B_m &= \int_{-\pi}^{0} f(\theta)\sin m\theta d\theta + \int_{0}^{\pi} f(\theta)\sin m\theta d\theta \\
&= \int_{\pi}^{0} f(-\theta)\sin m\theta d\theta + \int_{0}^{\pi} f(\theta)\sin m\theta d\theta \\
&= -\int_{0}^{\pi} f(\theta)\sin m\theta d\theta + \int_{0}^{\pi} f(\theta)\sin m\theta d\theta \\
&= 0.
\end{aligned}$$

Therefore the relation

$$f(\theta) = \frac{1}{\pi}\int_0^\pi f(\phi)d\phi + \frac{2}{\pi}\sum_{n=1}^{\infty} \cos n\theta \int_0^\pi f(\phi) \cos n\phi d\phi \quad . \quad (24.7)$$

applies in the case of even functions.

All the A_m-coefficients will vanish when $f(\theta)$ is an *odd* function, that is when $f(-\theta) = -f(\theta)$ holds between the limits $-\pi \leq \theta \leq \pi$, because equation (24.3) then enables us to write

$$\begin{aligned} A_m &= \frac{1}{\pi}\int_{-\pi}^0 f(\theta) \cos m\theta d\theta + \frac{1}{\pi}\int_0^\pi f(\theta) \cos m\theta d\theta \\ &= -\frac{1}{\pi}\int_0^\pi f(\theta) \cos m\theta d\theta + \frac{1}{\pi}\int_0^\pi f(\theta) \cos m\theta d\theta \\ &= 0 ; \end{aligned}$$

also $A_0 = 0$. Hence for odd functions we have

$$f(\theta) = \frac{2}{\pi}\sum_{n=1} \sin n\theta \int_0^\pi f(\phi) \sin n\phi d\phi \quad . \quad . \quad . \quad (24.8)$$

Ex. 1. Find a Fourier series for a function $f(\theta)$ that is constant and equal to -1 between the limits $\theta = -\pi$ and $\theta = 0$, equal to zero when $\theta = 0$, then equal to unity for the next half-period, and so on, as indicated by the graph shown dotted in Fig. 54. It may

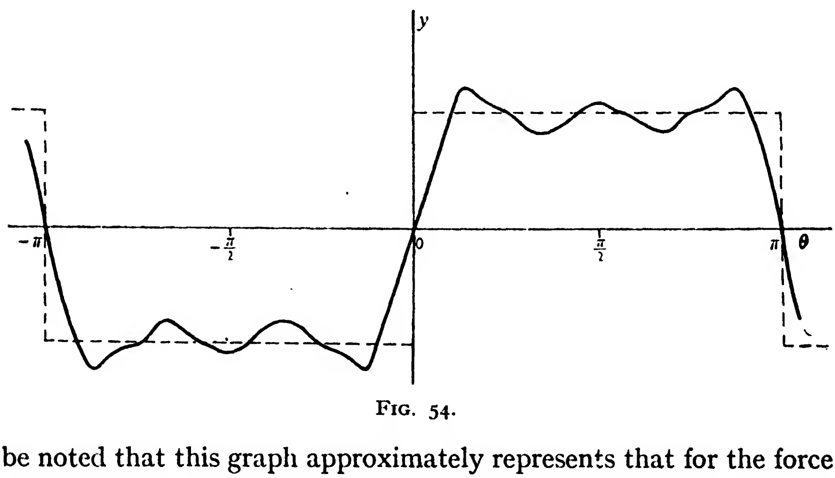


FIG. 54.

be noted that this graph approximately represents that for the force produced by the friction between two solid bodies when they are moved harmonically backwards and forwards relatively to one another, with a period equal to 2π.

Here

$$f(\theta) = \begin{cases} -1 \text{ for } -\pi \leq \theta < 0, \\ 0 \text{ for } \quad \theta = 0, \\ 1 \text{ for } \quad 0 < \theta \leq \pi, \end{cases}$$

from which it appears that $f(\theta)$ is an odd function, so that the general

coefficient A in the series (24.1) is zero. Our analysis also discloses the relation

$$B_m = \frac{2}{\pi}\int_0^\pi \sin m\theta d\theta$$
$$= 0 \text{ if } m \text{ is even, and}$$
$$= \frac{4}{m\pi} \text{ if } m \text{ is odd.}$$

Thus, inserting these values in equation (24.8),

$$f(\theta) = \frac{4}{\pi}(\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \ldots)$$

is the required series. We may readily verify that this result satisfies the specified conditions in certain particulars, for both the series and the function have zero values when $\theta = 0$; also $f(-\pi) = -1$ and $f(\pi) = 1$, and when $\theta = \pm\pi$ the series has the mean of these two values, namely zero.

The sum of these three terms in the series for $f(\theta)$ has been used in plotting the graph indicated by the full line in Fig. 54, to illustrate the manner in which this graph approximates to that for the original function. It is easily seen that a still closer approximation to the graph of $f(\theta)$ would be obtained if more terms in the series were used for the purpose.

Ex. 2. Evaluate the series for the graph shown dotted in Fig. 55,

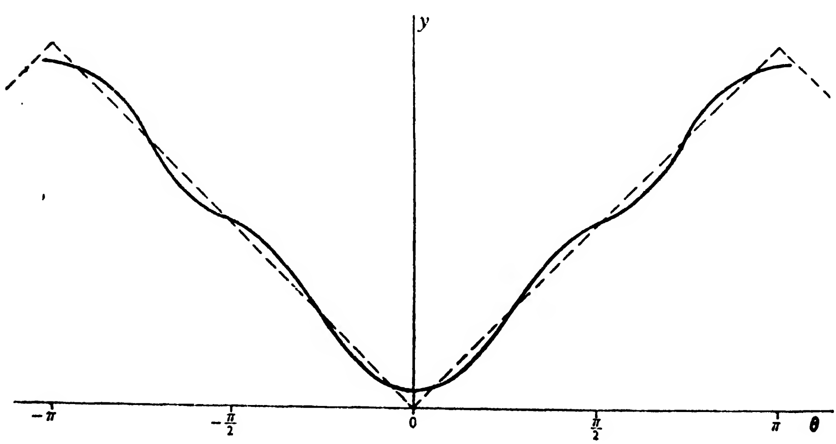


FIG. 55.

which relates to an even function such that $f(\theta) = -\theta$ between the limits $-\pi \leq \theta \leq 0$, and $f(\theta) = \theta$ between the limits $0 \leq \theta \leq \pi$.

Since the given function is even, it follows from the preceding

results that all the B_m-coefficients in the series (24.1) will vanish, in which circumstance we have

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta.$$

On proceeding further, it will be found that

$$A_0 = \frac{1}{\pi}\int_0^{\pi} \theta d\theta,\ A_m = \frac{2}{\pi}\int \theta \cos m\theta d\theta,$$

which lead to the values

$$A_0 = \frac{\pi}{2},\ A_m = \frac{2}{\pi}\left[\theta\frac{\sin m\theta}{m} + \frac{\cos m\theta}{m^2}\right]_0^{\pi} = \begin{cases} 0 \text{ when } m \text{ is even, and} \\ -\dfrac{4}{m^2\pi} \text{ when } m \text{ is odd.} \end{cases}$$

Consequently

$$A_2 = A_4 = A_6 = \ldots = 0,$$

$$A_1 = -\frac{4}{\pi},\ A_3 = -\frac{4}{3^2\pi},\ A_5 = -\frac{4}{5^2\pi}, \ldots,$$

and therefore

$$f(\theta) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos\theta + \frac{1}{3^2}\cos 3\theta + \frac{1}{5^2}\cos 5\theta + \ldots\right)$$

is the series in question.

The sum of these three terms is shown graphically by the full line in Fig. 55, and a graph plotted with the aid of more terms would lead, as in the previous example, to still closer approximation between the graphs of the function and its series.

These results show that the graph of a specified series consists of short waves, the amplitudes of which decrease with increase in the number of terms used. It is also evident that although the ordinates of the graphs for the function and its series are approximately equal, the slopes at a prescribed point for the two graphs are not generally equal. Therefore the derivative of a function cannot usually be derived by a term to term differentiation of the corresponding series, even when the areas under the two graphs are nearly equal.

CHAPTER II

BALANCING OF LOCOMOTIVES

25. We have now to apply the preceding analysis to the case of locomotives, in the process of which the results will be arranged with reference to the structural system formed by the engine, its track, and the bridges over which it may travel. Due attention must on this account be given to the unbalanced components of force in the direction at right-angles to that of the track, since these components operate on structural members which are comparatively slender in this direction. It is, moreover, essential to secure the best possible balance in the line of traction, so as to minimize the stress on the couplings situated between the carriages or wagons of a train. Although we shall in the main confine ourselves to the disturbed motion of locomotives, it will be seen that the results obtained in this and subsequent chapters elucidate the running characteristics of rolling stock generally, which have an important bearing on the damage done to certain kinds of freight.[1]

Our treatment for the locomotive type of engine is naturally based on that given in Chapter I, and we shall accordingly assume that the cranks proper are initially balanced.

26. Inside Cylinders. Let Fig. 56 represent the driving axle on an engine of this type, having cranks arranged 90 deg. apart and disposed symmetrically about the mid-length of the axle, with equal rotating and reciprocating masses assigned to each of the cylinders.

On actual engines it is, for a number of reasons, impracticable to arrange matters so that the moving parts and their balance-weights reciprocate and rotate in the same plane, the usual practice being that of placing the counter-weights in the wheels, as indicated in Fig. 57. These weights sometimes consist of built-up plates, between which is inserted the requisite amount of lead. It is manifest that under these conditions the axle will be subjected to bending stresses that vary in a periodic manner, owing to the method of construction leading to an unbalanced couple of the type denoted by $\mathfrak{C}$ in Fig. 4.

[1] *Engineering*, vol. 139, page 244 (1935).

To investigate the distribution of the inertia effects over the driving wheels shown in the figure, suppose for the present that M

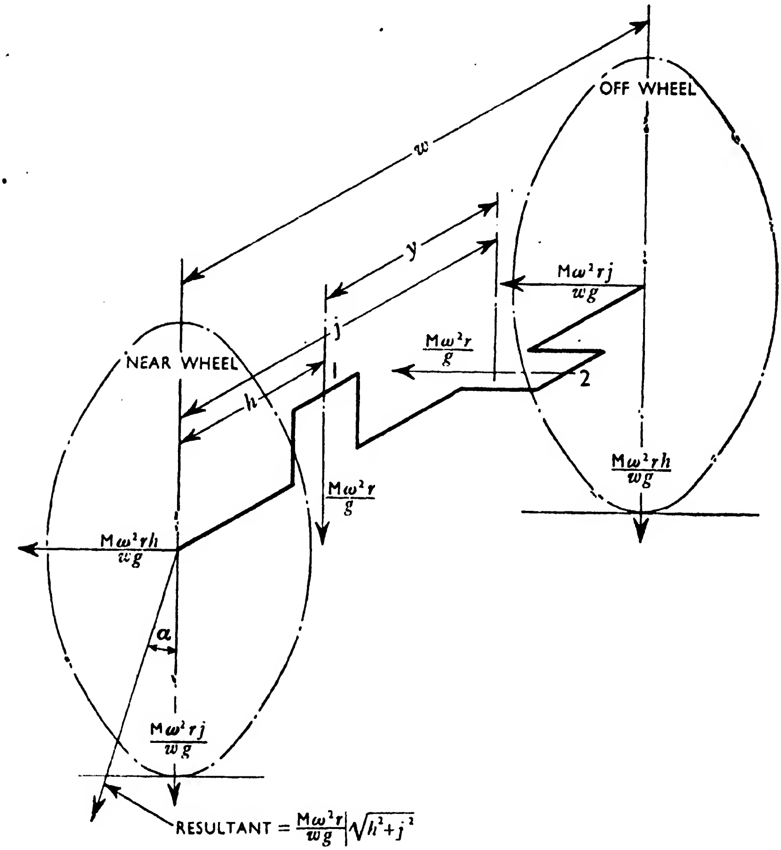


FIG. 56.

is the total weight of the unbalanced parts attached to each of the cranks, of throw r, which revolve with angular velocity ω. Considering first the crank marked 1, it is readily seen that the unbalanced effect associated with this crank is divided in the proportions of $\frac{j}{w}$ on the 'near' wheel, and $\frac{h}{w}$ on the 'off' wheel. The primary harmonic component of the disturbances arising from this source

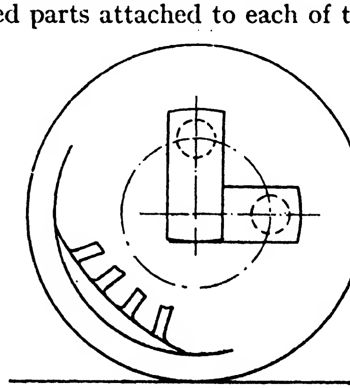


FIG. 57.

may, therefore, be neutralized with the aid of balance-weights that exert a pull amounting to $\frac{Mr\omega^2}{g}.\frac{j}{w}$ on the 'near' wheel, and one equal to $\frac{Mr\omega^2}{g}.\frac{h}{w}$ on the 'off' wheel. The direction of these centrifugal forces must be opposite that of the crank concerned.

From the symmetry of the mechanism it is to be inferred that the unbalanced effect on the crank numbered 2 could likewise be neutralized by means of a balance-weight placed opposite this crank, and heavy enough to produce pulls equal to $\frac{Mr\omega^2}{g}.\frac{h}{w}$ and $\frac{Mr\omega^2}{g}.\frac{j}{w}$ on the 'near' and 'off' wheels, respectively.

Hence if this measure of balance is to be secured with the help of a single mass attached to each of the wheels, its magnitude must be such as to induce a pull of $\frac{Mr\omega^2}{g}\ \frac{\sqrt{h^2+j^2}}{w}$, on the 'near' wheel, in the direction defined by α in Fig. 56, where $\tan\alpha = \frac{h}{j}$. A balance-weight of similar magnitude, but placed at a corresponding angle α in relation to the crank marked 2, would be required for the 'off' wheel, owing to like masses being assigned to each of the cranks. Practice in this country with inside-cylinder locomotives leads to the approximate relation $w = 2{\cdot}5y$, according to which $\alpha = 23$ deg., nearly.

27. The inertia force acting in the line of traction may next be examined by taking Fig. 58 to represent the above-mentioned

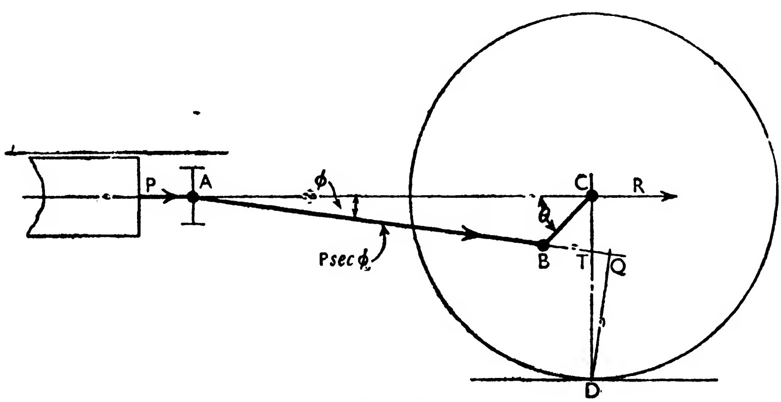


FIG. 58.

system. If then P denotes the force exerted by the working fluid on one of the pistons, its connecting rod will be subjected to a thrust of $P \sec\phi$. To complete the figure shown, draw at right-

angles to the line of stroke a line through C, to meet the rail in D. Also, produce the line AB to cut CD in T; and draw a line perpendicular to AB to pass through D and cut AB produced in Q.

Writing R for the horizontal component of the force exerted on the main bearing, on taking moments about a line through the point of contact D and at right-angles to the plane of the paper, we have

$$R.CD = P \sec \phi.DP$$
$$= P.DT,$$

whence $$R = P\frac{DT}{CD} \quad \ldots \ldots \quad (27.1)$$

Hence the propulsive force on the frame of the locomotive amounts to

$$P - P\frac{DT}{CD},$$

that is $$P\frac{CT}{CD}$$

on the forward stroke, and the same on the return stroke, since then

$$R - P = P\frac{DT}{CD} - P$$
$$= P\frac{CT}{CD}.$$

An analytical expression for R may easily be stated in terms of the components H and V given by equations (9.1) and (10.3).

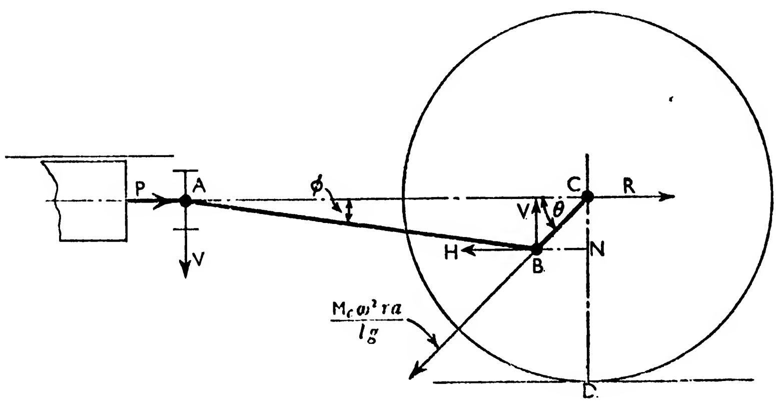


FIG. 59.

For this purpose draw in Fig. 59 a line through B and parallel to that of the stroke, to cut CD in N. The required relation follows on

taking moments of the forces about a line through D and at right-angles to the plane of the paper, thus

$$\begin{aligned} R.CD &= H.DN - V.BN \\ &= H.CD - (H.CN + V.BN), \end{aligned}$$

or

$$\begin{aligned} R &= H - \frac{CN}{CD}(H \sin \theta + V \cos \theta) \\ &= H - \frac{2r}{d}(H \sin \theta + V \cos \theta), \quad . \quad . \quad . \quad (27.2) \end{aligned}$$

where r denotes the throw of the crank, and d the diameter of the driving wheel.

The troublesome nature of this inertia effect in the way of balance arises from the fact that counter-weights attached to the wheels can neutralize only the primary harmonic component of the force R.

Taken together, equations (9.1) and (10.3) in this manner yield

$$R = \frac{M}{g}r\omega^2\left\{\cos \theta + \frac{r}{d}(\tfrac{1}{2}\gamma + \tfrac{1}{8}\gamma^3 + \tfrac{15}{256}\gamma^6 + \ldots) \sin \theta\right\} \quad (27.3)$$

for the primary harmonic component of R, where, as previously, $M = (M_p + \frac{b}{l}M_c)$. This amount of unbalance clearly necessitates the use of a counter-weight having a magnitude and position such as will produce the components $\frac{Mr\omega^2}{g}$ along the crank in the direction CB, and $\frac{Mr^2\omega^2}{gd}(\frac{1}{2}\gamma + \frac{1}{8}\gamma^3 + \frac{15}{256}\gamma^5 + \ldots)$ at right-angles to the crank in the outward direction, as shown in Fig. 60. The resultant of these components accordingly determines the value and

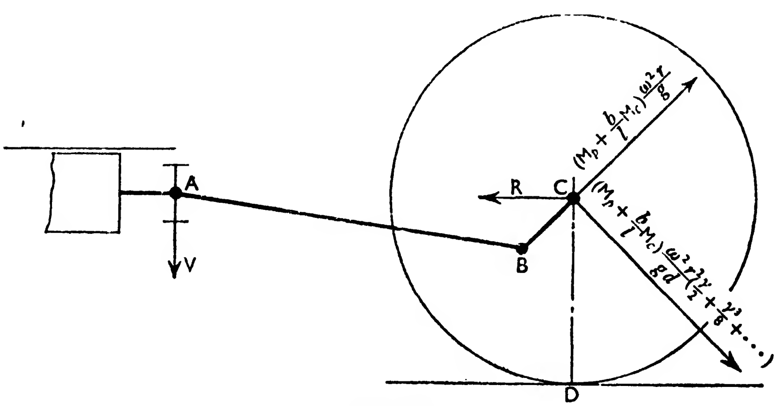


FIG. 60.

relative position of the requisite balance-weight, once its centre of gravity has been fixed with respect to the centre C. The other

cylinder and driving wheel in the system can be treated in a like manner.

28. A practical significance is attached to the consequences of the method of balancing just described, for if the component of the force specified in equation (27.3) were completely neutralized, the balance-weights would exert a vertical force equal to $\frac{Mr\omega^2}{g}\cos\theta$ for one cylinder, and $\frac{Mr\omega^2}{g}\cos\left(\theta - \frac{\pi}{2}\right)$ for the other. Thus a total upward force of magnitude

$$\frac{Mr\omega^2}{g}(\cos\theta + \sin\theta)$$

would operate on the revolving axle, and so subject the wheels to a load having a maximum value of $\sqrt{2}\frac{Mr\omega^2}{g}$ in each revolution of the axle. If this lifting effect at any instant exceeded the static load on the axle, the wheels would momentarily leave the rails and thereby make for slipping at the tread. But the minimum value of the load thus imposed on the wheels does not occur simultaneously, for which reason slipping does not necessarily take place at the tread of a particular wheel when the effective load on it tends towards small values, since the load on the other wheel may at the same instant be sufficiently large in value to ensure adhesion for the axle as a whole. From the present point of view the adhesion is unaffected by the balanced system of rotating parts on the locomotive under examination. Slipping at the tread of a wheel is, nevertheless, possible, and it may be avoided by counterbalancing only a fraction of the force given by equation (27.3), as is sometimes done in practice, though it is well to remember that partial balance causes a corresponding variation in the pull on the draw-bar.

When a wheel actually 'lifts' its subsequent contact with the rail imposes on the track a force that resembles a *hammer-blow* when the angular velocity ω is great. In addition to causing damage to the permanent way, such impacts are, under certain conditions, liable to initiate objectionable vibrations in the railway bridges concerned, in accordance with the treatment of Chapter III. Readers interested in this aspect of the general problem may also be referred to the *Report* [1] of the Bridge Stress Committee, and to Professor C. E. Inglis's special study [2] of the matter.

29. Coupled Wheels. An obvious method of mitigating the effect of hammer-blow is that of connecting the wheels by the

[1] H.M. Stationery Office (1928).

[2] *Mathematical Treatise on Vibrations in Railway Bridges.*

coupling rods indicated in Fig. 61 and, at the same time, distributing over the driving and trailing wheels the mass of the balance-weight assigned to each side of a given locomotive. The distribution of

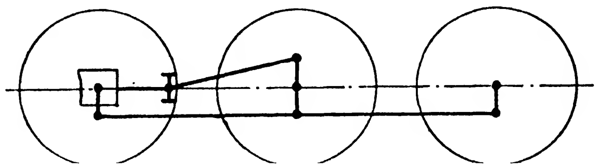


FIG. 61.

these weights might well be carried out in such a manner as will result in the greater proportion of the weight being carried by the trailing wheels, on account of the fact that the leading wheels act also as guides.

Moreover, coupled wheels offer a noteworthy advantage in connection with railway bridges, for the use of such systems affects the natural frequency of oscillation for a loaded structure of specified dimensions. One aspect of the implied modification is examined in Arts. 54–56, where it is shown that the period of vibration is influenced, to an extent which depends on the degree of stiffness afforded by the various joints, by the way in which a prescribed total load is distributed over a given structure. The natural frequency of oscillation for a loaded bridge thus varies from instant to instant during the interval taken by a locomotive to cross the structure, within limits that depend on the relative size and weight of the mechanical and structural systems involved, and this sometimes gives rise to the phenomenon of *interference* discussed in Art. 44. The combined effect of these factors must be included in a full analysis of the motion of a bridge.

30. Outside Cylinders. The foregoing results are applicable to this type of engine, provided the notation is accordingly modified. If, then, Fig. 62 be taken to represent the driving axle with similar masses attached to each of the cranks, it follows that a balance-weight equal to $\dfrac{Mr\omega^2}{g}\dfrac{\sqrt{h^2+j^2}}{w}$ would suffice to neutralize the force referred to the 'near' wheel, where M denotes the same quantity as in equation (27.3). Here the dimension w may vary within the approximate range of $0{\cdot}8y$ and $0{\cdot}9y$ so far as practice in this country is concerned, whence the angle defined by $\tan\alpha = \dfrac{h}{j}$ in Fig. 56 lies within the limits of $\alpha = 3$ deg. and $\alpha = 7$ deg.

Equations corresponding with those already found for the inside-cylinder type of engine can therefore readily be derived for

the case of outside cylinders, as may be explained with the help of a numerical example.

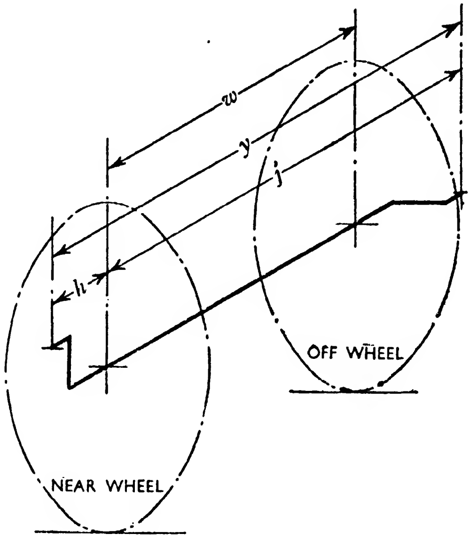


FIG. 62.

Ex. Apply the preceding results to the system shown in Fig. 63, involving two outside-cylinders and four coupled wheels, on the supposition that 80 miles an hour is the maximum speed of the

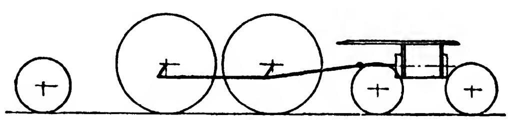


FIG. 63.

locomotive, and that the necessary balance-weights for the reciprocating parts are divided in the ratio 43 : 48 between the front and rear wheels. The specification of the system is as follows:

diameter of each wheel (d) = 6 ft. 8 in.;
stroke of the engine ($2r$) = 26 in.;
length of the connecting-rod (l) = 10 ft. 10 in.;
distance between the centre-lines of the cylinders (y) = 6 ft. 10 in.;
distance between the centre-lines of the coupling rods = 6 ft. 2 in.;
distance between the planes in which the balance-weights revolve (w) = 5 ft. 2 in.;
distance from the axis of rotation to the centre of gravity of each balance-weight = 3 ft.;
static load on a front wheel = 43,000 lb.;
static load on a rear wheel = 48,000 lb.;

weight of the rotating part of a connecting-rod $\left(\frac{a}{l}M_c\right) = 332$ lb.;

weight of the reciprocating part denoted by $\left(M_p + \frac{b}{l}M_c\right) = 818$ lb.;

weight of the part denoted by $\left(\frac{ab - k^2}{l^2}M_c\right) = 25$ lb.;

weight of the unbalanced part of a crank and its pin $= 95$ lb.;
weight of a coupling rod and its connections on each wheel $= 190$ lb.
The dimensions of the driving axle are given in Fig. 64.

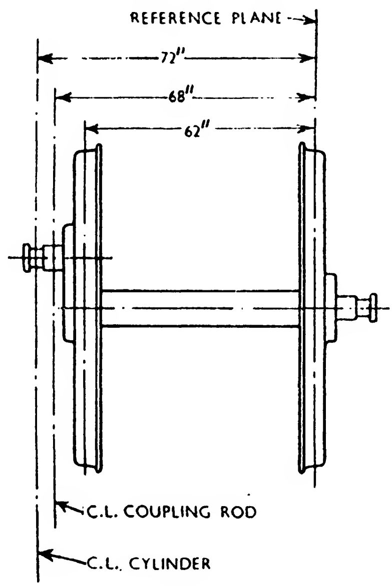


FIG. 64.

In the following calculations we shall confine our attention to the forces associated with one of the sets of moving parts, say on the 'near' crank, as it is a simple matter to refer the resulting quantities to the 'off' crank.

Consider first the rotating parts, then an application of elementary principles shows that:

(i) the rotating parts on the crankpin, amounting to $(332 + 95)$ lb., impose a load of

$$\frac{427 \times 72}{62} \text{ lb.}$$

on the 'near' wheel, and one of

$$\frac{427 \times 10}{62} \text{ lb.}$$

on the 'off' wheel;

(ii) the rotating mass of the coupling rod and its connections, amounting to 190 lb., imposes an additional load of

$$\frac{190 \times 68}{62} \text{ lb.}$$

on the 'near' wheel, and one of

$$\frac{190 \times 6}{62} \text{ lb.}$$

on the 'off' wheel. Disturbances from this source consequently give rise to unbalanced effects of 704·4 lb. on the 'near' wheel, and 87·4 lb. on the 'off' wheel. Taking moments of all these forces about the axis of rotation, bearing in mind that the above-mentioned masses act at a radius, of 13 in. and the balance-weights at a radius of 36 in., we realize that the rotating parts can be counter-balanced by weights placed opposite the 'near' crank and of magnitudes

$$\frac{704{\cdot}4 \times 13}{36} \text{ lb.}$$

or 254 lb. on the 'near' wheel, and

$$\frac{87{\cdot}4 \times 13}{36} \text{ lb.}$$

or 31·6 lb. on the 'off' wheel.

Turning now to the reciprocating parts, it has been demonstrated that the force defined in equation (27.3) necessitates the use of balance-weights that will exert a pull having the components $\left(M_p + \frac{b}{l}M_c\right)\frac{r\omega^2}{g}$ in a direction opposite that of the 'near' crank, and

$$\left(M_p + \frac{b}{l}M_c\right)\frac{r^2\omega^2}{gd}\left(\tfrac{1}{2}\gamma + \tfrac{1}{8}\gamma^3 + \tfrac{15}{256}\gamma^5 + \ldots\right)$$

at right-angles to that direction. These expressions give, on inserting the known values for the various terms,

$\frac{886{\cdot}3}{g}\omega^2$ lb. outwards along the 'near' crank, and

$\frac{7{\cdot}2}{g}\omega^2$ lb. at right-angles to that direction.

Remembering that the counter-weights for the reciprocating parts are to be divided in the proportion 43 : 48 between the front and rear wheels, and that these weights rotate in a circle of 3 ft. radius, an application of the principle of moments leads to the result that the front wheel must carry a weight of

$$\frac{43 \times 13 \times 886{\cdot}3}{48 \times 36 \times g}\omega^2 \text{ lb.}$$

placed opposite the 'near' crank, and one of

$$\frac{43 \times 13 \times 7{\cdot}2}{48 \times 36 \times g}\omega^2 \text{ lb.}$$

placed at right-angles to that direction. Also, the rear wheel on the same side must carry a weight of

$$\frac{48 \times 13 \times 886{\cdot}3}{43 \times 36 \times g}\omega^2 \text{ lb.}$$

disposed opposite the crank in question, and one of

$$\frac{48 \times 13 \times 7{\cdot}2}{43 \times 36 \times g}\omega^2 \text{ lb.}$$

disposed at right-angles to that direction.

Combining these results in a vectorial sense, it is seen that the reciprocating parts attached to the 'near' crank can be balanced by placing:

(i) on the front wheel a weight of 139·6 lb. opposite the given crank, and one of 1·1 lb. arranged at right-angles to that direction;

(ii) on the rear wheel a weight of 155·8 lb. opposite the given crank, and one of 1·3 lb. arranged at right-angles to that direction.

A second application of vectors leads further to the resultant which determines the single counter-weight needed for each of the wheels under examination. This operation involves so small an angle that the resultant gives practically the same values, namely a weight of 139·6 lb. on the front wheel, and one of 155·8 lb. on the rear wheel, but these are not exactly opposite the particular crank, being displaced about 27 minutes of arc from that direction.

If we, finally, combine the weights required for both the rotating and reciprocating parts and, for brevity in working, neglect the short distance between the plane containing the main bearings and that in which the weights rotate, it will be seen that our degree of balance is secured by fitting to the front wheel a weight of

$$\sqrt{(254 + 139{\cdot}6)^2 + 31{\cdot}6^2} \text{ lb.}$$

or 395 lb. in a position defined by $\tan \alpha = \dfrac{31{\cdot}6}{393{\cdot}6}$, that is $\alpha = 4$ deg. – 35 min. in the notation of Fig. 56.

A repetition of the procedure with reference to the balance-weights for the rear wheels of the coupled system shows that these are equivalent to the resultant of:

(i) a weight 155·8 lb. displaced 27 minutes of arc from the position opposite that of the crank concerned; and

(ii) a weight of $\frac{13}{36}\sqrt{208{\cdot}4^2 + 18{\cdot}4^2}$ lb., that is $75{\cdot}5$ lb., placed so that $\tan\alpha = \dfrac{18{\cdot}4}{208{\cdot}4}$, to counterbalance a coupling rod and its attachments. Thus the magnitude and position of the single weight needed for the purpose is specified by $231{\cdot}3$ lb. and $\alpha = 5$ deg. 15 min.

There remains to be estimated the maximum value of the upward force or pull tending to lift the locomotive, which has been shown to equal

$$\sqrt{2}\frac{r\omega^2}{g}\left(M_p + \frac{b}{l}M_c\right).$$

When the prescribed engine is travelling at 80 miles an hour, we thus ascertain

$$\frac{\sqrt{2} \times 13 \times 35{\cdot}2^2 \times 818}{12 \times 32{\cdot}2}\text{ lb.},$$

or approximately 48,200 lb., to be the greatest value of the force tending to lift the four coupled wheels. The load on the rail accordingly varies between the range of $(91{,}000 \pm 48{,}200)$ lb., or about $62{\cdot}22$ tons and $19{\cdot}15$ tons, during each revolution of the cranks. The impact-factor of nearly 50 per cent. involved here would be correspondingly reduced were only one-half or one-third of the reciprocating parts balanced in this manner, as is sometimes done in practice.

Although we have limited our investigation to the primary harmonic component of the force defined in equation (27.2), for the obvious reason that this alone can be neutralized by weights fitted to the wheels of locomotives, it is not to be inferred that relatively small values are always associated with the secondary harmonic component of the unbalanced force R. A case in point is presented in the above problem, as is seen on inserting the appropriate data in the expression for this component, namely

$$\frac{r\omega^2}{g}\left[\left(M_p + \frac{b}{l}M_c\right)\left\{(\gamma + \tfrac{1}{4}\gamma^3 + \ldots)\cos 2\theta - \frac{r}{d}\sin 2\theta\right\} - \frac{r}{d}\cdot\frac{ab - k^2}{l^2}M_c \sin 2\theta\right],$$

for it then reduces to

$$(2{\cdot}54 \cos 2\theta - 4{\cdot}26 \sin 2\theta) r\omega^2 \text{ lb.},$$

or about 3 tons when the speed of the engine is 80 miles an hour. Although considerable in magnitude, this unbalanced effect is applied with a relatively high frequency compared with the natural frequency of oscillation for railway bridges generally. It is

therefore common practice to neglect all but the primary harmonic component of the unbalanced force which may operate on a railway bridge.

31. Motion of Locomotives. Without in any way affecting the analysis of the general problem of motion, we shall for the present suppose the locomotive to consist of a single cylinder, and specify the system by writing:

M = weight of the reciprocating parts connected to the crank;
M_w = weight of all the wheels and any balance-weights which may be fitted to them;
M_e = weight of the remaining parts of the locomotive, such as the frame, boiler, tender, water, fuel, etc.
k_w = radius of gyration of a wheel (including balance-weights) about the axis of rotation;
l = length of the connecting rod;
r = throw of the crank;
d = diameter of the wheels, which are assumed to be similar in this connection;
x = displacement of the locomotive with respect to the track;
y = displacement of the piston with respect to the cylinder, reckoned from the dead-centre position;
z = displacement of the reciprocating parts with respect to the track.

We have to decide at the outset, for reasons stated in our previous discussion on Lagrange's method, on the variable or co-ordinate which best serves the purpose in view. To fix ideas on the point, let the motion of the given system be referred to the crank-angle θ

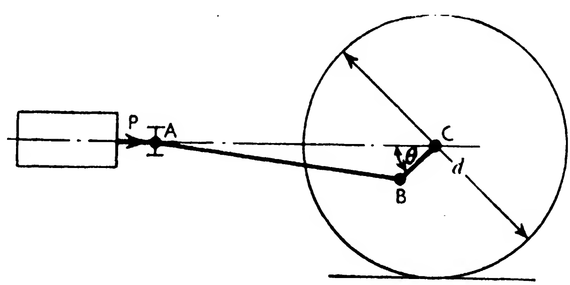


FIG. 65.

indicated in Fig. 65, then the corresponding form of equation (19.8) is obtained by substituting θ for q_r, whence

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta,$$

where T denotes the kinetic energy of the system, and Q_θ the

generalized component of force. If a_{11} be written for the corresponding coefficient of inertia, it has been shown that

$$2T = a_{11}\dot{\theta}^2$$

holds in the above formula, where it is understood that a_{11} must include, in one form or another, terms for all the masses in the prescribed system. In this way our choice of the several variables leads to the expression

$$a_{11}\ddot{\theta} + \tfrac{1}{2}\frac{da_{11}}{d\theta}\dot{\theta}^2 = Q_\theta \quad . \quad . \quad . \quad . \quad . \quad (31.1)$$

Taking account of all the specified quantities, we may at once write down

$$a_{11} = \frac{1}{g}\left\{M\left(\frac{dz}{d\theta}\right)^2 + M_w\left(\frac{dx}{d\theta}\right)^2 + M_w k_w^2 + M_e\left(\frac{dx}{d\theta}\right)^2\right\} \quad (31.2)$$

in terms of the variable θ, where k_w is, for convenience in working, taken to apply to each of the wheels. The use of the symbol for 'total' differentiation implies that all the variables are expressed in terms of θ alone. To effect this transformation we employ the geometrical relations for the mechanism, namely

$$x = \tfrac{1}{2}\theta d, \; z = r(1 - \cos\theta) + \tfrac{1}{2}\frac{r^2}{l}\sin^2\theta + x,$$

to the first approximation. These in turn lead to

$$\frac{dx}{d\theta} = \tfrac{1}{2}d, \; \frac{dz}{d\theta} = r(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) + \tfrac{1}{2}d,$$

where $\gamma = \dfrac{r}{l}$, by virtue of which we can write

$$a_{11} = \frac{1}{g}[M\{r(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) + \tfrac{1}{2}d\}^2 + M_w(\tfrac{1}{4}d^2 + k_w^2) + \tfrac{1}{4}M_e d^2] \quad (31.3)$$

and, by differentiation,

$$\frac{da_{11}}{d\theta} = \frac{Mr^2}{g}\{\cos\theta(2\sin\theta + \frac{d}{r}) + \gamma(2\sin\theta\cos 2\theta + \sin 2\theta\cos\theta$$
$$+ \frac{d}{r}\cos 2\theta) + \tfrac{1}{2}\gamma^2\sin 4\theta\} \quad . \quad . \quad . \quad (31.4)$$

for the coefficients in equation (31.1).

If the obliquity of the connecting-rod is neglected as of secondary importance in a problem involving the complete locomotive, the relation

$$\{M(r\sin\theta + \tfrac{1}{2}d)^2 + M_w(\tfrac{1}{4}d^2 + k_w^2) + \tfrac{1}{4}M_e d^2\}\ddot{\theta}$$
$$+ \tfrac{1}{2}Mr\cos\theta(2r\sin\theta + d)\dot{\theta}^2 = gQ_\theta \quad . \quad . \quad . \quad (31.5)$$

follows on inserting the expressions (31.3) and (31.4) in equation (31.1).

To state Q_θ in terms of known quantities, let P = total force exerted by the working fluid on the piston, and R = pull on the

draw-bar when θ defines the position of the crank. With this notation we have, on calculating the work done by the forces during an infinitesimal change in the configuration of the system,

$$Q_\theta = P\frac{\partial y}{\partial \theta} - R\frac{\partial x}{\partial \theta}.$$

Now the geometrical relation for y in terms of θ is, as may easily be verified,

$$y = r(1 - \cos\theta) + \tfrac{1}{2}\frac{r^2}{l}\sin^2\theta,$$

approximately, so that

$$\frac{dy}{d\theta} \doteq r(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)\,;$$

we can write also

$$\frac{dx}{d\theta} = \tfrac{1}{2}d$$

from the preceding results. Substituting these values in the expression for Q_θ and neglecting the obliquity of the connecting-rod in accordance with the procedure, we arrive at

$$Q_\theta = Pr\sin\theta - \tfrac{1}{2}Rd.$$

Equation (31.5) may consequently be written in the form

$$\{M(r\sin\theta + \tfrac{1}{2}d)^2 + M_w(\tfrac{1}{4}d^2 + k_w^2) + \tfrac{1}{4}M_e d^2\}\ddot{\theta}$$
$$+ \tfrac{1}{2}Mr\cos\theta(2r\sin\theta + d)\dot{\theta}^2 = g(Pr\sin\theta - \tfrac{1}{2}Rd) \quad . \quad (31.6)$$

The motion is determined by this relation, since it contains, along with the given dimensions of the locomotive, terms for quantities which can be evaluated from diagrams that exhibit the variations in the steam-pressure on the piston and the pull on the draw-bar during a working 'cycle'. The terms on the right-hand side together represent a Fourier series having the form

$$A_0 + \sum_{n=1}^{\infty} A_n(\cos n\theta + B_n\sin n\theta),$$

where $-\pi \leq \theta \leq \pi$ as in Art. 24. If the force $(Pr\sin\theta - \tfrac{1}{2}Rd)$ follows a simple harmonic law of variation having a frequency that approximately coincides with the natural frequency of oscillation for the system formed by a locomotive and a number of carriages or wagons, the treatment of Art. 46 shows that the phenomenon of *resonance* is then a probable condition. Only the terms up to the first harmonic in the Fourier series should be used in the process of solving the last equation, owing to the implied degree of approximation.

At this stage of the work it is a simple matter to express equation (31.6) in terms of any other of the several variables. Suppose, by way of illustration, that we wish to obtain the motion in terms

of x and its derivatives. The transformation follows from the expressions

$$\frac{dy}{dx} = 2\frac{r}{d}\sin\theta,\ \dot{\theta} = \frac{2}{d}\dot{x},\ \ddot{\theta} = \frac{2}{d}\ddot{x}$$

given by the geometrical relations stated above, since on substituting these values in equation (31.6), we have

$$\left\{\frac{4M}{d^2}(r\sin\theta + \tfrac{1}{2}d)^2 + M_w\left(1 + 4\frac{k^2_w}{d^2}\right) + M_e\right\}\ddot{x}$$
$$\cdot\quad \frac{4Mr\cos\theta}{d^3}(2r\sin\theta + d)\dot{x}^2 = g\left(2\frac{r}{d}P\sin\theta - R\right) \quad . \quad (31.7)$$

for the required equation of motion. This is expressed in terms of forces, compared with moments of forces or couples in the case of the previous equation.

The corresponding equations for uniform speed follow as a matter of course on writing either $\ddot{\theta} = 0$ or $\ddot{x} = 0$ in the foregoing results.

32. We may now examine the effect of the reciprocating parts on the motion of the engine already specified, on the assumption that the rotating parts are initially balanced. In order to make convenient substitutions for the terms referring to the mass of the wheels and frame, write

P_1 = horizontal reaction on the crankpin,
F = tractive force at the tread of each wheel,
F_1 = horizontal reaction on the main bearing,

as indicated in Fig. 66.

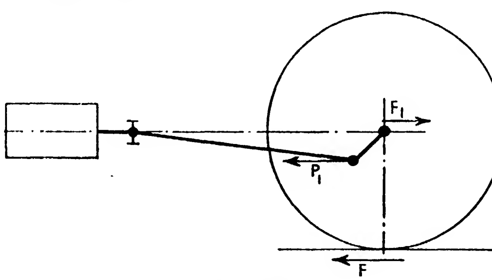


FIG. 66.

In equation (31.6) we now have

$$\tfrac{1}{2}\frac{M_w d}{g}\ddot{\theta} = \frac{M_w}{g}\ddot{x} = F + P_1 - F_1,$$
$$\frac{M_w k^2_w}{g}\ddot{\theta} = \frac{2M_w k^2_w}{gd}\ddot{x} = P_1 r\sin\theta - \tfrac{1}{2}Fd,$$
$$\tfrac{1}{2}\frac{M_e d}{g}\ddot{\theta} = \frac{M_e}{g}\ddot{x} = F_1 - P - R,$$

whence, on adding,

$$\{M_w(\tfrac{1}{4}d^2 + k^2_w) + \tfrac{1}{4}M_e d^2\}\ddot{\theta} = g\{P_1(r\sin\theta + \tfrac{1}{2}d) - \tfrac{1}{2}Pd - \tfrac{1}{2}Rd\}.$$

Making these substitutions and rearranging the terms, it will be found that

$$M(r\sin\theta+\tfrac{1}{2}d)^2\ddot{\theta}+Mr\cos\theta(r\sin\theta+\tfrac{1}{2}d)\dot{\theta}^2=g(P-P_1)(r\sin\theta+\tfrac{1}{2}d),$$

or $$M(r\sin\theta+\tfrac{1}{2}d)\ddot{\theta}+Mr\cos\theta\dot{\theta}^2=g(P-P_1)\ . \quad . \quad (32.1)$$

determines the effect of the reciprocating parts on the motion of the system.

It therefore appears, since P can always be expressed as a function of θ, that our equation of motion is of the type

$$f(\theta)\ddot{\theta}+\tfrac{1}{2}f'(\theta)\dot{\theta}^2+G(\theta)=0,$$

where $f'(\theta)=\frac{d}{d\theta}f(\theta)$, and $f(\theta)$, $G(\theta)$ are known functions in the case of a given engine. The solution is, as remarked in connection with equation (21.4), of the form

$$t=\pm\int\sqrt{\frac{f(\theta)}{c_1-2\int G(\theta)d\theta}}\,d\theta+c_2,$$

where c_1 and c_2 signify constants which depend on the initial conditions of motion.

When a locomotive consists of a number of cylinders and cranks, we take account of the phase-angles and combine the indicator diagrams so that they refer to the particular crank whose position is defined by θ, in a manner which will be explained in Art. 35.

33. The general equations (31.6) and (31.7) can be used for a number of purposes other than those relating to the problem of balance, such as, for instance, that of estimating the effect of high boiler-pressure and of early cut-off on the motion of prescribed locomotives, since these factors enter into the analysis by way of the P-terms. The advantage offered by high-tensile steels for the construction of parts having a minimum weight affords another application. Further discussion of these matters would lead to a problem in economics which lies beyond the limits of the present work.

34. In the particular case of an engine having two cranks arranged 90 deg. apart, with similar sets of moving parts, we have expressions of the form $A\cos 2\theta$ and $A\cos 2(\theta+\pi)$ for the secondary components of the force R in equation (27.2) when referred to the respective cranks. Notwithstanding the fact that these components are thus mutually balanced with reference to the pull on the draw-bar, they together produce a couple which acts about a vertical axis, with a periodicity equal to that of the engine.

35. Arrangement of the Cranks. The procedure to be followed in applying equation (32.1) to systems of cranks arranged in different ways will be understood from the following examples,

where it is assumed, for brevity in working, that the engines are moving with the uniform speed defined by $\dot{\theta} = \omega$. Then, with Q written for the inertia force produced by the reciprocating parts,

$$Q = \frac{Mr\omega^2}{g} \cos \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35.1)$$

to our order of approximation.

(*a*) *Two Cylinders*. Taking M to represent the weight of the reciprocating parts attached to each of the cranks shown in Fig. 67,

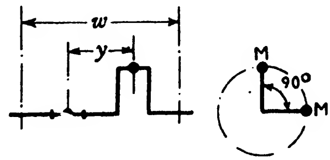


FIG. 67.

and writing θ, $\left(\theta - \frac{\pi}{2}\right)$ for the position of the cranks, the last equation gives

$$Q = \frac{Mr\omega^2}{g}\left\{\cos \theta + \cos \left(\theta - \frac{\pi}{2}\right)\right\}$$

$$= \frac{Mr\omega^2}{g}(\cos \theta + \sin \theta)$$

for the complete system in question. Hence the maximum value of this disturbance, occurring when θ is equal to either 45 deg. or 225 deg., amounts to $\sqrt{2}\frac{Mr\omega^2}{g}$. Consequently a couple of magnitude $\frac{1}{\sqrt{2}}\frac{Mr\omega^2}{g}y$, acting in the horizontal plane, is an inherent characteristic of this type of engine. There is also an unbalanced couple due to the secondary harmonic component of the inertia forces; these components are anti-phased, and therefore balanced in the direction of the draw-bar pull.

(*b*) *Three Cylinders*. Under this heading we shall investigate three classes, arranged according to the relative positions of the cranks and the ratios of the masses assigned to each of the cylinders.

Equal masses, Fig. 68. If M be the weight of the reciprocating

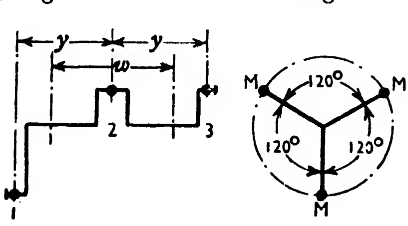


FIG. 68.

parts associated with each of the cylinders, and θ the position of the crank numbered *1* in the figure, we have

$$Q = \frac{Mr\omega^2}{g}\left\{\cos \theta + \cos \left(\theta + \frac{2\pi}{3}\right) + \cos \left(\theta + \frac{4\pi}{3}\right)\right\}$$

as the expression for the unbalanced force according to equation (35.1). Since the bracketed terms cancel, the engine is inherently balanced to this extent.

An unbalanced couple is, however, present in the system, as may readily be seen by taking moments of the forces about lines in the plane of the crank marked *3*, when it will be found that the horizontal couple is equal to

$$\frac{Mr\omega^2}{g}y\{2\cos\theta + \cos(\theta + \frac{2\pi}{3})\},$$

and that its maximum value is $\sqrt{3}\dfrac{Mr\omega^2}{g}y$.

Equal masses, Fig. 69. Repeating the process, with M denoting the weight of the reciprocating parts here assigned to each of the

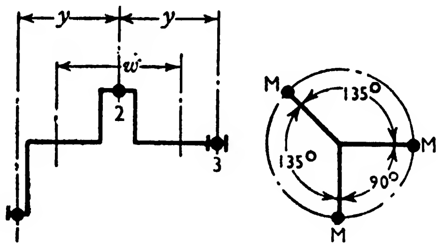


FIG. 69.

cylinders, and θ defining the position of the crank numbered *1*, it follows that

$$Q = \frac{Mr\omega^2}{g}\{\cos\theta + \cos(\theta + 135°) + \cos(\theta + 270°)\}$$

$$= \tfrac{1}{2}(2 - \sqrt{2})\frac{Mr\omega^2}{g}(\cos\theta + \sin\theta),$$

the greatest value of which is $0{\cdot}415\dfrac{Mr\omega^2}{g}$.

To evaluate the related couple, we refer the moments of the forces to the plane of the crank marked *3*, and so obtain

$$\frac{Mr\omega^2}{g}y\{2\cos\theta + \cos(\theta + 135°)\}$$

for the quantity in question. The maximum value of this unbalanced couple, amounting to $\sqrt{2}\dfrac{Mr\omega^2}{g}y$, is therefore less than that found for the system indicated in Fig. 68.

Unequal masses, Fig. 70. This arrangement, in which reciprocating parts of weights $\dfrac{1}{\sqrt{2}}M$, M, $\dfrac{1}{\sqrt{2}}M$ are associated in turn with

the cranks marked *1*, *2*, *3* in the figure, improves the balance of the engine as a whole. The extent of this improvement may be

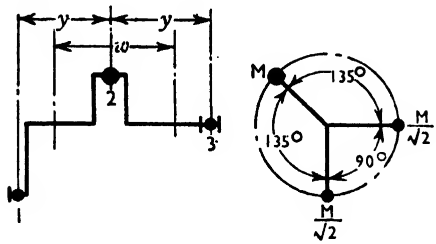


FIG. 70.

estimated by applying equation (35.1) and writing θ for the position of the crank numbered *1*, whence

$$Q = \frac{Mr\omega^2}{g}\left\{\frac{1}{\sqrt{2}}\cos\theta + \cos(\theta + 135°) + \frac{1}{\sqrt{2}}\cos(\theta + 270°)\right\}$$

$$= 0.$$

Further, when the moments of the unbalanced forces are referred to the plane of the crank marked *3*, the periodic couple in the horizontal plane is given by

$$\frac{Mr\omega^2}{g}y\left\{\frac{2}{\sqrt{2}}\cos\theta + \cos(\theta + 135°)\right\};$$

hence $\frac{Mr\omega^2}{g}y$ is the greatest value of the disturbing couple.

The centre-line of the middle cylinder in this type of locomotive is sometimes inclined to the horizontal, to enable the connecting rod to clear the front axle, but the usual practice does not greatly affect the foregoing results, as may be proved without much difficulty in a given case. The obliquity of the connecting rod may likewise be taken into account, when it will be found that the forces acting on the draw-bar of the system shown in Fig. 69 are inherently balanced up to the third harmonic component.

(*c*) *Four Cylinders*. It will be convenient here to combine the two driving-axles as indicated in the corresponding figures.

Equal masses, Fig. 71. Thus this system may be reduced to one having two pairs of cranks arranged 180 deg. apart, with the

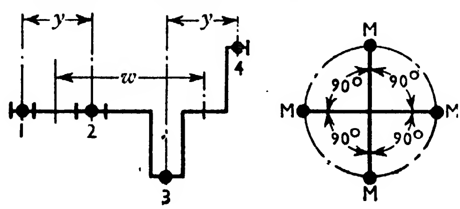


FIG. 71.

pairs mutually at right-angles, so that the combined axle contains four cranks disposed symmetrically in the axial direction.

The present arrangement leads to inherent balance in the sense of the force expressed by equation (35.1), because on writing M for the reciprocating parts fitted to each of the cylinders, and defining the position of the crank numbered *1* by θ, we obtain

$$Q = \frac{Mr\omega^2}{g}\left\{\cos\theta + \cos\left(\theta + \frac{\pi}{2}\right) + \cos(\theta + \pi) + \cos\left(\theta + \frac{3\pi}{2}\right)\right\}$$
$$= 0.$$

Hence, since the pairs (*1*, *4*) and (*2*, *3*) of cranks are actually on opposite sides of the frame,

$$\frac{Mr\omega^2}{g}y(\sin\theta + \cos\theta)$$

is the horizontal couple, so that $\sqrt{2}\frac{Mr\omega^2}{g}y$ is the maximum effect of this disturbance.

Unequal masses, Fig. 72. As indicated in the figure, the weight of the reciprocating parts amounts to M for each of the cranks

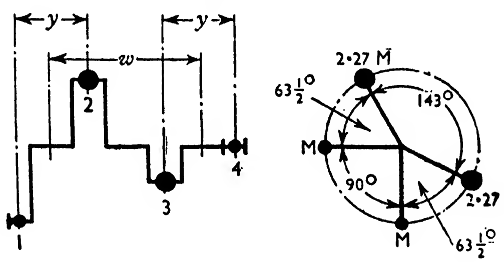


FIG. 72.

marked *1*, *4*, and to $2{\cdot}27M$ for each of the cranks numbered *2*, *3*. Particular interest is attached to this arrangement of the masses and crank-angles because it results in zero values for the force and the couple associated with equation (35.1).

To verify the point, let θ define the position of the crank marked *1*, then the usual procedure leads to

$$Q = \frac{Mr\omega^2}{g}\{\cos\theta + 2{\cdot}27\cos(\theta + 63\tfrac{1}{2}^\circ) + 2{\cdot}27\cos(\theta + 206\tfrac{1}{2}^\circ)$$
$$+ \cos(\theta + 270^\circ)\} = 0.$$

Moreover, on taking moments of the forces about lines perpendicular to the axle and passing through the mid-point of the outer-bearings, the expression for the disturbing couple becomes

$$\frac{Mr\omega^2}{g}y\{\cos\theta + 2{\cdot}27\cos(\theta + 63\tfrac{1}{2}^\circ) + 2{\cdot}27\cos(\theta + 206\tfrac{1}{2}^\circ)$$
$$+ \cos(\theta + 270^\circ)\} = 0,$$

as might have been inferred from the corresponding result for Q.

36. We shall now approach the problem considered in the previous Article by way of equation (31.7), in a manner which enables us to investigate the resulting disturbance in the motion of the train as a whole. The following applications will suffice to explain the general method. With the same notation as before, in each of the examples taken we have $(M + M_w + M_e)$ = total weight of the locomotive.

Two cylinders, Fig. 67. If θ and $\left(\theta + \frac{\pi}{2}\right)$ denote the respective crank-angles, then the terms

$$\sin^2\theta + \sin^2\left(\theta + \frac{\pi}{2}\right) = 1,$$

$$\sin\theta + \sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta + \cos\theta,$$

$$\cos\theta + \cos\left(\theta + \frac{\pi}{2}\right) = \cos\theta - \sin\theta,$$

$$\cos\theta\sin\theta + \cos\left(\theta + \frac{\pi}{2}\right)\sin\left(\theta + \frac{\pi}{2}\right) = 0$$

enter into equation (31.7). On effecting these substitutions, the equation of motion becomes

$$\left[4\frac{Mr}{d^2}\{r + d(\sin\theta + \cos\theta)\} + 4M_w\frac{k^2{}_w}{d^2} + \text{total weight of locomotive}\right]\ddot{x}$$

$$+ 4\frac{Mr}{d^2}(\cos\theta - \sin\theta)\dot{x}^2 = g\left(P\frac{dy}{dx} - R\right)$$

where, as previously, $\frac{dy}{dx} = 2\frac{r}{d}\sin\theta$.

Three cylinders, Fig. 68. Writing in equation (31.7) θ, $\left(\theta + \frac{2\pi}{3}\right)$, $\left(\theta + \frac{4\pi}{3}\right)$ for the respective crank-angles, we have

$$\left\{\frac{4M}{d^2}\left(\tfrac{3}{2}r^2 + \tfrac{3}{4}d^2\right) + M_w\left(1 + 4\frac{k^2{}_w}{d^2}\right) + M_e\right\}\ddot{x} = g\left(P\frac{dy}{dx} - R\right),$$

or $$\left\{6M\left(\frac{r}{d}\right)^2 + 4M_w\left(\frac{k_w}{d}\right)^2 + \text{total weight of locomotive}\right\}\ddot{x}$$

$$= g\left(P\frac{dy}{dx} - R\right)$$

as the equation of motion. Thus the effect of the reciprocating parts on the pull at the draw-bar is seen to be approximately equivalent to that of one-half their mass concentrated at the crank-pin.

Four cylinders, Fig. 71. With θ, $(\theta + \frac{1}{2}\pi)$, $(\theta + \pi)$, $(\theta + \frac{3}{2}\pi)$ written in turn for the crank-angles, an application of equation

(31.7) shows that the motion of the specified engine is determined by

$$\left\{8M\left(\frac{r}{d}\right)^2 + 4M_w\left(\frac{k_w}{d}\right)^2 + \text{total weight of locomotive}\right\}\ddot{x}$$
$$= g\left(P\frac{dy}{dx} - R\right)$$

The effect on the pull at the draw-bar of the parts under consideration is therefore, again, equal to that of one-half their mass concentrated at the crank-pin.

In the foregoing examples we must, for reasons previously stated, introduce only the primary harmonic terms of the Fourier series in the right-hand side of the equations of motion. The accuracy of the result may subsequently be improved by extending the analysis so as to include the second, third, . . . harmonic terms, taken in order.

37. Valve-Gears. An examination of all types of gear is here rendered impossible by limitations of space, and unnecessary by reason of the fact that generalized methods will be applied to representative forms of gear which are distinguished by the property that the relation between the displacement (x) of the valve from its mid-position and the angle (θ) of the main crank may be approximately represented by the expression

$$x = A \cos \theta - B \sin \theta, \quad . \quad . \quad . \quad . \quad (37.1)$$

where the coefficients A and B depend on the type of gear under consideration.

Once these coefficients are known for a prescribed gear having an effective weight M, Lagrange's method affords a means of solving the practical problem involved, because the kinetic energy T of the mechanism is then given by

$$2T = \frac{M}{g}\dot{x}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.2)$$

The weight denoted by M must be evaluated for each case, since it is naturally influenced by the form, as well as the size, of a given gear or link-motion. Analytical methods may be used for the purpose of estimating M, but an experimentally determined value is to be preferred, and this might be found during the process of measuring the frictional forces that oppose the motion in particular circumstances. Consequently the quantity indicated by M consists of the weight of the valve and its spindle *plus* a certain proportion of the weight of the operating gear. This proportion may be as small as $\frac{1}{15}$ in the case of Stephenson's link-motion, but it varies between a comparatively wide range of values, and we shall at the outset assume that an appropriate value can be assigned to M.

Since instances are met with in which the inertia effect of the

gear causes excessive wear on the keys, pins and joints, what remains of this chapter will be devoted to an examination of the inertia forces and couples.

(*a*) *Stephenson's Link-motion.* Consider an engine having a stroke equal to $2r$, on to which is fitted the mechanism specified by the dimensions shown in Fig. 73, where the crank-angle θ corresponds

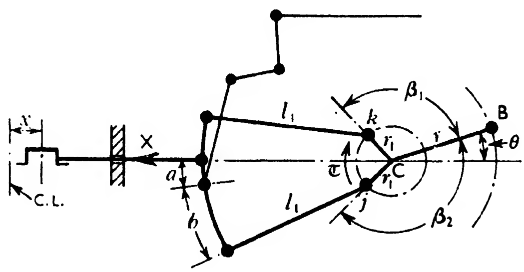


FIG. 73.

with the displacement x of the valve from its mid-position. But as it is common practice to make $Ck = Cj$, and $\beta_1 = \beta_2$, these proportions will be assumed here, by writing $Ck = Cj = r_1$, and $\beta_1 = \beta_2 = \beta$. It may be added that the symbol b represents one-half the length of the link, and that the position of the link, defined by a, is supposed to remain unchanged during the motion now to be examined.

With l_1 denoting the common length of the eccentric rods, the introduction of the geometrical relations for the system gives the coefficients of equation (37.1), in the form

$$\left.\begin{aligned} A &= r_1\left(\cos\beta - \frac{b^2 - a^2}{bl_1}\sin\beta\right), \\ B &= \frac{a}{b}r_1\sin\beta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (37.3)$$

Now if we substitute x for q_r, and $-X$ for Q_r in the formula (19.8), it follows from the preceding discussion on generalized co-ordinates that the resulting expression, namely

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = -X \quad . \quad . \quad . \quad . \quad (37.4)$$

determines the inertia-reaction X on the valve-spindle.

In view of equation (37.2) and the fact that the coefficients A and B in (37.3) are both independent of x and θ, we therefore have

$$\begin{aligned} \frac{\partial T}{\partial \dot{x}} &= \frac{M}{g}\dot{x} \\ &= -\frac{M}{g}(A\sin\theta + B\cos\theta)\dot{\theta}, \end{aligned}$$

whence, on differentiating with respect to the time,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{M}{g}(A \sin \theta . \ddot{\theta} + A \cos \theta . \dot{\theta}^2 + B \cos \theta . \ddot{\theta} - B \sin \theta . \dot{\theta}^2);$$

also
$$\frac{\partial T}{\partial x} = 0.$$

In view of this information, equation (37.4) implies that the reaction

$$X = \frac{M}{g}\{(A \sin \theta + B \cos \theta)\ddot{\theta} + (A \cos \theta - B \sin \theta)\dot{\theta}^2\}, \quad (37.5)$$

where A and B are specified in equations (37.3).

In the formula (19.8), we next write θ for q_r, and $\mathfrak{T}$ for Q_r, then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \mathfrak{T}, \quad . \quad . \quad . \quad . \quad . \quad (37.6)$$

and previous considerations show that $\mathfrak{T}$ is the driving torque involved in the motion of the gear.

To evaluate $\mathfrak{T}$ in terms of the known coefficients A and B, we first note from the above relations that

$$\begin{aligned} 2T &= \frac{M}{g}\dot{x}^2 \\ &= \frac{M}{g}(A^2 \sin^2 \theta + A.B \sin 2\theta + B^2 \cos^2 \theta)\dot{\theta}^2, \end{aligned}$$

whence, on effecting the appropriate differentiations,

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{M}{g}(A^2 \sin^2 \theta + A.B \sin 2\theta + B^2 \cos^2 \theta)\dot{\theta},$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{M}{g}\{&(A^2 \sin^2 \theta + A.B \sin 2\theta + B^2 \cos^2 \theta)\ddot{\theta} \\ &+ (A^2 \sin 2\theta + 2A.B \cos 2\theta - B^2 \sin 2\theta)\dot{\theta}^2\}; \end{aligned}$$

further $$\frac{\partial T}{\partial \theta} = \frac{M}{2g}(A^2 \sin 2\theta + 2A.B \cos 2\theta - B^2 \sin 2\theta)\dot{\theta}^2.$$

Thus, on making the substitutions in equation (37.6), it is seen that

$$\begin{aligned} \mathfrak{T} = \frac{M}{g}\{&(A^2 \sin^2 \theta + A.B \sin 2\theta + B^2 \cos^2 \theta)\ddot{\theta} \\ &+ \tfrac{1}{2}(A^2 \sin 2\theta + 2A.B \cos 2\theta - B^2 \sin 2\theta)\dot{\theta}^2\} \quad . \quad (37.7) \end{aligned}$$

gives the numerical value of the driving torque.

The maximum values of X and $\mathfrak{T}$, which alone are of account in the general process of design, may readily be found by the usual method of the differential calculus. In the special case when the

crank of the engine is rotating with the uniform velocity defined by $\dot{\theta} = \omega$, our results reduce to

$$-\frac{M}{g}\omega^2(B \sin \theta - A \cos \theta)$$

for the force X, and

$$\frac{M}{2g}\omega^2(A^2 \sin 2\theta + 2AB \cos 2\theta - B^2 \sin 2\theta)$$

for the torque $\mathfrak{T}$.

A comparison of the form exhibited in (21.5) with equations (37.5) and (37.7) shows at once that the general solutions of the latter are of the type indicated in (21.6), as might have been inferred from the nature of the analysis used here and in Art. 21. This form of solution is, in fact, of common occurrence in matters pertaining to the disturbed motion of mechanisms and structures, for, as already noticed, equation (32.1) also is of the same type.

(*b*) *Walschaert's Valve-Gear.* The travel of the valve in this gear, represented by Fig. 74, is derived from two sources, namely:

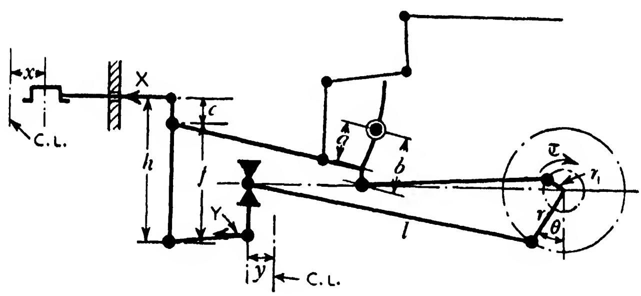


FIG. 74.

(i) an arm fixed to the crosshead of the engine;
(ii) an eccentric in the case of inside cylinders, or a return crank in the case of outside cylinders, arranged 90 deg. out of phase with the main crank.

Both of these driving agencies influence the coefficients A and B in equation (37.1) in a way which will now be investigated with reference to an engine having a stroke equal to $2r$, and a connecting rod of length l.

If y denote the displacement of the crosshead reckoned from its mid-position at the instant when the crank-angle is θ, as indicated in the diagram, the geometrical relations for the mechanism enable us to write

$$y = r \cos \theta - \frac{1}{2}\frac{r^2}{l} \sin^2 \theta. \quad . \quad . \quad . \quad . \quad (37.8)$$

Further inspection of the figure shows that the valve undergoes a component displacement of

$$\frac{c}{h}y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.9)$$

due to the arm fixed on the crosshead, and, on neglecting the obliquity of the eccentric rod as of the second order, an additional displacement of

$$\frac{a}{b}\cdot\frac{f}{h}r_1 \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.10)$$

due to the eccentric rod or return-crank, of equivalent radius r_1. Because the displacement x of the valve measured from its mid-position is equal to the vectorial sum of the last two expressions, and the positions of the main crank and the eccentric are 90 deg. out of phase, it follows that

$$x = \frac{c}{h}y - \frac{a}{b}\cdot\frac{f}{h}r_1 \sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.11)$$

or

$$x = \frac{c}{h}r\cos\theta - \frac{a}{b}\cdot\frac{f}{h}r_1\sin\theta - \tfrac{1}{2}\frac{c}{h}\cdot\frac{r^2}{l}\sin^2\theta.$$

If, as a first approximation, the obliquity of the connecting rod be neglected, the last relation reduces to

$$x = \frac{c}{h}r\cos\theta - \frac{a}{b}\cdot\frac{f}{h}r_1\sin\theta$$

$$= A\cos\theta - B\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.12)$$

where $A = \frac{c}{h}r$, $B = \frac{a}{b}\cdot\frac{f}{h}r_1$. These results are based on assumptions which clearly indicate the degree of accuracy involved in the relation (37.1).

Hence, writing T for the kinetic energy of the gear shown in Fig. 74, and M for the effective weight of its valve and moving parts, we have, from the last expression,

$$2T = \frac{M}{g}\dot{x}^2$$

$$= \frac{M}{g}(A^2\sin^2\theta + A.B\sin 2\theta + B^2\cos^2\theta)\dot{\theta}^2 \quad . \quad (37.13)$$

To proceed, suppose that the inertia effects cause a reaction X on the pin of the valve-spindle, a reaction Y on the pin connecting the union-link and crosshead, and a torque-reaction $\mathfrak{T}$ on the

crankshaft. From equation (19.8) it follows that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = X,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = Y,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\mathfrak{T},$$

whence we can without difficulty derive expressions for the quantities thus specified.

With reference to the first of these formulae, we have, from equation (37.13),

$$\frac{\partial T}{\partial \dot{x}} = \frac{M}{g}\dot{x}$$
$$= -\frac{M}{g}(A \sin \theta + B \cos \theta)\dot{\theta},$$

so that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = -\frac{M}{g}\{(A \sin \theta + B \cos \theta)\ddot{\theta} + (A \cos \theta - B \sin \theta)\dot{\theta}^2\};$$

and $\dfrac{\partial T}{\partial x} = 0$. Therefore the reaction

$$X = -\frac{M}{g}\{(A \sin \theta + B \cos \theta)\ddot{\theta} + (A \cos \theta - B \sin \theta)\dot{\theta}^2\} \quad (37.14).$$

To evaluate the reaction Y, use is first made of equation (37.11), to obtain

$$2T = \frac{M}{g}\dot{x}^2$$
$$= \frac{M}{g}\left(\frac{c^2}{h^2}\dot{y}^2 - 2\frac{c}{h}B \cos \theta.\dot{y}\dot{\theta} + B^2 \cos^2 \theta.\dot{\theta}^2\right)$$

Consequently

$$\frac{\partial T}{\partial \dot{y}} = \frac{M}{g}\left(\frac{c^2}{h^2}\dot{y} - \frac{c}{h}B \cos \theta.\dot{\theta}\right),$$

whence

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = \frac{M}{g}\left(\frac{c^2}{h^2}\ddot{y} - \frac{c}{h}B \cos \theta.\ddot{\theta} + \frac{c}{h}B \sin \theta.\dot{\theta}^2\right)$$
$$= \frac{M}{g}\left\{-\left(\frac{c^2}{h^2}r \sin \theta + \frac{c}{h}B \cos \theta\right)\ddot{\theta} + \left(\frac{c}{h}B \sin \theta - \frac{c^2}{h^2}r \cos \theta\right)\dot{\theta}^2\right\},$$

since equation (37.8) gives

$$\ddot{y} = -r(\sin \theta.\ddot{\theta} + \cos \theta.\dot{\theta}^2)$$

when the obliquity of the connecting rod is neglected; moreover,

$$\frac{\partial T}{\partial y} = 0.$$

Inserting these values in the formula for Y thus leads to

$$Y=\frac{M}{g}\left\{-\left(\frac{c^2}{h^2}r\sin\theta+\frac{c}{h}B\cos\theta\right)\ddot{\theta}+\left(\frac{c}{h}B\sin\theta-\frac{c^2}{h^2}r\cos\theta\right)\dot{\theta}^2\right\} \quad (37.15)$$

Turning, finally, to the relation for $\mathfrak{T}$, it is clear that

$$\frac{\partial T}{\partial\dot{\theta}}=\frac{M}{g}(A^2\sin^2\theta+A.B\sin 2\theta+B^2\cos^2\theta)\dot{\theta},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)=\frac{M}{g}\{(A^2\sin^2\theta+A.B\sin 2\theta+B^2\cos^2\theta)\ddot{\theta}$$
$$+(A^2\sin 2\theta+2A.B\cos 2\theta-B^2\sin 2\theta)\dot{\theta}^2\},$$

and $$\frac{\partial T}{\partial\theta}=\frac{1}{2}\frac{M}{g}(A^2\sin 2\theta+2A.B\cos 2\theta-B^2\sin 2\theta)\dot{\theta}^2$$

are the terms in the expression for $\mathfrak{T}$. Hence

$$\mathfrak{T}=\frac{M}{g}[(A^2\sin^2\theta+A.B\sin 2\theta+B^2\cos^2\theta)\ddot{\theta}$$
$$+\tfrac{1}{2}\{(A^2-B^2)\sin 2\theta+2A.B\cos 2\theta\}\dot{\theta}^2] \quad . \quad (37.16)$$

determines the torque-reaction, the coefficients A and B being defined in equation (37.12).

In the particular case of an engine rotating with uniform angular velocity $\dot{\theta}=\omega$, equations (37.14), (37.15), (37.16) reduce to

$$-\frac{M}{g}\omega^2(A\cos\theta-B\sin\theta)$$

for the reaction X,

$$\frac{M}{g}.\frac{c}{h}\omega^2(B\sin\theta-A\cos\theta)$$

for the reaction Y, and

$$\tfrac{1}{2}\frac{M}{g}\omega^2\{(A^2-B^2)\sin 2\theta+2A.B\cos 2\theta\}$$

for the driving torque $\mathfrak{T}$.

The negative signs arise from our notation, and they may be treated as irrelevant in the work of calculating the stresses due to these inertia forces and couples.

Ex. Apply the preceding analysis, on the basis of the implied assumptions, to the case of a Walschaert type of valve-gear specified by the following dimensions:

$a=8$ in.,
$b=14\tfrac{1}{16}$ in.,
$r=13$ in.,
$r_1=6\tfrac{1}{4}$ in.,
$l=11$ ft.
$f=28\tfrac{5}{8}$ in.,
$c=4\tfrac{3}{8}$ in.,

total weight of the valve and its attachment $(M)=202{\cdot}2$ lb.

Suppose the engine to be running in full forward gear, at a speed of 4 revolutions a second.

Using pound-foot-second units, we have

$$\omega^2 = 631{\cdot}5,$$
$$A = 0{\cdot}1438,$$
$$B = 0{\cdot}2575,$$
$$\frac{M}{g}\omega^2 = 3962,$$
$$\frac{c}{h}\frac{M}{g}\omega^2 = 525,$$
$$A^2 - B^2 = -0{\cdot}0456,$$
$$2AB = 0{\cdot}0742,$$

hence, inserting these in equations (37.15) and (37.16),

$$Y = 75{\cdot}60(1{\cdot}791 \sin \theta - \cos \theta) \text{ lb.},$$
$$\mathfrak{T} = 90{\cdot}40(\sin 2\theta - 1{\cdot}622 \cos 2\theta) \text{ lb.-ft.},$$

approximately.

The graphical representation of these expressions is exhibited in Fig. 75, where the full line refers to the variation in the reaction

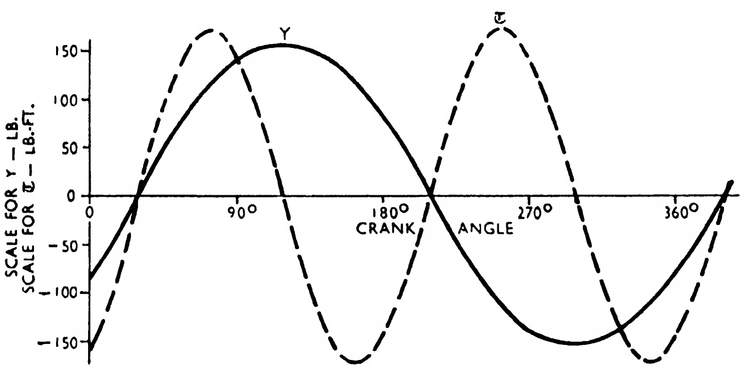


FIG. 75.

Y, and the dotted line to that in the driving torque $\mathfrak{T}$, during one revolution of the main crank. It will be observed that the maximum values of these quantities do not occur simultaneously, and the frequency of one is twice that of the other.

CHAPTER III

THEORY OF VIBRATIONS

38. Simple Harmonic Motion. When the inertia forces associated with engines act on elastic beams and foundations generally, the oscillations thus induced in the structure may be examined as if the system consisted of a number of very stiff springs. Our chief aim here is that of investigating the motion by way of evaluating the stresses produced under such conditions, provided always that the material remains elastic throughout the disturbed motion of the structural system. Hence Hooke's law is implied in the following treatment, which is therefore limited to the case of slight disturbances about the corresponding position of rest.

(*a*) *Linear Oscillations.* Let Fig. 76 represent a light spring which is free to describe oscillatory motion about the position of rest denoted by the line OO, under the influence of a mass of weight M attached to its lower end. Suppose the stiffness of the spring to be such that a force of magnitude c is required to extend it unit distance in the vertical direction, then a force amounting to cy must be applied to effect an extension equal to y. For the present we shall neglect any frictional forces which may act on the system.

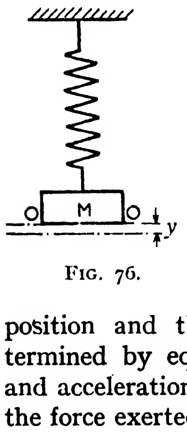


FIG. 76.

If the spring be suddenly given a vertical displacement y reckoned from the equilibrium-position and then released, the subsequent motion can be determined by equating the effective force to the product of mass and acceleration. In this manner we obtain, bearing in mind that the force exerted by the spring always opposes the motion,

$$\frac{M}{g}\ddot{y} = -cy,$$

or

$$\ddot{y} + p^2 y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38.1)$$

where $p^2 = \frac{cg}{M}$. The motion is given by the solution to this equation, therefore

$$y = A\cos pt + B\sin pt, \quad . \quad . \quad . \quad . \quad (38.2)$$

where the constants A and B depend on the initial conditions.

Thus if, by way of illustration, the mass be deflected a distance a from the position OO and then released, the circumstances of the motion are

$$\dot{y} = 0 \text{ and } y = a \text{ at the time } t = 0.$$

But according to the last equation

$$\dot{y} = -Ap \sin pt + Bp \cos pt$$

defines the velocity; hence $A = a$, $B = 0$ and, consequently,

$$y = a \cos pt. \qquad (38.3)$$

To elucidate this equation, suppose that the point P in Fig. 77 describes with uniform angular velocity p a circular path of radius a, starting at the instant $t = 0$ from the lower end of the diameter DOD'. With N denoting the projection of the point P, at the time t, we have the angle $D'OP = pt$ and $ON = a \cos pt$, from which it is evident that the motion given by equation (38.3) is the orthogonal projection of uniform circular motion.

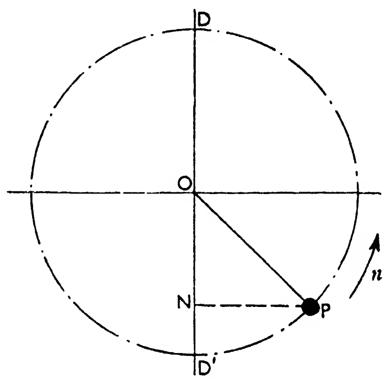


FIG. 77.

This is called *simple harmonic motion*, of *amplitude* a, and of *period* $\frac{2\pi}{p}$; the reciprocal of the period is known as the *frequency*, so that $p = 2\pi$ (frequency). We may also write equation (38.2) in the form

$$y = a \cos (pt + \alpha), \qquad (38.4)$$

when the time t is to be measured from some position of the mass other than that taken above. Here α represents the *phase*, and its value at the instant $t = 0$ is called the *epoch* of the vibration.

In this motion, therefore, the period is independent of the amplitude, and the frequency is raised by increase in the value of c, which we shall refer to as the *coefficient of stiffness*.

To present another view of the matter, multiply both sides of equation (38.1) by $\dot{y}dt$, and so obtain

$$d\left(\tfrac{1}{2}\frac{M}{g}\dot{y}^2\right) + d(\tfrac{1}{2}cy^2) = 0,$$

whence, integrating with respect to the time,

$$\tfrac{1}{2}\frac{M}{g}\dot{y}^2 + \tfrac{1}{2}cy^2 = \text{const.} \qquad (38.5)$$

Since by definition the first and second terms respectively represent

the kinetic and the potential energies of the system, it appears that the sum of these two forms of energy remains constant throughout the prescribed motion. Hence this type of *free* oscillation involves a continuous interchange between the kinetic and potential energies. Moreover, the potential energy is a maximum when pt is either zero or π, while the greatest value of the kinetic energy occurs when pt is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

For many practical purposes equation (38.1) is most conveniently arranged in the form

$$\frac{\ddot{y}}{y} = p^2,$$

where the negative sign is taken as irrelevant, for this leads to the useful relations

$$\begin{aligned} \text{period} &= 2\pi\sqrt{\frac{y}{\ddot{y}}} \\ &= 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} \\ &= 2\pi\sqrt{\frac{M}{cg}} \qquad \qquad (38.6) \end{aligned}$$

A wide range of problems can be solved with the aid of these formulae.

Ex. 1. Apply equations (38.6) to the systems shown in Fig. 78, illustrating alternative methods of supporting a mass of weight

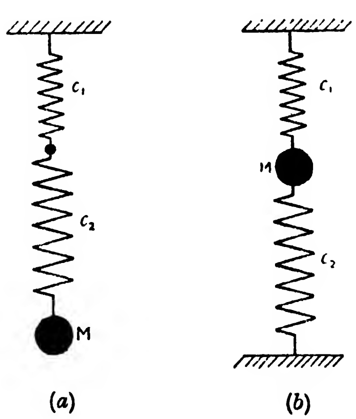


FIG. 78.

M by means of two light springs having stiffnesses specified by the coefficients c_1 and c_2; M=70 lb., c_1 = 60 lb. and c_2 = 100 lb. per inch extension of the springs.

From these data we have, bearing in mind the electrical analogues,

$$2\pi\sqrt{\frac{M(c_1 + c_2)}{c_1 c_2 g}}$$

$$= 2\pi\sqrt{\frac{70 \times 160}{6{,}000 \times 12 \times 32{\cdot}2}} \text{ sec.}$$

$$= 0{\cdot}437 \text{ sec.}$$

as the periodic time for the 'parallel' arrangement (*a*), and

$$2\pi\sqrt{\frac{M}{(c_1 + c_2)g}}$$

$$= 2\pi\sqrt{\frac{70}{160 \times 12 \times 32{\cdot}2}} \text{ sec.}$$

$$= 0{\cdot}212 \text{ sec.}$$

as the periodic time for the 'series' arrangement (*b*). The corresponding frequencies are 2·29 cycles and 4·72 cycles per second.

These alternative methods of support differ in one important particular, in that a constraint has been added to (*a*) in order to obtain (*b*), and this has naturally contributed to the change in frequencies. We have here, in fact, a simple illustration of the general result that the addition of constraints makes for increase in the 'stiffness' of a structural system.

When the weight of a spring cannot be neglected in comparison with the other quantities involved in a problem, account must be taken of the result obtained in Ex. 2 of Art. 68.

Ex. 2. Find the natural period of oscillation when a vertical load of 1,120 lb. is applied at the point P on the light frame indicated

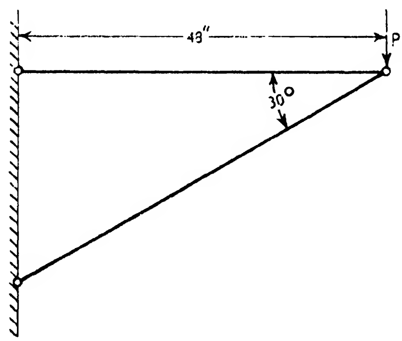


FIG. 79.

in Fig. 79, constructed of bars having a cross-sectional area of 1 square inch; the direct modulus of elasticity for the material amounts to 30,000,000 lb. per square inch. The frame is, as shown, supposed to be pin-jointed.

It is readily seen, on neglecting the weight of the frame, that in the formula

$$\text{period} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

the displacement is equivalent to the deflection of the point P produced by the static load of 1,120 lb., and the acceleration is that due to gravity. Since any one of the usual methods may be used to show that this displacement is 0·0172 inch, it follows that

$$\text{period} = 2\pi\sqrt{\frac{0{\cdot}0172}{12 \times 32{\cdot}2}} \quad \text{sec.}$$
$$= 0{\cdot}042 \text{ sec.,}$$

approximately, whence 23·81 cycles per second is the natural frequency of the given system.

Ex. 3. Calculate, on the assumption that the frictional agencies may be neglected, the period of oscillation for the liquid in the U-tube represented by Fig. 80(*a*), the tube being of uniform bore.

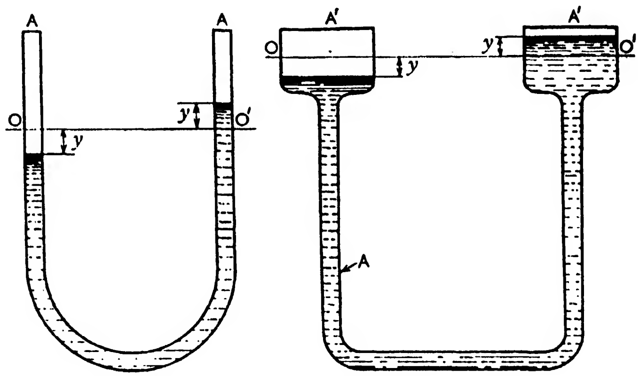


FIG. 80.

The free surface of the fluid is initially in the position of equilibrium defined by the line OO', and the disturbance is caused by the surfaces being given a small displacement y, measured from that position.

Writing ρ for the weight of liquid contained in a unit length of the tube, of cross-sectional area A, the force then tending to restore equilibrium amounts to $2\rho yA$, or $2\rho A$ per unit length of displacement about the line OO'. If, further, the length of the column of liquid be denoted by L, the weight of the fluid is equal to ρAL.

Hence we have, in equation (38.6), $M = \rho AL$ and $c = 2\rho A$, so that the

$$\text{period} = 2\pi\sqrt{\frac{M}{cg}}$$
$$= 2\pi\sqrt{\frac{L}{2g}}.$$

Now suppose, by way of comparison, that both ends of the tube are enlarged, so to give the arrangement shown in Fig. 80(b), where in the position of rest the liquid in the enlargement has a effective length L', and a (constant) sectional area A'.

Repeating our analysis, with L written for the length of the column in the tube of section A, it will be found that in the new conditions the periodic time is expressed by

$$2\pi\sqrt{\frac{\frac{A'}{A}L + L'}{2g}}.$$

Thus the effect of enlarging the free ends of the tube is to increase the period. This method has been applied by H. Frahm [1] in his anti-rolling tanks for ships, though it is well to notice here that the friction between the water and tube or pipe actually used for the purpose is so large that the oscillations soon die away.

(b) *Torsional or Angular Oscillations.* By effecting obvious changes in our notation we may without difficulty adapt the foregoing equations to the system shown in Fig. 81, representing a light

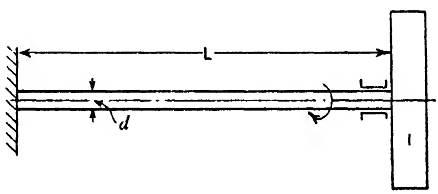


FIG. 81.

shaft of uniform cross-section. The shaft is rigidly fixed at one end and free at the other, and to the latter is attached a pulley having a polar moment of inertia I. We shall assume, in accordance with the present theory, that the bearing indicated in the figure is frictionless.

To specify the system, let L and d be respectively the free length and the diameter of the shaft, and c the coefficient of stiffness; that is c equals the couple required to twist the shaft through unit angular displacement. Then if J denote the polar moment of inertia of the shaft, and N the modulus of rigidity for the material, it follows from elementary considerations that

$$c = \frac{NJ}{L}.$$

With a given initial angular displacement θ reckoned from its position of rest, the pulley will describe oscillations under the influence of a restoring couple $c\theta$, and a torque-reaction $I\ddot{\theta}$ in the

[1] *Trans. Inst. N.A.*, vol. 53, page 183 (1911).

case of a light shaft. These opposing forces accordingly produce the motion

$$I\ddot{\theta} = -c\theta,$$

or

$$\ddot{\theta} + \frac{c}{I}\theta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38.7)$$

The solution of this equation is, as might be anticipated from the previous results, of the type

$$\theta = A \cos pt + B \sin pt,$$

where $p^2 = \frac{c}{I}$, and the constants A, B depend on the initial conditions. From these results it is to be inferred that the relations

$$\text{period} = 2\pi\sqrt{\frac{\theta}{\ddot{\theta}}}$$

$$= 2\pi\sqrt{\frac{\text{angular displacement}}{\text{angular acceleration}}}$$

$$= 2\pi\sqrt{\frac{I}{c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38.8)$$

apply in the case of torsional vibrations.

Pass now to the consideration of the arrangement shown in Fig. 82, having the stated diameters d_1 and d_2 over the free lengths

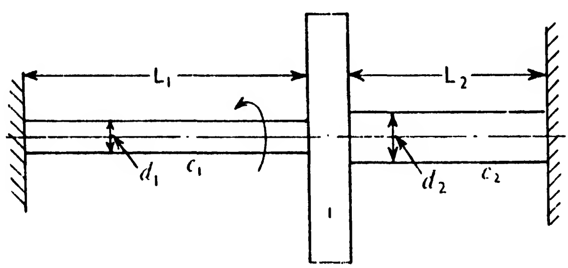


FIG. 82.

L_1 and L_2; the symbols c_1 and c_2 in the figure refer to the coefficients of stiffness for these lengths of shafting. Consequently, if J_1 and J_2 be the corresponding moments of inertia for the circular sections of the shaft, then

$$c_1 = \frac{NJ_1}{L_1}, \quad c_2 = \frac{NJ_2}{L_2},$$

assuming that the same value of N applies throughout the shaft.

To determine the period for this system, we notice that the arrangements shown in Figs. 81 and 82 bear the same relation to one another as do those shown in Fig. 78(*a*) and (*b*), in that the same type of constraint must be imposed on the first arrangement to give

the second in both instances. In the present case, with the shaft fixed at both ends, it is also clear that the restoring couple is equal to the sum of the torques exerted by the two parts of the shaft, namely

$$(c_1 + c_2)\theta,$$

for a small displacement θ measured from the position of rest. Since I is the polar moment of inertia for the pulley, the equation of motion is

$$I\ddot{\theta} + (c_1 + c_2)\theta = 0,$$

whence we can at once write

$$\text{period} = 2\pi\sqrt{\frac{I}{c_1 + c_2}}$$

$$= 2\pi\sqrt{\left(\frac{L_1L_2}{J_1L_2 + J_2L_1}\right)\frac{I}{N}}$$

in the case of the system specified by Fig. 82.

When a solid shaft of uniform diameter d is involved, a simple calculation suffices to prove that

$$J = \frac{\pi d^4}{32},\ c = \frac{\pi N d^4}{32L}.$$

Ex. 4. Evaluate the natural period for a hollow shaft 30 ft. in length, with external and internal diameters of 5 in. and $2\frac{1}{2}$ in. respectively, and loaded in the manner represented by Fig. 81. The pulley in question weighs 3 tons, and its polar radius of gyration is 4·5 ft.

Taking N as 13,000,000 lb. per square inch and neglecting the friction on the bearing, we have, in pound-foot-second units,

$$I = \frac{3 \times 2{,}240 \times 20{\cdot}25}{32{\cdot}2}$$

$$= 4{,}230,$$

$$c = \frac{\pi N(d_1^4 - d_2^4)}{32L}$$

$$= \frac{\pi \times 13 \times 10^6 \times 144 \times 586}{32 \times 30 \times 12^4}$$

$$= 173{,}400,$$

hence the $\text{period} = 2\pi\sqrt{\frac{4{,}230}{173{,}400}}$ sec.

$$= 0{\cdot}9830 \text{ sec.}$$

Ex. 5. Determine the natural frequency of vibration for the system shown in Fig. 83, such as is sometimes used for testing

engines. Passing over the pulley, of radius $r = 8$ in. and polar moment of inertia $I = 5{\cdot}5$ lb.-ft.^{2}, is a slender cord, one end of which is fixed to a spring having a coefficient of stiffness $c = 10$ lb. per inch extension, while its free end is attached to a mass M of weight 10 lb.

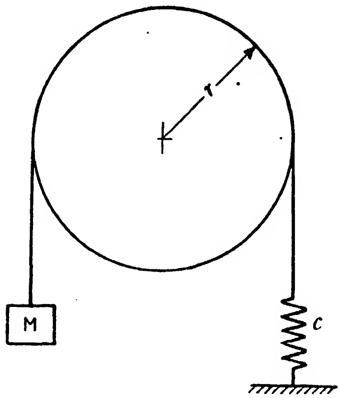


FIG. 83.

Denoting the effective moment of inertia by I', and the corresponding coefficient of stiffness by c', it is readily seen that $I' = I + \frac{M}{g}r^2$, $c' = cr^2$. Since, in pound-foot-second units,

$$
\begin{aligned}
I' &= \frac{5{\cdot}5}{32{\cdot}2} + \frac{10 \times 4}{32{\cdot}2 \times 9} \\
&= 0{\cdot}3088, \\
c' &= \frac{10 \times 12 \times 4}{9} \\
&= 53{\cdot}30,
\end{aligned}
$$

a slight disturbance about the position of equilibrium will initiate a vibration of

$$
\begin{aligned}
\text{period} &= 2\pi\sqrt{\frac{I'}{c'}} \\
&= 2\pi\sqrt{\frac{0{\cdot}3088}{53{\cdot}30}} \text{ sec.} \\
&= 0{\cdot}478 \text{ sec.},
\end{aligned}
$$

corresponding to a frequency of approximately 2·09 cycles per second.

Ex. 6. Examine the oscillatory motion which would follow a small displacement from the position of rest of the arrangement represented in Fig. 84, formed by a light shaft having an effective length L and a uniform diameter d, on the assumption that the friction on the bearings may be neglected. As indicated in the figure, flywheels with polar moments of inertia I_1 and I_2 are attached to the free ends of the shaft.

Since the flywheels move in opposite senses during the prescribed disturbance, there will be a section on the shaft for which the displacement is zero, say at N on the displacement graph. Such a section or point is known as a *node*. It is also manifest that the node must divide the shaft so that the frequency of vibration for

the left-hand part is the same as that for the right-hand. Let the point N divide L into the lengths l_1 and l_2, and write in succession

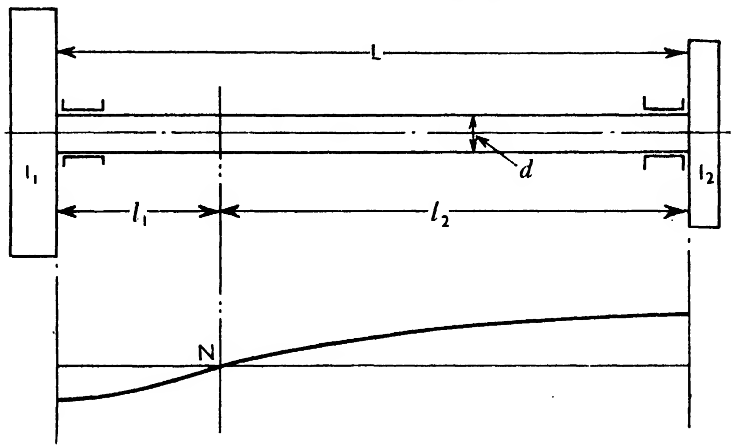


FIG. 84.

c_1 and c_2 for the corresponding coefficients of stiffness. Then, on equating the frequencies, we obtain

$$\sqrt{\frac{c_1}{I_1}} = \sqrt{\frac{c_2}{I_2}},$$

or

$$\sqrt{\frac{NJ}{I_1 l_1}} = \sqrt{\frac{NJ}{I_2 l_2}},$$

where J denotes the polar moment of inertia for a shaft of diameter d. From the resulting relation

$$I_1 l_1 = I_2 l_2$$

it follows that the node divides the length L inversely as the moments of inertia of the masses, and

$$l_1 = \frac{I_2}{I_1 + I_2} L.$$

The shaft to the left of the point N therefore describes vibratory motion with a

$$\text{period} = 2\pi\sqrt{\frac{I_1}{c_1}}$$

where $c_1 = \dfrac{NJ}{l_1}$; thus, in other terms, the

$$\text{period} = 2\pi\sqrt{\frac{I_1 l_1}{NJ}}$$

$$= 2\pi\sqrt{\frac{I_1 I_2 L}{(I_1 + I_2)NJ}}.$$

This necessarily applies also to the other part of the shaft.

A special significance is associated with the position of nodes on a shaft, since they are the most suitable points at which to fix gearing, owing to the obvious fact that the presence of 'play' or 'backlash' renders the teeth liable to severe stresses due to the vibratory motion on the shaft at any point other than a node. On this account it is sometimes necessary to modify the period of oscillation for a system of given overall dimensions, by making appropriate changes in the constraints.

Ex. 7. Suppose that we wish to carry out the modification just mentioned, by inserting a spring-coupling between the shaft and the flywheel specified by I_2 in Fig. 84, to obtain the arrangement shown in Fig. 85. Here r represents the mean distance between

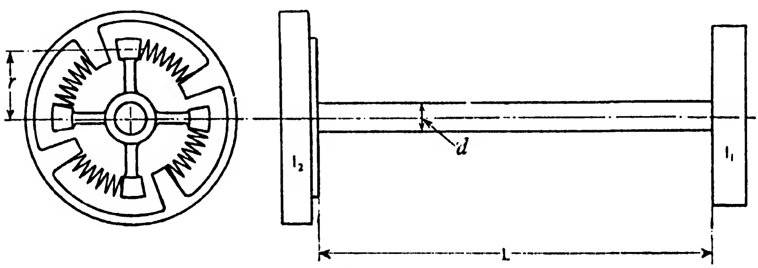


FIG. 85.

each of the four springs and the axis of rotation in a system having the following dimensions:

$L = 36$ in.,
$d = 1{\cdot}5$ in.,
$r = 5$ in.,
$N = 13{,}000{,}000$ lb. per square inch,
$c = 5{,}000$ lb. per inch compression for each of the springs,
$I_1 = 12$ lb.-in.^{2}
$I_2 = 24$ lb.-in.^{2}

Compare the frequencies of vibration for the arrangement without and with the prescribed spring-coupling, on the assumption that the weight of the springs and the shaft may be neglected.

To determine the position of the node in relation to the flywheel having a moment of inertia I_1, suppose the spring-coupling to be absent and let l_1 be as indicated in Fig. 84. Then

$$l_1 = \frac{I_2}{I_1 + I_2}L$$

$$= \frac{2 \times 36}{3} \text{ in.}$$

$$= 24 \text{ in.}$$

Proceeding, with J written for the polar moment of inertia of the shaft having a coefficient of stiffness c_1, we have, in pound-inch-second units,

$$\begin{aligned} c_1 &= \frac{NJ}{l_1} \\ &= \frac{0{\cdot}4970 \times 13 \times 10^6}{24} \\ &= 0{\cdot}2694 \times 10^6, \end{aligned}$$

whence the period $= 2\pi\sqrt{\dfrac{I_1}{c_1}}$

$$\begin{aligned} &= 2\pi\sqrt{\frac{12}{0{\cdot}2694 \times 10^6}} \text{ sec.} \\ &= 0{\cdot}042 \text{ sec.,} \end{aligned}$$

which corresponds to a frequency of 23·81 cycles per second.

Turning now to the system fitted with the spring-coupling, it is at once evident that the effect of the springs is to increase the original length L of the shaft. If l be this virtual increase in length, and c_2 the coefficient of stiffness for the four springs taken together, we have, with the same units as before,

$$\begin{aligned} c_2 &= 4cr^2 \\ &= 4 \times 5{,}000 \times 25 \\ &= 500{,}000, \end{aligned}$$

and

$$\begin{aligned} l &= \frac{NJ}{c_2} \\ &= \frac{13 \times 10^6 \times 0{\cdot}4970}{0{\cdot}500 \times 10^6} \\ &= 12{\cdot}92 \text{ in.} \end{aligned}$$

Hence the effective length of the shaft, namely $(L + l)$, amounts to 48·92 in., and the distance l_1 of the nodal point from the flywheel specified by I_1 is

$$\begin{aligned} l_1 &= \frac{I_2}{I_1 + I_2}(L + l) \\ &= \frac{2 \times 48{\cdot}92}{3} \text{ in.} \\ &= 32{\cdot}6 \text{ in.} \end{aligned}$$

Inserting these values in the formula

$$\text{period} = 2\pi\sqrt{\frac{I_1 l_1}{NJ}}$$

thus yields 0·0488 sec. as the periodic time for the arrangement fitted with the coupling, and this corresponds to a frequency of 20·46 cycles per second.

This example illustrates the way in which spring-couplings enable us to change the position of nodes on given shafts. It shows further that the introduction of spring leads to a reduction in the frequency of oscillation for a specified system, which agrees with the general result arrived at in Art. 55, where it is demonstrated that the effect of adding a partial constraint is to raise the gravest frequency in given circumstances.

A practical inference to be drawn from the foregoing applications of the theory is that problems relating to quite complicated arrangements of shafts can be solved with the help of models which include springs. The various parts of such models must be constructed in accordance with the principle of dynamical similitude, Art. 19 (e).

39. Superposition of Simple Harmonic Motions. In many instances of disturbed motion which are encountered in practice the observed movement is the resultant of a number of simple harmonic motions. This may, on the one hand, be due to the resultant displacement arising from the simultaneous operation of different disturbing agencies, a notable example being that of the effect on the tides of the sun and moon, which together produce tide-raising forces in the ratio of 3 : 7. In this particular sense much the same result follows when, on the other hand, a single source of disturbance acts on a structural system having different degrees of stiffness in different directions, as commonly occurs when buildings and structures vibrate on account of earthquakes.

Let us study the matter by combining two simple harmonic motions having equal frequencies in the first place, and unequal frequencies in the second.

Equal Frequencies. Consider the path described by a mass of weight M, attached to an elastic system with a coefficient of stiffness c, when the mass simultaneously executes in one plane two simple harmonic motions having equal frequencies, but different amplitudes and phases.

Suppose that

$$\frac{M}{g}\ddot{x} = -cx, \frac{M}{g}\ddot{y} = -cy$$

represent the equations of the resulting motion referred to the orthogonal axes Ox and Oy. The solutions are clearly of the type

$$\left.\begin{aligned} x &= A_1 \cos (pt - \alpha_1), \\ y &= A_2 \cos (pt - \alpha_2), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (39.1)$$

where A_1, A_2, α_1, α_2 relate to arbitrary constants.

An equation for the path can be derived by elimination of the time t from these expressions, which leads to

$$y = A_2 \cos \{pt - \alpha_1 + (\alpha_1 - \alpha_2)\}$$
$$= \frac{A_2}{A_1}x \cos (\alpha_1 - \alpha_2) - A_2\left(1 - \frac{x^2}{A^2_1}\right)^{\frac{1}{2}} \sin (\alpha_1 - \alpha_2).$$

If, for convenience in working, we write $\alpha = \alpha_1 - \alpha_2$, the result may be rearranged in the form

$$(A_1y - A_2x \cos \alpha)^2 = A^2_2(A^2_1 - x^2) \sin^2 \alpha,$$
or
$$A^2_2x^2 - 2A_1A_2xy \cos \alpha + A^2_1y^2 = A^2_1A^2_2 \sin^2 \alpha \quad . \quad (39.2)$$

It is to be noted, first, that this is the general equation of a conic, and, secondly, that the conic is an ellipse. The shape of the path is naturally influenced by the values of the variables, in a manner which may be explained by reference to three important cases.

(i) If $\alpha = 0$ or π, the ellipse degenerates into one of the two straight lines defined by

$$A_2x + A_1y = 0 \text{ and } A_2x - A_1y = 0.$$

(ii) If $\alpha = \pm \frac{\pi}{2}$, equation (39.2) reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1,$$

which represents an ellipse whose axes coincide with Ox and Oy.

(iii) If the amplitudes of the component motions are equal, then the principal axes of the conic or path make an angle of 45 deg. with Ox and Oy; also the lengths of the axes of the conic amount to $2A \sin \frac{1}{2}\alpha$ and $2A \cos \frac{1}{2}\alpha$, where $A = A_1 = A_2$.

Unequal Frequencies. The path followed by the given mass is not in general an ellipse when the component motions have different frequencies. To prove the point, let the combined motion be the resultant of the oscillations defined by the relations

$$\left.\begin{aligned} x &= A_1 \cos (p_1t - \alpha_1), \\ y &= A_2 \cos (p_2t - \alpha_2). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (39.3)$$

We must also assume that the values of the frequencies are commensurable, for it is readily shown that the motion would not be periodic if this condition were not satisfied. If, then, p' and q' denote two integers without a common factor, on writing $p_1 = \beta p'$ and $p_2 = \beta q'$, it appears that after β units of time the x- and y-displacements will in turn have completed p' and q' oscillations. Consequently the path described by the mass will be a closed one, and the motion will be periodic.

To determine the path, we eliminate t from equations (39.3) and so obtain

$$q' \cos^{-1} \frac{x}{A_1} - p' \cos^{-1} \frac{y}{A_2} = p'\alpha_2 - q'\alpha_1 \quad . \quad . \quad (39.4)$$

The graph of this expression is known as a *Lissajous figure.* Hence, for specified amplitudes and phases of the component motions, one of these figures is assigned to each pair of the integers p' and q', so that the form of the figure depends on the ratio $\frac{p_1}{p_2}$.

Ex. Investigate the path described by a mass of weight M when its motion, referred to orthogonal axes Ox and Oy, is completely defined by the equations

$$\frac{M}{g}\ddot{x} = - c_1 x,$$
$$\frac{M}{g}\ddot{y} = - c_2 y,$$

together with the conditions that the amplitudes of the component motions are similar, and the coefficients of stiffness c_1, c_2 have different values.

If the component motions, with unequal frequencies, be taken as in phase at the time $t = 0$, a comparison between the present equations and (38.1) shows that here

$$x = A \cos p_1 t,$$
$$y = A \cos p_2 t,$$

where $p_1{}^2 = \frac{c_1 g}{M}$, $p_2{}^2 = \frac{c_2 g}{M}$.

It will make for conciseness if we next let $p_1 = p$ and $p_2 = p + \varepsilon$, implying that ε is the difference between the angular velocities associated with the component motions, for the procedure leads to

$$\left.\begin{aligned} x &= A \cos pt, \\ y &= A \cos (p + \varepsilon)t. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (39.5)$$

Further simplification will follow, regard being had to case (iii) above, on rotating through 45 deg. the principal axes of the path given by the last equations. Using accents for the new co-ordinates, the operation is effected by making the substitutions

$$x = \frac{\sqrt{2}}{2}(x' - y'),$$

$$y = \frac{\sqrt{2}}{2}(x' + y'),$$

when it will be found that equations (39.5) transform into

$$(1 - \cos \varepsilon t)x'^2 + (1 + \cos \varepsilon t)y'^2 = A^2 \sin^2 \varepsilon t,$$

or

$$\frac{x'^2}{2A^2 \cos^2 \frac{1}{2}\varepsilon t} + \frac{y'^2}{2A^2 \sin^2 \frac{1}{2}\varepsilon t} = 1 \quad . \quad . \quad . \quad . \quad . \quad (39.6)$$

We may illustrate the general character of the motion by reference to the horizontal disturbance which was produced by an earthquake on a particular building, when it was estimated that $\frac{\varepsilon}{p} = \frac{1}{7}$ approximately. Inserting this numerical ratio in the last equation and plotting the corresponding graph for different values of the time t, results in Fig. 86, where we may imagine the 'cycle'

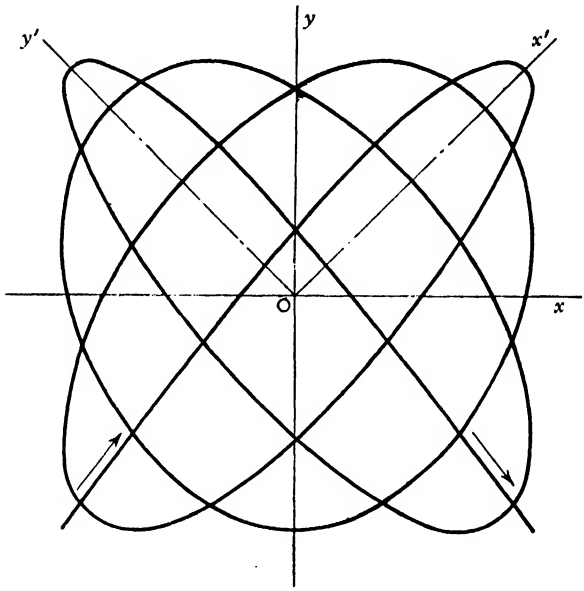


FIG. 86.

as starting in the lower left-hand quarter and finishing in the opposite quarter, as indicated by arrows in the figure. The diagram makes manifest the practical advantage to be gained by rotating the axes, since we thereby facilitate the process of finding the maximum amplitude, which is the only value of any account in the work of design.

From the practical point of view the graph of equation (39.6) is, therefore, equivalent to an ellipse with rotating axes. Had the frequencies of the component vibrations been nearly, but not exactly, equal, Fig. 86 would have still more closely resembled an ellipse with a continuously changing configuration. This can be verified by generalizing equations (39.5) and writing $p_2 = p_1 + \varepsilon$, then

$$x = A_1 \cos (p_1 t - \alpha_1),$$
$$y = A_2 \cos \{p_1 t - (\alpha_2 - \varepsilon t)\},$$

where ε is accordingly small compared with p_1. Now the term $(\alpha_2 - \varepsilon t)$ remains almost constant throughout a 'cycle', and to

this extent the Lissajous figure coincides with an ellipse. Also, owing to the fact that the axes of the figure make an angle

$$\tfrac{1}{2}\tan^{-1}\frac{2A_1A_2\cos(\alpha_1-\alpha_2+\varepsilon t)}{A^2{}_1-A^2{}_2}$$

with the orthogonal axes Ox and Oy, the varying difference $(\alpha_1 - \alpha_2 + \varepsilon t)$ in phase causes the shape of the approximate ellipse to change continuously throughout the period $\frac{2\pi}{\varepsilon}$, after which the cycle is repeated. It is of interest to notice that in the general case the figure is executed in the counter-clockwise direction when $A_1 < A_2$, compared with the clockwise direction when $A_1 > A_2$.

40. Forced Oscillations. The disturbed motion of structures is commonly caused by the application of periodic forces, an important example of which is presented by the inertia forces of engines. Although such oscillations themselves afford an interesting study, our chief aim is that of evaluating the additional stresses which are thus produced on the structures concerned.

To take a simple case, impose on the system implied in equation (38.1) the periodic force defined by $P\cos(\omega t+\varepsilon)$. The equation of motion then becomes

$$\frac{M}{g}\ddot{y}+cy=P\cos(\omega t+\varepsilon),$$

or

$$\ddot{y}+p^2y=Q\cos(\omega t+\varepsilon), \quad . \quad . \quad . \quad . \quad (40.1)$$

where $p^2=\frac{cg}{M}$, $Q=\frac{Pg}{M}$.

By the usual method of finding the 'particular integral' we write $y = B\cos(\omega t+\varepsilon)$, and so obtain

$$B=\frac{Q}{p^2-\omega^2},$$

whence

$$y=\frac{Q}{p^2-\omega^2}\cos(\omega t+\varepsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40.2)$$

is a solution; and, therefore,

$$y=A\cos(pt+\alpha)+\frac{Q}{p^2-\omega^2}\cos(\omega t+\varepsilon) \quad . \quad (40.3)$$

is the complete solution of equation (40.1), where A, α are arbitrary constants.

This relation for the amplitude of the forced motion shows that the displacement becomes infinitely large when ω approaches p in magnitude, that is when the imposed frequency approximately coincides with the natural frequency of the system. If $\omega = p$, the displacement amounts to

$$\frac{Q}{2p}t\sin pt,$$

which obviously means that the displacement increases indefinitely with the time t, and that the phase of the oscillation lags 90 deg. behind that of the disturbing force. In actual systems, however, there is always present a certain amount of friction, and this will subsequently be shown to exert a considerable effect on the value of y when $\omega = p$.

Before proceeding, attention should be drawn to the fact that in a given equation of motion, such as, for example, (40.3), the "particular integral' and the 'complementary function' determine the *forced* and the *free* oscillations, respectively.

Ex. Calculate, without regard to frictional agencies, the maximum stress on beams loaded in the following manner. The system consists of two standard beams 10 in. × 6 in., simply supported at points 24 ft. apart, at the mid-lengths of which is fixed an engine weighing 2 tons; the engine is symmetrically arranged on the beams. When working at its normal speed of 400 revolutions a minute, the engine produces in a vertical direction an unbalanced effect equivalent to that of a mass weighing 26 lb. and revolving at a radius of 6 in. The direct modulus of elasticity for the material may be taken as $E = 29{,}000{,}000$ lb. per square inch.

Confining our attention to one of the beams, and neglecting its weight, we have, from equation (38.6), the free period given by

$$\text{period} = 2\pi\sqrt{\frac{M}{cg}},$$

where M is the effective weight imposed on the beam, and c the force necessary to deflect the beam through unit distance. But it is known from the theory of structures that, in the case of a simply supported beam,

$$c = \frac{48EI}{L^3}$$

for a member with a length L and a cross-section specified by the moment of inertia I; in view of this we can write the previous equation in the form

$$\text{period} = 2\pi\sqrt{\frac{ML^3}{48EgI}}.$$

Inserting in this expression the above data, namely

$M = 2{,}240$ lb.,
$L = 288$ in.,
$I = 204{\cdot}8$ inch units, with
$g = 12 \times 32{\cdot}2$ in. per sec. per sec.,

in this manner leads to $0{\cdot}139$ second for the natural period of the system.

Consider next the forced motion of the same beam. If we write

$P \cos \omega t$ for the inertia force in question, and y_1 for the maximum displacement produced on the beam by that force, then, from equation (40 3),

$$y_1 = \frac{P}{\left(c - \frac{M}{g}\omega^2\right)}.$$

For either of the beams P is equal to the centrifugal effect of 13 lb. acting at a radius of 6 in., that is 355 lb. when the speed is 400 revolutions a minute; also

$$c = \frac{48EI}{L^3}$$

$$= 11{,}940 \text{ lb. per inch deflection,}$$

and
$$\frac{M}{g}\omega^2 = \frac{2{,}240}{386}\left(\frac{2\pi \times 20}{3}\right)^2$$

$$= 10{,}180 \text{ lb.}$$

Hence
$$y_1 = \frac{355}{1{,}760} \text{ in.}$$

$$= 0{\cdot}202 \text{ in.}$$

represents the greatest amplitude of the forced motion.

Since the static load required to produce this deflection is

$$\frac{11{,}940 \times 0{\cdot}202}{2} \text{ lb. or } 1{,}206 \text{ lb.,}$$

the total load on either of the beams is equivalent to

$$(2{,}240 + 1{,}206) \text{ lb., i.e. } 3{,}446 \text{ lb.}$$

The corresponding stress in the material is given by the well-known formula

$$\text{Bending moment} = fZ,$$

where f is the maximum value of the stress, and Z the modulus of the beam. Hence, as

$$\frac{3{,}446 \times 288}{4} \text{ lb.-in.}$$

is the bending moment, and 40·96 inch units is Z,

$$f = \frac{3{,}446 \times 288}{4 \times 40{\cdot}96} \text{ lb. per square inch}$$

$$= 6{,}070 \text{ lb. per square inch}$$

is the stress induced in each of the beams by the combined effect of the static and dynamic loads.

In these circumstances the weight of the beam can, for reasons which will be explained later, be taken into account by adding one-quarter of its weight to the value of M. But from the practical

point of view this modification is here of minor importance, because the weight of the specified beams is only about 10 per cent. that of the static load.

41. Dissipative Forces. The vibrations of actual systems are more or less affected by frictional or dissipative forces of various kinds, and the origin of these forces is usually known. As the laws of dry- and fluid-friction are both commonly involved in a given problem, it follows that the resistance to motion is not in general proportional to the velocity in question. In what follows we shall, nevertheless, assume that the damping resistances vary as the velocities.

Thus if the velocity be $\dot{y}$, the frictional force can be expressed by $K\dot{y}$, where K denotes a coefficient of friction which may be determined by experimental means. If, alternatively, the frictional force is proportional to the *relative* velocities, say $\dot{y}_1$ and $\dot{y}_2$, then on this supposition its value is given by $K(\dot{y}_1 - \dot{y}_2)$, where K is, again, an appropriate coefficient of friction.

Although, as already observed, the coefficient K does not generally remain constant over a wide range of velocities in the case of actual machines and structures, it is often permissible to use a mean value of the coefficient proper to the small oscillations under examination.

We can now without difficulty modify the foregoing analysis in accordance with our assumed law of friction.

42. If a frictional force of this type be imposed on the system implied in equation (38.1), the *damped free oscillations* which would follow are defined by

$$\frac{M}{g}\ddot{y} + K\dot{y} + cy = 0,$$

or

$$\ddot{y} + R\dot{y} + p^2 y = 0, \quad . \quad . \quad . \quad . \quad . \quad (42.1)$$

where $R = \dfrac{Kg}{M}$, $p^2 = \dfrac{cg}{M}$, remembering that the frictional force always opposes the motion. Our equation, involving one degree of freedom, consequently refers to the damped motion of a mass of weight M connected to an elastic system having a coefficient of stiffness c, when executing periodic motion under the influence of a force that varies as the displacement and a resistance that is proportional to the velocity $\dot{y}$.

Making the usual substitution $y = e^{\lambda t}$, we have

$$\lambda^2 + R\lambda + p^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad (42.2)$$

whence, with λ_1 and λ_2 denoting the roots of this quadratic in λ,

$$y = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \quad . \quad . \quad . \quad . \quad . \quad (42.3)$$

is the general solution of equation (42.1), where

$$\lambda_1 = \tfrac{1}{2}\{-R + (R^2 - 4p^2)^{\frac{1}{2}}\},\ \lambda_2 = \tfrac{1}{2}\{-R - (R^2 - 4p^2)^{\frac{1}{2}}\}.$$

Mention should be made of the motion described when the value of K is large in the first place, and small in the second.

Large Resistance. If the coefficient K be very large in magnitude, such as is the case when a pendulum moves through a viscous fluid, $R^2 > 4p^2$, equation (42.3) becomes

$$y = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t}, \quad . \quad . \quad . \quad . \quad . \quad (42.4)$$

where the constants A_1 and A_2 depend on the initial conditions. The corresponding time-displacement graph is readily shown to be of the type indicated by Fig. 87(1), where the displacement decreases indefinitely with the time.

Non-periodic motion is also involved in the related, but rather uncommon, case of $R^2 = 4p^2$, because the solution of equation (42.1) then takes the form

$$y = e^{-\frac{1}{2}Rt}(A_1 + A_2 t).$$

There are here three principal kinds of time-displacement graphs, depending on the relative values of the constants A_1 and A_2. For example, Fig. 87(1) illustrates the graph for the conditions

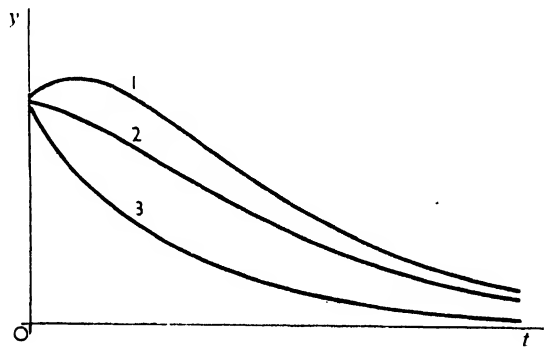


FIG. 87.

$A_1 < \dfrac{2A_2}{R}$, so that the maximum displacement does not take place at the instant $t = 0$. Fig. 87(2) corresponds with the condition $A_1 = \dfrac{2A_2}{R}$, when, on the contrary, the greatest displacement occurs at the time $t = 0$. Finally, Fig. 87(3) shows the graph obtained when $A_1 > \dfrac{2A_2}{R}$, in which circumstance the motion dies away from the start. When the disturbance gradually dies away in the manner indicated by these graphs, the system is sometimes referred to as being *critically damped*.

Small Resistance. Since this implies the condition $R^2 < 4p^2$, the solution of equation (42.1) is of the type

$$y = Ae^{-\frac{1}{2}Rt}\cos(p_1 t + \varepsilon), \quad . \quad . \quad . \quad . \quad (42.5)$$

where A, ε are arbitrary constants, and $p_1^2 = p^2 - \frac{1}{4}R^2$. Hence the maximum amplitude amounts to $Ae^{-\frac{1}{2}Rt}$, and the displacement decreases progressively with the time, as illustrated by Fig. 88. This, as is to be expected, is of the same type as the graph shown dotted in Fig. 50.

It appears, then, from the relation for the periodic time of oscillation, namely

$$\text{period} = \frac{2\pi}{p_1}\,.$$

$$= \frac{2\pi}{\sqrt{p^2 - \frac{1}{4}R^2}}, \quad . \quad . \quad . \quad . \quad (42.6)$$

bearing in mind that R is now small in value, that slight damping does not greatly affect the frequency, the effect being mainly that of decreasing the amplitude.

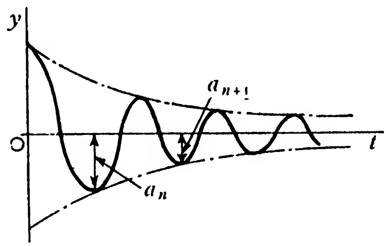


FIG. 88.

If in Fig. 88 we let a_n and a_{n+1} represent the amplitudes of two successive maxima on the same side of the equilibrium-position, then

$$\frac{a_n}{a_{n+1}} = e^{\frac{\pi R}{p_1}},$$

i.e. $$\log_e a_n - \log_e a_{n+1} = \frac{\pi R}{\sqrt{p^2 - \frac{1}{4}R^2}}, \quad . \quad . \quad . \quad . \quad (42.7)$$

the constant $\dfrac{\pi R}{p_1}$ being known as the *logarithmic decrement.*

The practical significance of this decrement is to be found in the fact that in the damped motion under examination at any instant the dissipation of energy per cycle is equal to twice the total energy multiplied by the logarithmic decrement.

Ex. To illustrate the point just mentioned, consider first an electrical instrument with a rotating system, of polar moment of inertia 0·10 gramme-centimetre units, which is controlled by a spring that exerts a couple of 100 gramme-centimetre units for each angular degree of displacement about the equilibrium-position. In our notation the damping is defined by $\dfrac{a_n}{a_{n+1}} = 10$, and this may be taken as constant throughout the motion to be investigated.

In small oscillations of the instrument we have

$$I\ddot{\theta} + K\dot{\theta} + c\theta = 0,$$

where I is the moment of inertia for the rotating parts, K the damping coefficient, and c the couple exerted by the spring for unit displacement. It has also been shown that

$$\log_e \frac{a_n}{a_{n+1}} = \frac{\pi K}{\sqrt{4Ic - K^2}}.$$

Now, with

$$\log_e \frac{a_n}{a_{n+1}} = 2{\cdot}303,$$

it appears that, in gramme-centimetre-second units,

$$K^2 = \frac{4Ic(2{\cdot}303)^2}{\pi^2 + (2{\cdot}303)^2}$$

$$= \frac{4 \times 0{\cdot}1 \times 100(2{\cdot}303)^2}{\pi^2 + (2{\cdot}303)^2}$$

$$= 14{\cdot}0,$$

whence $K = 3{\cdot}74$ gramme-centimetre units. If the appropriate values be substituted in the expression

$$\text{period} = \frac{4I \log_e 10}{K},$$

it will be found that 0·247 second is the periodic time for the specified instrument.

Now imagine the instrument as damaged and, subsequently, repaired in such a way as to leave K and c unchanged, but with the relevant moment of inertia increased to 0·15 gramme-centimetre units.

By way of evaluating the constants for the repaired instrument, let δ and δ_1 denote the logarithmic decrements before and after the modification, respectively. By the relation

$$\frac{\delta_1}{\delta} = \frac{\sqrt{4Ic - K^2}}{\sqrt{4I_1c - K^2}}$$

where I_1 refers to the new moment of inertia, we have, on inserting the known values,

$$\delta_1 = 2{\cdot}303 \frac{\sqrt{4 \times 0{\cdot}10 \times 100 - (3{\cdot}74)^2}}{\sqrt{4 \times 0{\cdot}15 \times 100 - (3{\cdot}74)^2}}$$

$$= 1{\cdot}73,$$

and $$\text{period} = \frac{4 \times 1{\cdot}73 \times 0{\cdot}15}{3{\cdot}74} \text{ sec.}$$

$$= 0{\cdot}278 \text{ sec.}$$

As a consequence of the repairs the periodic time of the instrument has, therefore, changed from 0·247 second to 0·278 second.

43. To derive an equation for the *damped forced oscillations* which would ensue from applying the force $P \cos \omega t$ to the system in mind, we add the term for this periodic force to the right of equation (42.1), and thus obtain

$$\frac{M}{g}\ddot{y} + K\dot{y} + cy = P \cos \omega t,$$

or

$$\ddot{y} + R\dot{y} + p^2 y = Q \cos \omega t, \quad . \quad . \quad . \quad . \quad (43.1)$$

where $R = \dfrac{Kg}{M}$, $p^2 = \dfrac{cg}{M}$, $Q = \dfrac{Pg}{M}$.

It is shown in works devoted to differential equations that, with D written for $\dfrac{d}{dt}$, a 'particular solution' of this equation is given by the real part of

$$\frac{1}{D^2 + RD + p^2} Qe^{i\omega t},$$

i.e.

$$\frac{1}{p^2 - \omega^2 + iR\omega} Qe^{i\omega t}$$

or

$$\frac{p^2 - \omega^2 - iR\omega}{(p^2 - \omega^2)^2 + R^2\omega^2} Qe^{i\omega t},$$

which is

$$\frac{(p^2 - \omega^2) \cos \omega t + R\omega \sin \omega t}{(p^2 - \omega^2)^2 + R^2\omega^2} Q$$

or

$$\frac{Q \cos (\omega t + \beta)}{\{(p^2 - \omega^2)^2 + R^2\omega^2\}^{\frac{1}{2}}},$$

where $\tan \beta = \dfrac{R\omega}{\omega^2 - p^2}$.

Therefore, introducing the 'complementary function', the complete solution of equation (43.1) is

$$y = Ae^{-\frac{1}{2}Rt} \cos (p_1 t + \varepsilon) + \frac{Q \cos (\omega t + \beta)}{\{(p^2 - \omega^2)^2 + R^2\omega^2\}^{\frac{1}{2}}}, \qquad (43.2)$$

where $p^2{}_1 = p^2 - \frac{1}{4}R^2$. Taken in order the right-hand members, as already pointed out, refer to the free and the forced oscillations.

Since β represents the phase-difference between the force and displacement, the last equation shows that the displacement is not generally in phase with the force, lagging behind it by less than one-quarter period if $\tan \beta$ is negative, that is when the natural frequency of the system is greater than the imposed frequency. If, on the contrary, the natural frequency is less than the imposed, $\tan \beta$ is positive and the displacement lags from one-quarter to one-half period behind the force. It is, moreover, seen that $\tan \beta$ tends to infinitely large values as the natural and imposed fre-

quencies approach equality. When ω actually coincides with p, the maximum displacement occurs when the imposed force is zero and, therefore, lags behind the force by 90 deg.

In circumstances where the frictional force is so small that we may neglect R in comparison with the other quantities, the expression for the displacement y reduces to

$$y = \frac{Q \cos \omega t}{p^2 - \omega^2},$$

showing that the periods and the phases are both the same. Thus the maximum displacement attains the value

$$\frac{Q}{p^2 - \omega^2},$$

i.e.

$$\frac{Q}{p^2}\frac{p^2}{p^2 - \omega^2},$$

or

$$\frac{P}{c}\frac{p^2}{p^2 - \omega^2}.$$

Hence the *dynamic magnifying factor*, defined by the ratio of the maximum displacement in the forced vibration and the deflection which would be produced by the same load when steadily applied, amounts to

$$\frac{p^2}{p^2 - \omega^2},$$

or

$$\frac{1}{1 - \left(\frac{\omega}{p}\right)^2}.$$

44. Interference. Particular interest is attached to the motion represented by equation (43.2) when ω and p are nearly, but not quite, equal. To fix ideas in the process of discussing the matter, suppose the frictional coefficient K to be comparatively small, and the initial conditions to be

$$y = 0 \text{ and } \dot{y} = 0 \text{ at the time } t = 0.$$

Motion under these conditions is, as may easily be verified, approximately expressed by the equation

$$y = B\{\cos(\omega t + \beta) - e^{-\frac{1}{2}Rt}\cos(p_1 t + \beta)\} \quad . \quad . \quad (44.1)$$

Let us, for the moment, restrict our analysis to the early stages of this disturbance, when it will be permissible to neglect the time-factor $e^{-\frac{1}{2}Rt}$. The last equation then relates to the motion which would result from the superposition of two simple harmonic oscillations of nearly equal frequency, because the resulting expression may be arranged in the form

$$y = B_1 \cos(\omega t + \beta) + B_2 \cos\{(\omega + \Delta\omega)t + \varepsilon\},$$

where the small quantity $\Delta\omega$ denotes 2π times the frequency-difference. The fact that the second member on the right can be expressed as

$$B_2 \cos \{(\omega t + \beta) + t\Delta\omega - \beta + \varepsilon\},$$

or $\quad B_2\{\cos(\omega t+\beta)\cos(t\Delta\omega-\beta+\varepsilon) - \sin(\omega t+\beta)\sin(t\Delta\omega-\beta+\varepsilon)\}$

enables us next to write

$$y = C\cos(\omega t + \beta) + D\sin(\omega t + \beta),$$

where

$$C = B_1 + B_2\cos(t\Delta\omega - \beta + \varepsilon),\ D = -B_2\sin(t\Delta\omega - \beta + \varepsilon).$$

Or, more concisely,

$$y = E\cos(\omega t + \beta - \alpha), \quad . \quad . \quad . \quad . \quad (44.2)$$

where $E^2 = C^2 + D^2$, $\tan\alpha = \dfrac{D}{C}$; consequently

$$E = \{B_1^2 + B_2^2 + 2B_1B_2\cos(t\Delta\omega - \beta + \varepsilon)\}^{\frac{1}{2}}$$

in terms of the original symbols. It is easily seen that the motion given by equation (44.2) is equivalent to a harmonic vibration having an amplitude which varies between the limits $(B_1 + B_2)$ and $(B_1 - B_2)$, and a period of $\dfrac{2\pi}{\Delta\omega}$. In other words, the frequency is equal to the difference of the frequencies for the component oscillations, and the phase is variable, as indicated in the corresponding time-displacement graph of Fig. 89.

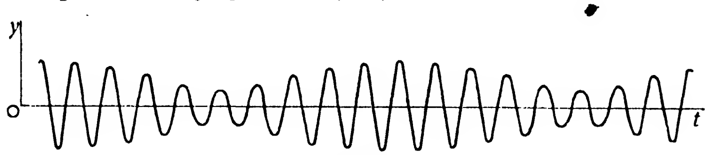


FIG. 89.

The phenomenon thus exhibited is known as *interference* or *beats*, and it is in general a characteristic of vibratory motion when the frequencies of two components are approximately equal.

Now extend the analysis so as to cover all the stages of the disturbance, by taking account of the time-factor $e^{-\frac{1}{2}Rt}$ in equation (43.2). The effect of this factor on the motion is, as already pointed out, to damp the free or natural oscillation, leaving only the impressed motion, as represented to the right of the graph in Fig. 90.

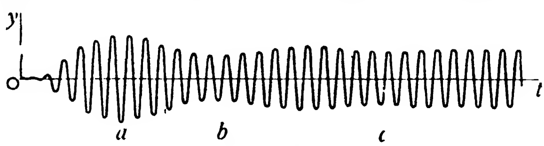


FIG. 90.

Records of vibrations executed under these conditions sometimes exhibit the *surging* phenomena shown at the points (*a*) and (*b*) in the figure, but continued operation of the disturbing force soon leads to the steady state marked (*c*).

Many instances of interference are to be found in engineering practice, such as, for example, on ships fitted with twin-screws, where a slight difference in the speeds of the shafts makes for this type of disturbance.

It is, however, not always easy to determine the chief causes of interference, because it commonly arises from a number of independent sources. A case in point is that of an aero-engine driving a two-blade airscrew through reduction-gearing having a ratio that is nearly, but not quite, equal to 2 : 1. Beats may well appear on the records taken at various points on the fuselage, but these will be due mainly to the combined effect of two independent causes. We may refer to these by supposing, for definiteness, that the engine rotates at 2,000 revolutions a minute and drives the airscrew through a reduction-gearing of ratio 49 : 26.

(i) The vibratory motion induced by the unbalanced parts of the engine and airscrew will undergo periodic variations in phase, since the speed of the airscrew is slightly greater than half that of the engine. This source will, by equation (44.2), produce in the fuselage beats with a frequency equal to the difference between the two component frequencies, that is the difference between the speed of the engine and twice that of the propeller. Thus

$$2{,}000(2 \times \tfrac{26}{49} - 1) \text{ cycles a minute, i.e. } 122{\cdot}5 \text{ cycles a minute}$$

represents the frequency of the beats referred to the first harmonic component of the engine. An interference phenomenon with a frequency of 245 cycles a minute will, for the same reason, be associated with the second harmonic components of the engine, and so on with respect to the higher harmonic components of the unbalanced effects.

A disturbance arising from this source may therefore take place with two-blade airscrews, but it can be avoided by substituting airscrews having three or four blades.

(ii) The unbalanced effect of the airscrew will simultaneously initiate oscillatory motion which interferes with that induced by the half-order harmonic component of the torque-reaction on the usual type of aero-engine. To this may be added the effect of interference between the vibrations due to the aerodynamic couples on the airscrew-blades in the first place, together with the primary harmonic components of the unbalanced force and torque-reaction on the engine in the second.

The interference which is experienced in actual machines will

generally arise from both of these sources, and give a disturbance of the type shown in Fig. 89, with the difference that in the resulting figure the horizontal axis assumes a wave-like form.

It will be shown later that these remarks on aircraft apply, strictly speaking, only so long as an aeroplane continues on a straight course.

45. Resonance or Synchronism. At the instant when the effect of the exponential term $e^{-\frac{1}{2}Rt}$ causes the free vibrations to vanish from equation (43.2), the expression for the displacement reduces to

$$y = \frac{Q \cos(\omega t + \beta)}{\{(p^2 - \omega^2)^2 + R^2\omega^2\}^{\frac{1}{2}}},$$

where $Q = \frac{Pg}{M}$, $p^2 = \frac{cg}{M}$, $R = \frac{Kg}{M}$; hence

$$y = \frac{P \cos(\omega t + \beta)}{c\left[\left\{1 - \left(\frac{\omega}{p}\right)^2\right\}^2 + \frac{K^2 g}{Mc}\left(\frac{\omega}{p}\right)^2\right]^{\frac{1}{2}}} \quad . \quad . \quad (45.1)$$

With a written for $\frac{K^2 g}{Mc}$, it follows that the maximum value of

$$y = \frac{P}{c\left[\left\{1 - \left(\frac{\omega}{p}\right)^2\right\}^2 + a\left(\frac{\omega}{p}\right)^2\right]^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (45.2)$$

where the impressed and natural frequencies are in succession denoted by $\frac{\omega}{2\pi}$ and $\frac{p}{2\pi}$.

To simplify further examination of the matter, let us assign the values

$$\begin{aligned} P &= 4, \\ c &= 1{,}440, \\ K &= 1, \\ a &= \tfrac{1}{447}, \\ p^2 &= 4{,}640, \end{aligned}$$

which are given in pound-foot-second units, to the quantities in equation (45.2), then

$$y = \frac{1}{360\left[\left\{1 - \left(\frac{\omega}{p}\right)^2\right\}^2 + \frac{1}{447}\left(\frac{\omega}{p}\right)^2\right]^{\frac{1}{2}}} \text{ feet.}$$

If this expression be used for evaluating the amplitude y which corresponds with various values of $\left(\frac{\omega}{p}\right)$, say between the range of

zero and 3, the graph representing y as a function of $\left(\frac{\omega}{p}\right)$ will be found to be as indicated in Fig. 91.

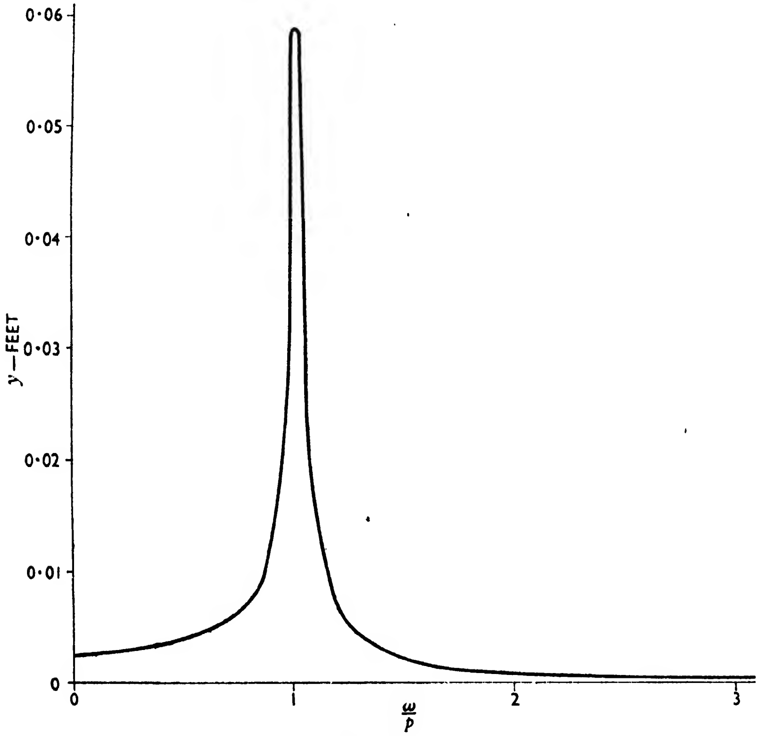


FIG. 91.

It is important to note that the amplitude tends to relatively large values as $\left(\frac{\omega}{p}\right)$ approaches unity, when the system is said to be in a state of *resonance* or *synchronism*. Actually the greatest displacement does not occur when $\omega = p$, but rather when the denominator in equation (45.2) attains a minimum or zero value. If the degree of damping is only slight, however, the condition $\omega = p$ approximately defines the state of resonance.

Regard being had to practical considerations, it is to be inferred from the foregoing treatment that the phenomenon of interference commonly precedes and follows that of synchronism.

In circumstances where resonance is a probable consequence, it is clearly advisable to remove the conditions that make for a state of motion in which a comparatively small force may cause dangerous stresses in a structural system. Each problem must be considered on its merits in this regard, since our results show that the remedial

measures to be taken depend on a number of factors, such as the quantities denoted by p, ω, c, K in equation (45.1). Also, supercharging offers a limited means of control over troublesome torque-reactions on the shafts of aero-engines, as will be pointed out later. In large buildings containing a number of storeys, to take another example, the value of c is usually different for different storeys, and this leads to a corresponding number of possible conditions of resonance. Moreover, in structures generally a state of synchronism is associated with each of the several parts, such as the floor or ceiling of a particular room, which may therefore independently execute synchronous oscillations.

46. General Theory of Vibrations. Our discussion on oscillations has, in the main, been limited to systems having one degree of freedom, but we commonly meet with in practice structures and machines which can oscillate simultaneously about a number of axes. With a view to examining such cases we proceed to a general treatment of the dynamics of systems making small vibrations, with any finite number n degrees of freedom, about the *equilibrium-position*, corresponding to the configuration in which a system can remain permanently at rest.

In this work we shall establish certain relations, in Arts. 47–51, by way of utilizing the generalized co-ordinates of Art. 19 for the purpose of deriving expressions for the kinetic and potential energies of specified systems. The oscillatory motion is then determined by Lagrange's method, and in this connection it may, again, be pointed out that a generalized component of force need not have the 'dimensions' of a mechanical force, it being only necessary that the *product* of a 'generalized' force and the related co-ordinate shall have the dimensions of work.

47. Consider a conservative system free from external forces, and let its position be defined by n independent co-ordinates q_1, q_2, . . . , q_n, so chosen that they vanish in the undisturbed configuration ; this adjustment can always be effected by the addition of suitable constants. Then the kinetic energy T is given by equation (19.5), namely

$$2T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots,$$

where the coefficients of inertia a_{rs} are functions of the co-ordinates. With V representing the potential energy, we have, by the formula (19.9), the equation of motion in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r}, \quad . \quad . \quad . \quad . \quad (47.1)$$

where, with $r = 1, 2, 3, \ldots, n$, q_r refers to any one of the co-ordinates $q_1, q_2, \ldots, q_n$.

To ensure that the configuration shall be one of equilibrium, equation (47.1) must be satisfied by

$$\dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n = 0 \quad . \quad . \quad . \quad . \quad (47.2)$$

for all values of the time t, so that

$$\frac{\partial V}{\partial q_r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47.3)$$

Hence the sufficient condition for equilibrium about the stated configuration is that the potential energy V shall have 'stationary' values for small variations in the co-ordinates.

When the potential energy is a *minimum* in the position of rest, as is commonly the case in engineering problems, it is readily seen that the equilibrium is *stable*. This is so because in the slight disturbances under consideration the total energy $(T + V)$ is, by equation (38.5), constant, and T is essentially positive, whence V cannot exceed its equilibrium-value by more than a small amount which depends on the energy of the disturbance. Our hypothesis therefore implies an upper limit to the deviation of each co-ordinate from its position of equilibrium, and P. L. Dirichlet [1] has shown that the limiting value diminishes indefinitely with the energy of the initial disturbance.

In mechanical and structural systems the total energy $(T + V)$ diminishes continually so long as the velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ differ from zero, due to the effect of dissipative forces. Here, too, the equilibrium-conditions (47.3) must be fulfilled, since if the system be started from rest in a position for which V is less than in the equilibrium-configuration, the potential energy would undergo further decrease, and either diminish indefinitely or result in the system attaining a new position of rest, as T can never be negative.

48. To deduce the *relations between statical forces and displacements*, impose on the system extraneous forces having the generalized components, $Q_1, Q_2, \ldots, Q_n$. The n equations for the resulting motion follow on adding the term Q_r to the right of equation (47.1), and if these be equated to zero, the n conditions of equilibrium are seen to be

$$-\frac{\partial V}{\partial q_r} + Q_r = 0,$$

or

$$Q_r = \frac{\partial V}{\partial q_r}, \quad . \quad . \quad . \quad . \quad (48.1)$$

where $r = 1, 2, \ldots, n$.

When all the co-ordinates have been arranged to vanish in the equilibrium-configuration, for most practical purposes the potential

[1] *Werke*, vol. 2, page 3.

energy V in a position which differs only slightly from that of rest can be written in the form

$$2V = c_{11}q_1^2 + c_{22}q_2^2 + \ldots + 2c_{12}q_1q_2 + \ldots \quad . \quad (48.2)$$

This arises from the fact that V, being a function of the co-ordinates, can be expanded in powers of those co-ordinates, in which operation we retain only the quadratic terms and neglect the higher powers. A constant term in the expansion would vanish from equation (47.1), and it is therefore irrelevant ; nor would terms linear in the co-ordinates be of any account, for the n conditions (47.3) must be satisfied by zero values of the co-ordinates $q_1, q_2, \ldots, q_n$.

Moreover, it is to be inferred from the foregoing remarks that the last expression must be essentially positive if *stability* is to be secured, and that if the position were one of unstable equilibrium the system would not vibrate about it. Consequently the coefficients c_{rs} are subject to algebraic restrictions similar to those imposed on the coefficients of inertia a_{rs} in equation (19.6).

To state the displacements in terms of the forces, and so find the forces required to maintain given displacements, apply equations (48.1) to (48.2), then the n relations become

$$Q_r = c_{1r}q_1 + c_{2r}q_2 + \ldots + c_{nr}q_n, \quad . \quad . \quad . \quad (48.3)$$

with $r = 1, 2, \ldots, n$. It is clear that the quantities indicated by the symbols c_{rs} represent the related *coefficients of stiffness*, and that these may be treated as constants throughout the small oscillations in question. In other words, equations (48.3) refer to a set of forces which are linear functions of the displacements reckoned from the equilibrium-position, so these equations imply Hooke's law of elasticity.

The potential energy is then equal to the *strain energy*, as may be shown in the following manner, where we suppose, for simplicity, that the forces are gradually increased from zero to their final values $Q_1, Q_2, \ldots, Q_n$, in such a way as to maintain the same ratios to one another. Multiply both sides of the equations (48.3) by q_r, and write down the set of n relations obtained when the values 1, 2, $\ldots$, n are in turn assigned to the suffix r. This yields, on summing,

$$Q_1q_1 + Q_2q_2 + \ldots + Q_nq_n = c_{11}q_1^2 + c_{22}q_2 + \ldots c_{nn}q_n^2,$$

whence, by equation (48.2),

$$V = \tfrac{1}{2}(Q_1q_1 + Q_2q_2 + \ldots + Q_nq_n), \quad . \quad . \quad . \quad (48.4)$$

indicating that the potential energy is equivalent to the total work done by the forces in displacing the system from zero to the maximum values of the co-ordinates $q_1, q_2, \ldots, q_n$, which is by definition the strain energy associated with the prescribed displacement.

49. The equations (48.3) may be used to derive important *reciprocal relations* for any system executing two states of motion through the same configuration, say that defined by $q_1, q_2, \ldots, q_n$.

If the forces and co-ordinates in one of the states be as indicated in equations (48.3), and the corresponding terms for the other state be represented by accented symbols, the expressions for the rth force in the two states become

$$Q_r = c_{1r}q_1 + c_{2r}q_2 + \ldots + c_{nr}q_n,$$
$$Q_r' = c_{1r}q_1' + c_{2r}q_2' + \ldots + c_{nr}q_n',$$

where $r = 1, 2, \ldots, n$, and the coefficients c_{rs} denote constants. On multiplying in succession the right-hand members of the first expression by $q_1', q_2', \ldots, q_n'$, and those of the second by $q_1, q_2 \ldots, q_n$, both operations yield

$$c_{11}q_1q_1' + c_{22}q_2q_2' + \ldots + c_{12}(q_1q_2' + q_1'q_2) + \ldots,$$

whence we infer that

$$Q_1q_1' + Q_2q_2' + \ldots Q_nq_n' = Q_1'q_1 + Q_2'q_2 + \ldots + Q_n'q_n. \quad (49.1)$$

This reciprocal relation between the forces and displacements in the two states of motion, which is due to Lord Rayleigh,[1] has many applications in the theory of structures. It shows, for example, that

$$\frac{Q_r}{q_s} = \frac{Q_s'}{q_r'} \quad \ldots \ldots \ldots \quad (49.2)$$

holds if all the forces vanish with the exception of Q_r and Q_s'. Thus if the co-ordinates have the same dimensions, both being either linear or angular, the displacement of type r produced by a force of type s is equal to the displacement of type s produced by a force of type r. Hence if r and s refer to any two points on a beam having any cross-section and supported in any manner, it follows from the last relation that:

(i) If a specified load applied at r causes a deflection y at s, then the same load applied at s will cause a deflection y at r.

(ii) If a given couple applied at r causes a rotation θ at s, then the same couple applied at s will cause a rotation θ at r.

(iii) If a bending moment or couple $\mathfrak{T}$ applied at r causes a deflection y at s, then a force $\frac{\mathfrak{T}}{\beta}$ at s will produce a rotation $\frac{y}{\beta}$ at r.

50. To investigate the *effect of constraints on the potential energy*, consider the potential energy induced by a given set of forces together with the energy acquired when the same system is constrained in any way. For this purpose let unaccented symbols

[1] *Scientific Papers*, vol. I, page 223.

refer to the unconstrained motion, and accented symbols to the constrained motion, then by equation (48.4) we have

$$V - V' = \tfrac{1}{2}\sum_r (Q_r q_r - Q_r' q_r')$$
$$= \tfrac{1}{4}\sum_r (Q_r + Q_r')(q_r - q_r') + \tfrac{1}{4}\sum_r (Q_r - Q_r')(q_r + q_r'),$$

where the summation signs apply to all the suffixes. But the foregoing reciprocal relations indicate that the two summations in this expression are equal, since $(Q_r + Q_r')$ and $(Q_r - Q_r')$ denote the forces associated respectively with $(q_r + q_r')$ and $(q_r - q_r')$. Therefore

$$V - V' = \tfrac{1}{2}\sum_r (Q_r - Q_r')(q_r + q_r')$$
$$= \tfrac{1}{2}\sum_r (Q_r - Q_r')(q_r - q_r') + \sum_r (Q_r - Q_r')q_r', \quad . \quad (50.1)$$
$$V' - V = \tfrac{1}{2}\sum_r (Q_r' + Q_r)(q_r' - q_r)$$
$$= \tfrac{1}{2}\sum_r (Q_r' - Q_r)(q_r' - q_r) + \sum_r (q_r' - q_r)Q_r \quad . \quad (50.2)$$

Suppose first that all the r-co-ordinates appear in the unconstrained motion, and that some of them vanish in the constrained motion. We then have $Q_r = Q_r'$ for the forces common to both states of motion, and all the co-ordinates q_r' vanishing for the remaining forces, hence, by equation (50.1),

$$V - V' = \tfrac{1}{2}\sum_r (Q_r - Q_r')(q_r - q_r') \quad . \quad . \quad . \quad (50.3)$$

As the right-hand expression is essentially positive, since it represents the work done on the system, we may on this account say: *The potential energy of the system when deformed by specified forces is greater than the energy acquired by the same system when constrained.*

Suppose next that the unconstrained system mentioned above is initially given r prescribed displacements, by the application of r suitable forces; also that some of the forces vanish when constraints are imposed on the same system. Under these conditions equation (50.2) yields

$$V' - V = \tfrac{1}{2}\sum_r (Q_r' - Q_r)(q_r' - q_r), \quad . \quad . \quad . \quad (50.4)$$

because $q_r = q_r'$ for the displacements common to both states of motion, and the forces Q_r vanish for the remaining co-ordinates. This result is implied in the statement: *The work required to produce a particular displacement is less than that which would be needed to give the same displacement if the same system were constrained.*

In connection with the analogous theorems of Thomson and Bertrand, Art. 19(*f*), Lord Rayleigh [1] has pointed out that both are included in the statement that the effect of additional constraints is to increase the inertia, or moment of inertia, of a system.

[1] *Theory of Sound*, vol. I, page 100.

The examples of Art. 38 contain simple illustrations of these theorems.

It is well to remember that constraints are, to the present degree of approximation, practically proportional to the difference between the co-ordinates corresponding to the free and constrained states of a given system.

51. If the n equations (48.3) be used for the purpose of writing the co-ordinates $q_1, q_2, \ldots, q_n$ in terms of the corresponding components of force $Q_1, Q_2, \ldots, Q_n$, on substituting the results in equation (48.4), we obtain a useful relation between the *potential energy and the disturbing forces*, in the form

$$2U = \alpha_{11}Q_1^2 + \alpha_{22}Q_2^2 + \ldots + 2\alpha_{12}Q_1Q_2 + \ldots, \quad (51.1)$$

where U corresponds with V in the original equations, and the coefficients α_{rs} represent constants if the c_{rs}-terms are constants. It is evident that the symbols α_{rs} may be treated as *coefficients of stiffness*, but they may equally well be regarded as *coefficients of stability*, since they appear in the problem of stability.

Further, by making $(Q_r' - Q_r)$ in equation (50.1) an infinitesimal quantity, we have

$$\begin{aligned} q_r &= \frac{\partial U}{\partial Q_r} \\ &= \alpha_{1r}Q_1 + \alpha_{2r}Q_2 + \ldots + \\ &= \sum_{r=1}^{n} \alpha_{rs}Q_r, \end{aligned} \quad (51.2)$$

which determines the displacements in terms of the forces. The *reactions* to the forces are given by

$$q_r = -\sum_{r=1}^{n} \alpha_{rs}Q_r \quad (51.3)$$

52. Free Vibrations about Equilibrium. In a system executing small oscillations about its equilibrium-configuration we have shown that the kinetic energy T is expressed by

$$2T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots, \quad (52.1)$$

where a_{rs} denotes the coefficient of inertia associated with the velocities $\dot{q}_r\dot{q}_s$. Also, with the co-ordinates so adjusted that they vanish in the position of equilibrium, the expression for the potential energy V is

$$2V = c_{11}q_1^2 + c_{22}q_2^2 + \ldots + 2c_{12}q_1q_2 + \ldots, \quad (52.2)$$

where the coefficient of stiffness c_{rs} is associated with the co-ordinates q_rq_s.

In the present treatment it will be assumed, for reasons already stated, that the coefficients a_{rs} and c_{rs} relate to quantities which

are constant and equal to their values in the equilibrium-configuration.

If the term $\dfrac{\partial T}{\partial q_r}$ be neglected as of the second order in the velocities, in the notation of equations (52.1) and (52.2) the Lagrangian formula for free oscillations is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial V}{\partial q_r} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52.3)$$

where $r = 1, 2, \ldots, n$. Hence, on making the substitutions (52.1) and (52.2) in this formula, we obtain a set of n equations of the type

$$(a_{1r}\ddot{q}_1 + c_{1r}q_1) + (a_{2r}\ddot{q}_2 + c_{2r}q_2) + \ldots + (a_{nr}\ddot{q}_n + c_{nr}q_n) = 0, \quad (52.4)$$

where $a_{rs} = a_{sr}$ and $c_{rs} = c_{sr}$, the complete solution of which will contain $2n$ arbitrary constants. Assuming that the relation

$$q_r = A_r e^{\lambda t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52.5)$$

holds in these expressions, in accordance with the usual method of solving such linear equations, we have the n equations

$$(a_{1r}\lambda^2+c_{1r})A_1+(a_{2r}\lambda^2+c_{2r})A_2+ \ldots +(a_{nr}\lambda^2+c_{nr})A_n = 0 \quad . \quad (52.6)$$

These give, finally, after eliminating the $(n - 1)$ ratios

$$A_1 : A_2 : \ldots : A_n,$$

the determinantal equation

$$\begin{vmatrix} a_{11}\lambda^2 + c_{11}, & a_{21}\lambda^2 + c_{21}, & \ldots, & a_{n1}\lambda^2 + c_{n1} \\ a_{12}\lambda^2 + c_{12}, & a_{22}\lambda^2 + c_{22}, & \ldots, & a_{n2}\lambda^2 + c_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n}\lambda^2 + c_{1n}, & a_{2n}\lambda^2 + c_{2n}, & \ldots, & a_{nn}\lambda^2 + c_{nn} \end{vmatrix} = 0 \quad . \quad (52.7)$$

This expression, of the nth degree in λ^2, may be briefly written and referred to as

$$\Delta(\lambda^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52.8)$$

Now it can be shown [1] that the n values of λ^2 in this symmetrical determinant are real and all negative if V is essentially positive, as is usually the case in engineering problems. We shall take the proof of these statements for granted, and suppose the potential energy to be a minimum in the equilibrium-configuration. If, for the present, we assume that the roots of equation (52.8) are all different, then the values of λ will appear in pairs and of the form

$$\lambda = \pm ip, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52.9)$$

where $i = \sqrt{-1}$.

By substituting any one of these values for λ in equation (52.6)

[1] H. Lamb, *Higher Mechanics*, page 220, second edition.

we obtain, in virtue of equation (52.7), the $(n-1)$ necessary relations between the $(n-1)$ ratios

$$A_1 : A_2 : \ldots : A_n,$$

where the absolute values of the constants alone are arbitrary. Thus if $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the minors of any one row of the determinant $\Delta(\lambda^2)$, it follows that

$$\frac{A_1}{\alpha_1} = \frac{A_2}{\alpha_2} = \ldots = \frac{A_n}{\alpha_n} = R, \quad . \quad . \quad . \quad . \quad (52.10)$$

where R refers to an arbitrary constant, and the α-terms are functions of λ^2, in consequence of which the minors are the same for values of λ which differ only in sign. Hence the 'real' part of the solution for any one pair of the roots (52.9) is of the form

$$\begin{aligned} q_r &= B\alpha_r \cos(pt + \varepsilon_r) \\ &= C_r \cos(pt + \varepsilon_r), \quad . \quad . \quad . \quad . \quad . \quad (52.11) \end{aligned}$$

with $C_r = B\alpha_r$, and the arbitrary constants B and ε the same for all the co-ordinates.

According to this particular solution each co-ordinate q_r appears in its own equation independently of the others, and every point on the system describes a simple-harmonic vibration of period $\dfrac{2\pi}{p}$. In other words, the several particles pass simultaneously through their positions of equilibrium, the direction and amplitude both being determinate once the constants B and ε are known from the initial conditions of motion. The system is then said to oscillate in its *normal* or *fundamental mode.*

When the values of λ^2 are all different and negative, it is clear that there are n such fundamental modes, and that the superposition of these gives the total displacement of the co-ordinate q_r. In the general case we therefore have

$$q_r = C_1 \cos(p_1t+\varepsilon_1) + C_2 \cos(p_2t+\varepsilon_2) + \ldots + C_n \cos(p_nt+\varepsilon_n), \; (52.12)$$

which suffice for the solution of any problem involving known values for the n co-ordinates $q_1, q_2, \ldots, q_n$ and the n velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$, since the $2n$ arbitrary constants $C_1, C_2, \ldots, C_n$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are available for the purpose.

It is readily inferred from the periodic form of equation (52.12), with small values assigned to $C_1, C_2, \ldots, C_n$, that the configuration differs only slightly from the position of equilibrium, whence a *minimum* value of V indicates *stability* in the normal modes. Taking another view of the matter, it is thus seen that we can resolve the most general small motion of a system into the normal modes.

In the special case where V is not an absolute minimum, equation (52.2) may become negative, when positive values of λ^2 will occur and give rise to solutions of the type

$$q_r = \alpha_r(F_1 e^{\lambda t} + F_2 e^{-\lambda t}) \quad . \quad . \quad . \quad . \quad . \quad (52.13)$$

Here the value of q_r would increase until it rendered invalid the assumptions on which our approximate solutions are based, and this is liable to take place, as was pointed out in Art. 22, with certain types of governors.

Ex. Consider the small oscillations in a normal mode about the equilibrium-position of the system shown in Fig. 92 (*a*), consisting of three concentrated small masses of equal weight M attached

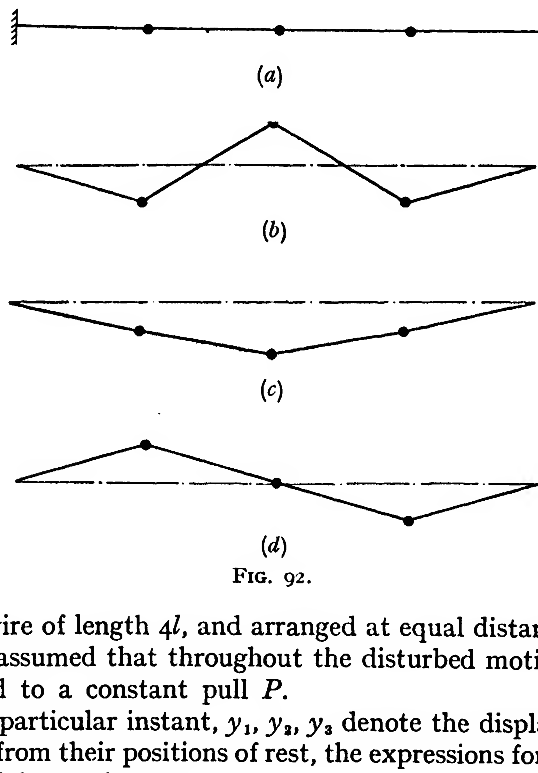


FIG. 92.

to a light wire of length $4l$, and arranged at equal distances l apart. It may be assumed that throughout the disturbed motion the wire is subjected to a constant pull P.

If, at a particular instant, y_1, y_2, y_3 denote the displacements of the masses from their positions of rest, the expressions for the kinetic and potential energies are

$$2T = \frac{M}{g}(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2),$$

$$2V = \frac{P}{l}\{y_1^2 + (y_2 - y_1)^2 + (y_3 - y_2)^2 + y_3^2\},$$

in that the components of force vary as the sine of the small angles made by the disturbed wire with the position of rest.

These expressions enable us at once to write

$$\frac{\partial T}{\partial \dot{y}_1} = \frac{M}{g}\dot{y}_1, \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) = \frac{M}{g}\ddot{y}_1,$$

$$\frac{\partial V}{\partial y_1} = \frac{P}{l}(2y_1 - y_2)$$

referred to the co-ordinate y_1 in the Lagrangian formula

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) + \frac{\partial V}{\partial y} = 0,$$

whence
$$M\ddot{y}_1 + \frac{Pg}{l}(2y_1 - y_2) = 0.$$

By the same procedure we have also

$$M\ddot{y}_2 + \frac{Pg}{l}(2y_2 - y_3 - y_1) = 0,$$

and
$$M\ddot{y}_3 + \frac{Pg}{l}(2y_3 - y_2) = 0$$

for the co-ordinates y_2, y_3.

On the assumption that the displacements vary according to the law implied in equation (52.11), we can write

$$y_1 = A_1 \cos pt, \ y_2 = A_2 \cos pt, \ y_3 = A_3 \cos pt$$

in the above equations, where A_1, A_2, A_3 are arbitrary constants, and $\frac{2\pi}{p}$ is the period of the normal vibration. These substitutions give, with β written for $\frac{Pg}{Ml}$,

$$\begin{aligned} (p^2 - 2\beta)y_1 + \beta y_2 &= 0, \\ \beta y_1 + (p^2 - 2\beta)y_2 + \beta y_3 &= 0, \\ \beta y_2 + (p^2 - 2\beta)y_3 &= 0, \end{aligned}$$

by reason of which, on eliminating the ratios $y_1 : y_2 : y_3$,

$$(p^4 - 4\beta p^2 + 2\beta^2)(p^2 - 2\beta) = 0.$$

The roots of the first member are

$$p^2 = (2 \pm \sqrt{2})\beta,$$

the negative sign giving $y_1 = \frac{1}{\sqrt{2}} y_2 = y_3$, and the positive sign $y_1 = -\frac{1}{\sqrt{2}} y_2 = y_3$. It is evident that the former set of values correspond to the mode indicated in Fig. 92(*c*), having the longest periodic time, while the latter set refer to the mode represented in Fig. 92(*b*), where two nodes are involved. The third root, namely

$$p^2 = 2\beta,$$

leads to the values $y_1 = -y_3$, $y_2 = 0$; this mode contains one node, and is shown in Fig. 92(*d*).

The three fundamental modes of oscillation for the specified system are thus exhibited in the figure, from which it is to be inferred that in the general case the number of normal modes is equal to the number of masses concerned.

53. Normal Co-ordinates. If the co-ordinates $q_1, q_2, \ldots, q_n$ in

$$\left.\begin{aligned} 2T &= a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots, \\ 2V &= c_{11}q_1^2 + c_{22}q_2^2 + \ldots + 2c_{12}q_1q_2 + \ldots, \end{aligned}\right\} \quad (53.1)$$

be transformed by n linear relations of the type

$$q_r = \beta_{1r}\theta_1 + \beta_{2r}\theta_2 + \ldots + \beta_{nr}\theta_n \quad . \quad . \quad (53.2)$$

with constant coefficients, where $\theta_1, \theta_2, \ldots, \theta_n$ are n new variables and $r = 1, 2, \ldots, n$, it is clear that the velocities will be transformed by the same set of relations. Hence the transformation will not affect either the quadratic character of the expressions for T and V or the form of the solution in the general case. If, further, the coefficients $\beta_{1r}, \beta_{2r}, \ldots, \beta_{nr}$ be determined so that equations (53.1) reduce to the sum of squares, say

$$\left.\begin{aligned} 2T &= \delta_1\dot{\theta}^2{}_1 + \delta_2\dot{\theta}^2{}_2 + \ldots + \delta_n\dot{\theta}^2{}_n, \\ 2V &= \gamma_1\theta^2{}_1 + \gamma_2\theta^2{}_2 + \ldots + \gamma_n\theta^2{}_n, \end{aligned}\right\} \quad . \quad . \quad . \quad (53.3)$$

it is manifest that the procedure will make for simplification in working. On the present assumptions the coefficients $\gamma_1, \gamma_2, \ldots, \gamma_n$ will be constants.

This algebraic operation can always be effected in dynamical problems, for it is shown in books [1] devoted to the subject that when we have two homogeneous quadratic functions of any number n variables, one of which is positive for all values of the variables (as we have seen is the case with the kinetic energy in dynamical problems), then it is possible by a real linear transformation to reduce both expressions to sums of squares. By the same theorem we may simultaneously arrange the coefficients of the square terms in one of the expressions so that each is equal to unity, by taking as variables the quantities $\sqrt{\delta_1}\theta_1, \sqrt{\delta_2}\theta_2, \ldots, \sqrt{\delta_n}\theta_n$ instead of $\theta_1, \theta_2, \ldots, \theta_n$, when the relations (53.3) become

$$\left.\begin{aligned} 2T &= \dot{\theta}^2{}_1 + \dot{\theta}^2{}_2 + \ldots + \dot{\theta}^2{}_n, \\ 2V &= \rho_1\theta^2{}_1 + \rho_2\theta^2{}_2 + \ldots + \rho_n\theta^2{}_n, \end{aligned}\right\} \quad . \quad . \quad (53.4)$$

where $\rho_r = \dfrac{\gamma_r}{\delta_r}$. This modification is equivalent to imposing the conditions

$$\delta_1 = \delta_2 = \ldots = \delta_n = 1, \quad . \quad . \quad . \quad . \quad (53.5)$$

which is the same as placing *unit* loads on the system concerned.

[1] E.g., M. Bôcher, *Introduction to Higher Algebra*, Chap. XIII (New York, 1905).

In the transformed system we call the new variables $\theta_1, \theta_2, \ldots, \theta_n$ the *normal* or *principal co-ordinates,* the coefficients $\delta_1, \delta_2, \ldots, \delta_n$ the *principal coefficients of inertia,* and $\gamma_1, \gamma_2, \ldots, \gamma_n$ the *principal coefficients of stiffness* (*stability*). The quantities $\delta_1, \delta_2, \ldots, \delta_n$ are always positive, owing to T being essentially positive ; $\gamma_1, \gamma_2, \ldots, \gamma_n$ also are positive if V is a minimum in the equilibrium-configuration about which the system is supposed to oscillate.

In terms of the normal co-ordinates of (53.3) the equations of motion are

$$\delta_1\ddot{\theta}_1 + \gamma_1\theta_1 = 0,\ \delta_2\ddot{\theta}_2 + \gamma_2\theta_2 = 0, \ldots, \delta_n\ddot{\theta}_n + \gamma_n\theta_n = 0, \quad (53.6)$$

by Lagrange's formula (52.3). Since these equations are independent, in the case of complete stability we have the solutions

$$\theta_1 = A_1 \cos (p_1t + \varepsilon_1),\ \theta_2 = A_2 \cos (p_2t + \varepsilon_2), \ldots,$$
$$\theta_n = A_n \cos (p_nt + \varepsilon_n), \quad . \quad . \quad (53.7)$$

where $p^2{}_1 = \dfrac{\gamma_1}{\delta_1}$, $p^2{}_2 = \dfrac{\gamma_2}{\delta_2}$, $\ldots$, $p^2{}_n = \dfrac{\gamma_n}{\delta_n}$, and the A- and ε-terms together constitute $2n$ arbitrary constants. Thus $\dfrac{2\pi}{p_1}, \dfrac{2\pi}{p_2}, \ldots, \dfrac{2\pi}{p_n}$ determine the periods of the n normal modes which may be executed by the system. That is to say, when the system is vibrating in a normal mode its motion is simple-harmonic, and the velocity of every point on it becomes zero at the same instant twice in every complete oscillation.

To obtain the normal co-ordinates in terms of the original co-ordinates $q_1, q_2, \ldots, q_n$, we have only to substitute $\theta_1, \theta_2, \ldots, \theta_n$ in succession for $\cos (p_1t + \varepsilon_1)$, $\cos (p_2t + \varepsilon_2)$, $\ldots$, $\cos (p_nt + \varepsilon_n)$ in the n equations (52.12) and solve them for the θ's in terms of the q's. By this means it is possible to resolve the general vibrations of a system into normal modes, in which the system will be stable if it is stable for small oscillations in general.

So far we have assumed that the roots of the determinant (52.7) are distinct, in which circumstance the above transformation is unique. Since the occurrence of multiple roots in that equation is, of course, a possible condition, a remark may be made on the point. The equality of two roots would usually lead to terms of the type $(A + Bt) \cos pt$, but we have demonstrated that stable vibrations are impossible with terms of this type. Nevertheless, the equality of roots does not indicate instability ; it merely indicates that two normal modes have the same period, and therefore that the solution is indeterminate. The following example shows how this accidental property may arise from the use of unsuitable co-ordinates for the examination of small vibrations about the equilibrium-configuration.

Ex. Consider the small oscillations in a normal mode for a

particle of weight M moving under gravity on the inner surface of a spherical bowl which is perfectly smooth.

Since the position of rest coincides with the lowest point on the spherical surface, let us attempt to solve the problem by substituting the usual polar co-ordinates θ, ϕ for the two co-ordinates q_1, q_2 of the centre of the particle, with the origin at the centre of the surface and the axis vertically downwards. Writing r for the radius of the surface, we then have, in the above notation, $\delta_1 = \frac{M}{g}r^2, \delta_2 = 0$,

and, therefore, $$2T = \frac{M}{g}r^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta),$$

since $\theta = 0$ in the position of equilibrium. According to this expression, however, the equilibrium-configuration is independent of the second co-ordinate ϕ, showing that there is an indeterminacy in the normal co-ordinates involved.

To overcome this difficulty we refer the motion to the tangent-plane at the lowest point on the spherical surface. If (x, y) define the centre of the particle when projected on that plane, and z its vertical distance from the centre of the sphere when measured downwards, so that $z^2 = r^2 - x^2 - y^2$, then the kinetic energy T of the particle is given by

$$2T = \frac{M}{g}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Now as $z\dot{z} = -(x\dot{x} + y\dot{y})$, and z is approximately equal to r for the small oscillations under consideration, it follows that the velocity $\dot{z}$ is of the second order of small quantities compared with either $\dot{x}$ or $\dot{y}$. If, on this account, we neglect the velocity $\dot{z}$, the relation for T reduces to

$$2T = \frac{M}{g}(\dot{x}^2 + \dot{y}^2);$$

to the same approximation the potential energy

$$V = M(r - z).$$

The next step consists in expanding V in terms of x and y near the point $x = 0, y = 0$, in which process it is to be noticed that $z = r$ at the stated point. As the first derivatives of V yield zero values at points near the position of rest, we must proceed to its second derivatives, which there have the values $\left(-\frac{1}{r}, 0, -\frac{1}{r}\right)$. Hence to this approximation

$$2V = \frac{M}{r}(x^2 + y^2)$$

holds in small vibrations about the equilibrium-position. Application

of formula (52.3) to these expressions for T and V, leads to the equations

$$\ddot{x} + \frac{g}{r}x = 0, \quad \ddot{y} + \frac{g}{r}y = 0,$$

whence, by Art. 39(*b*), the general motion of the particle referred to the horizontal plane is an ellipse.

Our solution for a perfectly smooth surface applies even when the bowl rotates about its vertical axis. But this is not necessarily so if there is the slightest friction between the particle and the bowl, as was pointed out by H. Lamb,[1] who showed that if the angular velocity of the bowl exceed a certain value, the particle would move outwards in a spiral path towards the position in which it rotates with the bowl.

54. Applications to Structural Frames. We shall now illustrate the results of Arts. 52 and 53 by reference to structural systems executing 'normal' vibrations about the equilibrium-configuration, with the object of estimating the free period of oscillation for a particular type of structure. This is a matter of primary importance, as has been demonstrated in Art. 45, in instances where approximate equality is likely to occur between the frequency of any disturbing force and the natural frequency of a given structural system.

Our applications must be confined to the *pin-jointed* type of structure, as the presence of riveted joints would make for continuous systems which do not satisfy the condition that finite values are assigned to n in the determinantal equation (52.7). We shall, for simplicity in working, limit the present treatment to the case of 'non-redundant' structures, and neglect the effect of friction at the pin-joints.

For definiteness in connection with a structure of the prescribed type, let Fig. 93 represent any one of its members which is specified by the joints J_r and J_s. A sufficient exemplification of the analysis will be afforded if we suppose that the member always lies in the plane of the paper. If the motion be referred to the orthogonal x- and y-axes, then $(\dot{x}_r, \dot{y}_r)$ and $(\dot{x}_s, \dot{y}_s)$ will denote in succession the velocities of the ends J_r and J_s. To these variables we may, for the present, add the angle θ indicated in the figure.

A remark is called for here concerning the kinetic energy of the member indicated by Fig. 93, in that we shall subsequently suppose it to be connected to a number of other members, to form a framed structure of the stated type. Having regard to the preceding theory and the dimensions of actual frames, it is manifest that the natural frequency of vibration for the member in question

[1] *Proc. Roy. Soc., A.*, vol. 53, page 168 (1908).

will usually be greater than that of the complete structure thus formed. Moreover, in any oscillatory motion the member will be constrained when it constitutes part of the complete structure, and

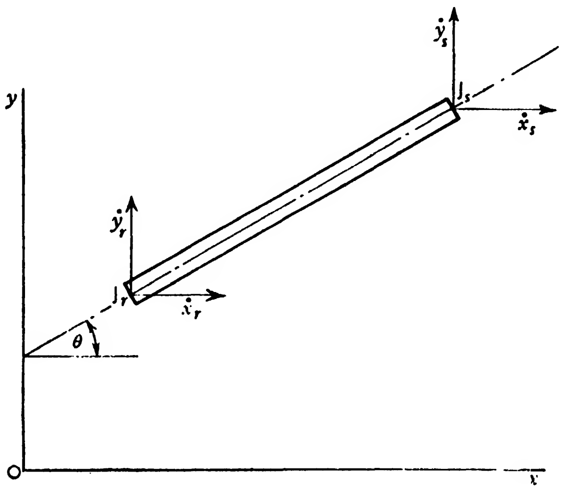


FIG. 93.

it will be shown, in Art. 55, that the effect of any constraint is to raise the frequency. Hence for modes such as are experienced in practice the frequency of oscillation for the specified member will usually be greater than the natural frequency of the complete frame. Since these conditions are fulfilled by structures in general, and those which we shall consider in particular, it is practicable to assume that the individual members of such systems vibrate as rigid bodies throughout the oscillatory motion of the complete frameworks in mind.

If, then, M_{rs} denote the weight of the member shown in Fig. 93, we may at once express its kinetic energy T_{rs} in the form

$$2T_{rs} = \frac{M_{rs}}{g}\left[\left(\frac{\dot{x}_r + \dot{x}_s}{2}\right)^2 + \left(\frac{\dot{y}_r + \dot{y}_s}{2}\right)^2 + \tfrac{1}{12}\{(\dot{y}_s - \dot{y}_r)\cos\theta - (\dot{x}_s - \dot{x}_r)\sin\theta\}^2\right], \quad . \quad . \quad . \quad (54.1)$$

for the right-hand members of this equation represent in order the energy of the linear motion referred to the centre of gravity, and that of the angular motion about the relevant axis through the same point.

It is scarcely necessary to say that no loss of generality will be incurred if the motion is referred to the joint J_r. On writing X_r and Y_r for the components of force acting on that joint due to inertia

effects, parallel respectively to the x- and y-axes, by Lagrange's formula (19.8) we have

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial T_{rs}}{\partial \dot{x}_r}\right) - \frac{\partial T_{rs}}{\partial x_r} &= X_r,\\ \frac{d}{dt}\left(\frac{\partial T_{rs}}{\partial \dot{y}_r}\right) - \frac{\partial T_{rs}}{\partial y_r} &= Y_r.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (54.2)$$

In the process of substituting equation (54.1) in these formulae we can treat θ as constant, owing to the fact that at any instant the configuration of the structure is completely defined by its n joints. Bearing this in mind, it follows that

$$\frac{\partial T_{rs}}{\partial \dot{x}_r} = \frac{M_{rs}}{g}[\tfrac{1}{4}(\dot{x}_r + \dot{x}_s) + \tfrac{1}{12}\{(\dot{y}_s - \dot{y}_r)\cos\theta\sin\theta + (\dot{x}_r - \dot{x}_s)\sin^2\theta\}]$$

and, on differentiating with regard to the time,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T_{rs}}{\partial \dot{x}_r}\right) &= \frac{M_{rs}}{g}[\tfrac{1}{4}(\ddot{x}_r + \ddot{x}_s) + \tfrac{1}{12}\{(\ddot{y}_s - \ddot{y}_r)\cos\theta\sin\theta + (\ddot{x}_r - \ddot{x}_s)\sin^2\theta]\\ &= \frac{M_{rs}}{4g}\{(1 + \tfrac{1}{3}\sin^2\theta)\ddot{x}_r + (1 - \tfrac{1}{3}\sin^2\theta)\ddot{x}_s\\ &\qquad + \tfrac{1}{3}(\ddot{y}_s - \ddot{y}_r)\cos\theta\sin\theta\},\end{aligned}$$

whence, by the first of equations (54.2),

$$\begin{aligned}X_r = \frac{M_{rs}}{4g}\{(1 + \tfrac{1}{3}\sin^2\theta)\ddot{x}_r &+ (1 - \tfrac{1}{3}\sin^2\theta)\ddot{x}_s\\ &+ \tfrac{1}{3}(\ddot{y}_s - \ddot{y}_r)\cos\theta\sin\theta\}, \quad . \quad . \quad (54.3)\end{aligned}$$

since $\dfrac{\partial T_{rs}}{\partial x_r} = 0$. A repetition of the procedure with respect to the y-co-ordinate further leads to

$$\begin{aligned}Y_r = \frac{M_{rs}}{4g}\{(1 + \tfrac{1}{3}\cos^2\theta)\ddot{y}_r &+ (1 - \tfrac{1}{3}\cos^2\theta)\ddot{y}_s\\ &+ \tfrac{1}{3}(\ddot{x}_s - \ddot{x}_r)\cos\theta\sin\theta\} \quad . \quad . \quad (54.4)\end{aligned}$$

These relations determine the inertia forces associated with the disturbed motion of the given member—or the corresponding reactions if negative signs are used—so that further discussion of the matter implies a knowledge of the structure of which the member is to form a part.

To proceed, suppose Fig. 93 to illustrate any one of the web-members of the frame shown in Fig. 94. When so connected, the

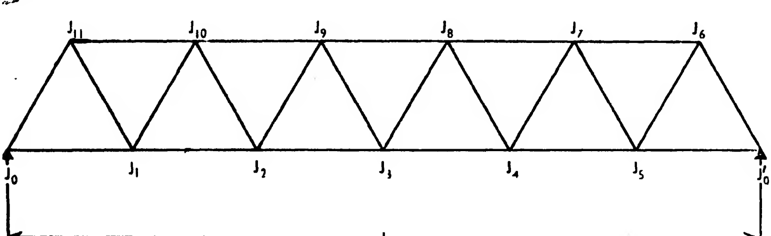


FIG. 94.

member is subjected to certain constraints which we have already identified with the coefficients of stiffness c_{rs} in equation (52.7). To express the necessary coefficients in terms of quantities which are commonly required for purposes other than those in view, let us impose the n conditions (53.5) and write

α_{rs} = x-deflection at joint J_r due to *unit* X-force at joint J_s,
β_{rs} = y-deflection at joint J_r due to *unit* X-force at joint J_s,
γ_{rs} = x-deflection at joint J_r due to *unit* Y-force at joint J_s,
δ_{rs} = y-deflection at joint J_r due to *unit* Y-force at joint J_s.

The work of calculating these quantities is greatly simplified if account is taken of the reciprocal relations (49.2), since they show that

$$\alpha_{rs} = \alpha_{sr}, \; \beta_{rs} = \gamma_{sr}, \; \gamma_{rs} = \beta_{sr}, \; \delta_{rs} = \delta_{sr} \quad . \quad (54.5)$$

Although no reference has been made to the statical loads which are usually imposed on such a structure, it is readily seen that they may be combined with the M_{rs}-terms, provided we know the distribution of loading and the manner in which the masses are attached to the structure in question. Suppose, by way of illustration, this process of combination to be effected for a given structural frame, and that the resulting system is equivalent to a set of loads rigidly fixed to a structure whose weight can be neglected in comparison with the other masses and forces involved in the problem. We might then, if we wished, examine the vibratory motion with the help of a model of the type indicated in Fig. 92. In small oscillations about the equilibrium-position of such a system, a load M_s at the joint J_s would at that point induce the inertia-reactions $-\frac{M_s}{g}\ddot{x}_s$ and $-\frac{M_s}{g}\ddot{y}_s$ in the horizontal and vertical directions, respectively. These forces will separately produce at the joint J_r x-displacements amounting to $-\alpha_{rs}\frac{M_s}{g}\ddot{x}_s$ and $-\gamma_{rs}\frac{M_s}{g}\ddot{y}_s$; and the combined effect at J_r is, by the principle of superposition of small vibrations discussed in Art. 53, equal to the sum of the separat displacements, which is thus seen to be

$$-\frac{M_s}{g}(\alpha_{rs}\ddot{x}_s + \gamma_{rs}\ddot{y}_s)$$

The related y-displacement at the joint J_r is, by a similar argument, equal to

$$-\frac{M_s}{g}(\beta_{rs}\ddot{x}_s + \delta_{rs}\ddot{y}_s)$$

Consequently, in the case of a structure having n joints loaded in this way we have, on summing over the n joints,

$$\left.\begin{aligned} x_r &= -\frac{1}{g}\sum_{s=1}^{n} M_s(\alpha_{rs}\ddot{x}_s + \gamma_{rs}\ddot{y}_s), \\ y_r &= -\frac{1}{g}\sum_{s=1}^{n} M_s(\beta_{rs}\ddot{x}_s + \delta_{rs}\ddot{y}_s) \end{aligned}\right\} \quad . \quad . \quad . \qquad (54.6)$$

as the expressions for the total displacements of the given joint. We might have deduced these equations direct from (51.3), and thus obtained them in the form

$$\left.\begin{aligned} x_r &= -\sum_{s=1}^{n}(\alpha_{rs}X_s + \gamma_{rs}Y_s), \\ y_r &= -\sum_{s=1}^{n}(\beta_{rs}X_s + \delta_{rs}Y_s). \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (54.7)$$

This completes the general treatment for structural systems of the prescribed type, and further discussion on the matter is best effected by exemplification of the foregoing results, in the course of which we shall extend the analysis to suit the given circumstances.

Ex. 1. Determine the natural frequency in a normal mode of (small) oscillation for the *pin-jointed* frame illustrated in Fig. 94, on the assumptions that the weight of the structure itself and the frictional forces may be neglected, when a total load of 28·5 tons is *uniformly distributed* over the lower boom, which is simply supported at the rigid abutments indicated in Fig. 95. As to the

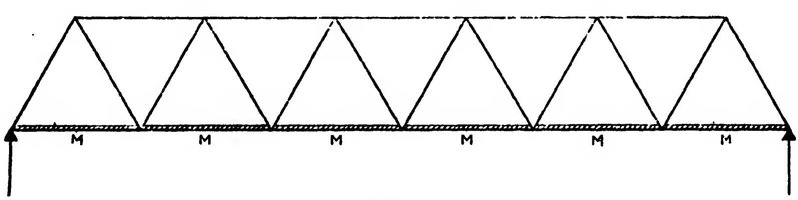


FIG. 95.

required dimensions: the span L is 55 ft.; the sectional area of each boom-member is 9 sq. in., and that of each web-member 5·3 sq. in.; the direct modulus of elasticity for the material is 12,000 tons per square inch. The load may be taken as executing the same motion as the structural member to which it is attached.

Seeing that the specified system of loading is equivalent to 4·75 tons evenly distributed over each of the members on the lower boom

of a structure whose weight is to be neglected, and that the load is rigidly fixed to the members, in the notation of equation (54.4) we can write $M_{rs} = 4{\cdot}75$ tons and consider only the joints J_0, J_1, J_2, J_3, J_4, J_5, J_0' in Fig. 94. We shall make the approximately true assumption that, with rigid abutments, the joints J_0, J_0' remain stationary throughout the prescribed disturbance, in consequence of which the abutments will be *nodes.*

If the numerical values of α_{rs}, β_{rs}, γ_{rs}, δ_{rs} for the given structure be found by one of the usual graphical methods, the data may be exhibited as in Tables 1, 2, 3, though in this particular problem we require only those quantities that correspond with the values of 1, 2, 3, 4, 5 of the suffixes r and s. Ton-inch-second units will be used in the following calculations.

TABLE 1

Values of $\alpha_{rs} \times 10^3$ in inches.

s \ r	1	2	3	4	5
1	0·7691	0·6153	0·4615	0·3077	0·1538
2	0·6153	1·230	0·9230	0·6153	0·3077
3	0·4615	0·9230	1·389	0·9230	0·4615
4	0·3077	0·6153	0·9230	1·230	0·6153
5	0·1538	0·3077	0·4615	0·6153	0·7691

TABLE 2

Values of $\beta_{rs} \times 10^3$, or $\gamma_{sr} \times 10^3$ in inches.

s \ r	1	2	3	4	5
1	0	−0·3552	−0·5328	−0·5328	−0·3552
2	−0·3552	0	−0·5328	−0·7104	−0·5328
3	−0·5328	−0·5328	0	−0·5328	−0·5328
4	−0·5328	−0·7104	−0·5328	0	−0·3552
5	−0·3552	−0·5328	−0·5328	−0·3552	0

TABLE 3

Values of $\delta_{rs} \times 10^3$ in inches

s \ r	1	2	3	4	5	6	7	8	9	10	11
1	6·923	8·000	7·435	5·641	3·022	1·538	4·411	6·667	7·898	7·693	3·590
2	8·000	14·35	13·64	10·46	5·641	2·872	8·206	12·31	14·35	11·49	4·102
3	7·435	13·64	17·38	13·64	7·435	3·795	10·76	15·90	15·90	10·76	3·795
4	5·641	10·46	13·64	14·35	8·000	4·102	11·49	14·35	12·31	8·206	2·872
5	3·022	5·641	7·435	8·000	6·923	3·590	7·693	7·898	6·667	4·411	1·538
6	1·538	2·872	3·795	4·102	3·590	0·7820	2·244	3·397	4·038	3·962	2·961
7	4·411	8·206	10·76	11·49	7·693	2·244	6·422	9·679	11·40	10·96	3·962
8	6·667	12·31	15·90	14·35	7·898	3·397	9·679	14·42	16·61	11·40	4·038
9	7·898	14·35	15·90	12·31	6·667	4·038	11·40	16·61	14·42	9·679	3·397
10	7·693	11·49	10·76	8·206	4·411	3·962	10·96	11·40	9·679	6·422	2·244
11	3·590	4·102	3·795	2·872	1·538	2·961	3·962	4·038	3·397	2·244	0·7820

At this stage of the work certain simplifications can be introduced into the general analysis, regard being had to the specified system. Here, for example, the coefficients α_{rs} and β_{rs} are irrelevant, owing to the fact that X-forces are absent. Moreover, it is manifest that for many practical purposes the numerical values of γ_{rs} are negligibly small compared with those of the δ_{rs}-coefficients. We shall here take this degree of approximation as sufficient, and therefore consider only those quantities which contain the δ_{rs}-coefficients.

Since the loads are applied to members on the lower boom alone, for each of which $\theta = 0$ in Fig. 93, it follows from equation (54.4) that the vertical reaction $-Y_r$ at the joint J_r, due to a *loaded* member, is determined by

$$-Y_r = -\frac{M}{g}(\tfrac{1}{3}\ddot{y}_r + \tfrac{1}{6}\ddot{y}_s), \quad . \quad . \quad . \quad . \quad (54.8)$$

with $M_{rs} = M$ in Fig. 95.

In the process of connecting the r-end of the member shown in Fig. 93 to an appropriate number of others meeting at the *loaded* joint J_r, it is to be noticed that we thereby fix the value of the suffix r in the expression (54.1) for the kinetic energy T_{rs}, and leave the suffix s free to assume certain other values. As J_{rs} may be any one of the loaded joints, with the suffix r fixed, the values $(r - 1)$ and $(r + 1)$ can be ascribed to the suffix s in equation (54.6) or (54.7). In this manner it appears that

$$-\frac{M}{g}(\tfrac{1}{3}\ddot{y}_r + \tfrac{1}{6}\ddot{y}_{r-1})$$

represents the reaction due to the left-hand member of J_r, and

$$-\frac{M}{g}(\tfrac{1}{3}\ddot{y}_r + \tfrac{1}{6}\ddot{y}_{r+1})$$

the reaction due to the right-hand member attached to the same joint, whence the total inertia-reaction $-Y_r$ at the joint J_r is determined by

$$-Y_r = -\frac{M}{g}(\tfrac{1}{6}\ddot{y}_{r-1} + \tfrac{2}{3}\ddot{y}_r + \tfrac{1}{6}\ddot{y}_{r+1}), \quad . \quad . \quad . \quad (54.9)$$

being the sum of the two components. Further, the symmetrical form of this equation enables us to write down

$$-Y_s = -\frac{M}{g}(\tfrac{1}{6}\ddot{y}_{s-1} + \tfrac{2}{3}\ddot{y}_s + \tfrac{1}{6}\ddot{y}_{s+1}) \quad . \quad . \quad . \quad (54.10)$$

as the expression for the related reaction $-Y_s$.

The end conditions, remembering our supposition that the joints J_0 and J_0' remain stationary throughout the motion, are

$$\ddot{y}_0 = 0 \text{ and } \ddot{y}_{r+1} = 0 \text{ when } r = 5,$$
$$\ddot{y}_0 = 0 \text{ and } \ddot{y}_{s+1} = 0 \text{ when } s = 5.$$

To obtain an expression for the total displacement produced at the rth joint by the reaction $-Y_s$, we have only to substitute equation (54.10) in (54.7), when it will be found, with the above end conditions, that

$$y_r = \frac{M}{g}\{(\tfrac{2}{3}\ddot{y}_1+\tfrac{1}{6}\ddot{y}_2)\delta_{1r}+(\tfrac{1}{6}\ddot{y}_1+\tfrac{2}{3}\ddot{y}_2+\tfrac{1}{6}\ddot{y}_3)\delta_{2r}+(\tfrac{1}{6}\ddot{y}_2+\tfrac{2}{3}\ddot{y}_3+\tfrac{1}{6}\ddot{y}_4)\delta_{3r}$$
$$+(\tfrac{1}{6}\ddot{y}_3+\tfrac{2}{3}\ddot{y}_4+\tfrac{1}{6}\ddot{y}_5)\delta_{4r}+(\tfrac{1}{6}\ddot{y}_4+\tfrac{2}{3}\ddot{y}_5)\delta_{5r}\} \quad . \quad . \quad . \quad (54.11)$$

follows by writing in order s equal to 1, 2, 3, 4, 5, because the system contains no X-forces. This is a particular form of the general equation (51.2), and it yields five expressions when r is given the values 1, 2, 3, 4, 5 in succession.

If we next assume that y_1, y_2, y_3, y_4, y_5 vary as e^{ipt}, as is implied in the relation (52.5), then it is permissible to write

$$y_r = A_r \cos(pt + \varepsilon) \quad . \quad . \quad . \quad . \quad . \quad (54.12)$$

in the last equation, and this leads, by the method of Art. 52, to a determinant of the 5th degree in p^2, corresponding with the determinantal expression (52.7) in the general theory.

In order to simplify the work of solving the resulting equation, attention may, again, be drawn to the fact that the present system is of the type illustrated in Fig. 92. This comparison at once shows, in the first place, that the structure under examination may be regarded as consisting of a loaded wire having elastic characteristics which are defined by the above-mentioned coefficients of stiffness, and, in the second, that Figs. 92(c), (d), (b) in turn illustrate the first three symmetrical and antisymmetrical modes of oscillation which the structure is liable to execute in the specified circumstances. By reason of this analogue we shall examine separately the symmetrical and antisymmetrical modes, and thus reduce the work to that of solving equations of only the 2nd and 3rd degree in p^2.

In the symmetrical mode corresponding to Fig. 92(c), for example, we have

$$y_1 = y_5, \; y_2 = y_4,$$

which lead to

$$\ddot{y}_1 = \ddot{y}_5, \; \ddot{y}_2 = \ddot{y}_4$$

in equations (54.11), hence

$$y_r = \frac{M}{g}\{(\tfrac{2}{3}\delta_{1r}+\tfrac{1}{6}\delta_{2r}+\tfrac{1}{6}\delta_{4r}+\tfrac{2}{3}\delta_{5r})\ddot{y}_1+(\tfrac{1}{6}\delta_{1r}+\tfrac{2}{3}\delta_{2r}+\tfrac{1}{3}\delta_{3r}+\tfrac{2}{3}\delta_{4r}+\tfrac{1}{6}\delta_{5r})\ddot{y}_2$$
$$+(\tfrac{1}{6}\delta_{2r}+\tfrac{2}{3}\delta_{3r}+\tfrac{1}{6}\delta_{4r})\ddot{y}_3\},$$

or

$$y_r = \frac{M}{6g}\{(4\delta_{1r}+\delta_{2r}+\delta_{4r}+4\delta_{5r})\ddot{y}_1+(\delta_{1r}+4\delta_{2r}+2\delta_{3r}+4\delta_{4r}+\delta_{5r})\ddot{y}_2$$
$$+(\delta_{2r}+4\delta_{3r}+\delta_{4r})\ddot{y}_3\}, \quad . \quad . \quad (54.13)$$

where $r = 1, 2, 3$. Since the displacements y_r are assumed to vary in accordance with equation (54.12), after making the substitutions for the various values of r in the last relation, and writing λ^2 for $\frac{6g}{Mp^2}$, we obtain

$$\left.\begin{aligned}
&(4\delta_{11}+\delta_{21}+\delta_{41}+4\delta_{51}-\lambda^2)A_1 + (\delta_{11}+4\delta_{21}+2\delta_{31}\\
&\qquad +4\delta_{41}+\delta_{51})A_2+(\delta_{21}+4\delta_{31}+\delta_{41})A_3 = 0,\\
&(4\delta_{12}+\delta_{22}+\delta_{42}+4\delta_{52})A_1 + (\delta_{12}+4\delta_{22}+2\delta_{32}+4\delta_{42}\\
&\qquad +\delta_{52}-\lambda^2)A_2+(\delta_{22}+4\delta_{32}+\delta_{42})A_3 = 0,\\
&(4\delta_{13}+\delta_{23}+\delta_{43}+4\delta_{53})A_1 + (\delta_{13}+4\delta_{23}+2\delta_{33}+4\delta_{43}\\
&\qquad +\delta_{53})A_2+(\delta_{23}+4\delta_{33}+\delta_{43}-\lambda^2)A_3 = 0
\end{aligned}\right\} \quad . \quad . \quad (54.14)$$

Finally, after eliminating the ratios $A_1 : A_2 : A_3$ between these equations, we arrive at the determinantal equation

$$\begin{vmatrix}
(4\delta_{11}+\delta_{21}+\delta_{41}+4\delta_{51}-\lambda^2), & (\delta_{11}+4\delta_{21}+2\delta_{31}+4\delta_{41}+\delta_{51}), & (\delta_{21}+4\delta_{31}+\delta_{41})\\
(4\delta_{12}+\delta_{22}+\delta_{42}+4\delta_{52}), & (\delta_{12}+4\delta_{22}+2\delta_{32}+4\delta_{42}+\delta_{52}-\lambda^2), & (\delta_{22}+4\delta_{32}+\delta_{42})\\
(4\delta_{13}+\delta_{23}+\delta_{43}+4\delta_{53}), & (\delta_{13}+4\delta_{23}+2\delta_{33}+4\delta_{43}+\delta_{53}), & (\delta_{23}+4\delta_{33}+\delta_{43}-\lambda^2)
\end{vmatrix} = 0.$$

Now the numerical part of the solution for the symmetrical modes consists in inserting the proper values of δ_{rs}, from Table 3, in this determinant, when it becomes

$$\begin{vmatrix}
53{\cdot}42 - \mu^2, & 79{\cdot}37, & 43{\cdot}38\\
79{\cdot}37, & 140{\cdot}1 - \mu^2, & 79{\cdot}37\\
86{\cdot}76, & 158{\cdot}8, & 96{\cdot}80 - \mu^2
\end{vmatrix} = 0,$$

where $\mu^2 = 1{,}000\lambda^2 = \dfrac{6{,}000g}{Mp^2}$.

Hence, on expanding in terms of μ^2,

$$(\mu^2)^3 - 290{\cdot}3(\mu^2)^2 + 3{,}557\mu^2 - 944{\cdot}9 = 0,$$

nearly. The solution of this cubic in μ^2 gives the approximate values $277{\cdot}0$, $12{\cdot}56$ and $0{\cdot}2715$ for the roots. With the above units and notation, the frequency ν is given by the relation

$$\nu = \frac{p}{2\pi},$$

where $p = \dfrac{1}{\mu}\sqrt{\dfrac{6{,}000g}{M}}$, $M = 4{\cdot}75$ tons, $g = 32{\cdot}2 \times 12$.

A successive substitution of the roots of μ^2 in the expression for ν thus shows that in the symmetrical modes the frequencies in cycles per second are :

$$\nu_{277} = 6{\cdot}670,$$
$$\nu_{12.56} = 31{\cdot}30,$$
$$\nu_{0.2715} = 213{\cdot}0.$$

Turning now to the antisymmetrical modes, in which the relations between the displacements are

$$y_1 = -y_5,\ y_2 = -y_4,\ y_3 = 0,$$

we can write

$$\ddot{y}_1 = -\ddot{y}_5,\ \ddot{y}_2 = -\ddot{y}_4,\ \ddot{y}_3 = 0$$

in equation (54.11), and in this way express the vertical displacement of the joint J_r in the form

$$y_r = \frac{M}{g}\{(\tfrac{2}{3}\delta_{1r} + \tfrac{1}{6}\delta_{2r} - \tfrac{1}{6}\delta_{4r} - \tfrac{2}{3}\delta_{5r})\ddot{y}_1 + (\tfrac{1}{6}\delta_{1r} + \tfrac{2}{3}\delta_{2r} - \tfrac{2}{3}\delta_{4r} - \tfrac{1}{6}\delta_{5r})\ddot{y}_2\}, \quad . \quad . \quad (54.15)$$

where $r = 1, 2$. It will be understood that the stated end conditions are to be satisfied in the work of finding this equation. On making the substitutions (54.12) and eliminating the ratios $A_1, : A_2$ between the resulting expressions, in a manner similar to that already explained, it will be seen that

$$\begin{vmatrix} (4\delta_{11}+\delta_{21}-\delta_{41}-4\delta_{51}-\lambda^2), & (\delta_{11}+4\delta_{21}-4\delta_{41}-\delta_{51}) \\ (4\delta_{12}+\delta_{22}-\delta_{42}-4\delta_{52}), & (\delta_{12}+4\delta_{22}-4\delta_{42}-\delta_{52}-\lambda^2) \end{vmatrix} = 0,$$

where, as before, $\lambda^2 = \dfrac{6g}{Mp^2}$.

When the numerical values of δ_{rs} are introduced into this equation, we obtain, with μ^2 written for $1{,}000\lambda^2$,

$$\begin{vmatrix} 17{\cdot}96 - \mu^2, & 13{\cdot}34 \\ 13{\cdot}33, & 17{\cdot}92 - \mu^2 \end{vmatrix} = 0,$$

so that, on expanding in terms of μ^2,

$$(\mu^2)^2 - 35{\cdot}88\ \mu^2 + 144{\cdot}2 = 0,$$

approximately. If the roots of this quadratic in μ^2, namely 31·30 and 4·630, are inserted in the relation

$$\nu = \frac{p}{2\pi} = \frac{1}{2\pi\mu}\sqrt{\frac{6{,}000g}{M}},$$

it is seen that in the antisymmetrical modes the frequencies in cycles per second are :—

$$\nu_{31\cdot30} = 19{\cdot}89$$
$$\nu_{4\cdot630} = 51{\cdot}70$$

Hence, when arranged in ascending order of magnitude, the frequencies of oscillation, in cycles per second, for the specified structure are :

6·670, 19·89, 31·30, 51·70, 213·0.

It is of some interest to notice here that for many practical purposes only the first two of these frequencies would be of any account, in which case it is necessary to find only the greatest values

of the roots in the above equations for the symmetrical modes on the one hand, and the antisymmetrical on the other.

Ex. 2. Consider the structural system specified in the foregoing problem, but with the same total load of 28·5 tons arranged so that a *concentrated* load of 4·75 tons is applied at each of the joints J_1, J_2, J_3, J_4, J_5, and one of 2·375 tons at each of the joints J_0, J_0' in Fig. 94.

It is at the outset evident that the loads here ascribed to the joints J_0, J_0' may be omitted from our solution, by reason of the fact that rigid abutments are involved in the system.

In order to explain the manner by which the coefficients α_{rs}, β_{rs}, γ_{rs}, δ_{rs} enter into the general problem of concentrated loads, we shall at first include all these terms in the analytical part of the work.

The horizontal and vertical components of the *total* displacement for the rth joint on the structural system are, to the second order of small quantities, obtained, as previously, by effecting the summations implied in equations (54.6), namely

$$x_r = -\frac{1}{g}\sum_{s=1}^{n} M_s(\alpha_{rs}\ddot{x}_s + \gamma_{rs}\ddot{y}_s),$$

$$y_r = -\frac{1}{g}\sum_{s=1}^{n} M_s(\beta_{rs}\ddot{x}_s + \delta_{rs}\ddot{y}_s),$$

where M_s relates to the load associated with the sth joint.

If we make the usual assumption that in the oscillatory motion the co-ordinates $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ vary as e^{ipt}, implying in terms of real quantities that

$$x_r = A_r \cos(pt + \varepsilon),$$
$$y_r = B_r \cos(pt + \varepsilon),$$

on inserting these relations in the above expressions for x_r, y_r and cancelling the common factor $\cos(pt + \varepsilon)$ in the operation of summing over the n joints, it is seen that

$$A_r = \frac{1}{g}\sum_{s=1}^{n} M_s(\alpha_{rs}A_s p^2 + \gamma_{rs}B_s p^2),$$

$$B_r = \frac{1}{g}\sum_{s=1}^{n} M_s(\beta_{rs}A_s p^2 + \delta_{rs}B_s p^2).$$

When $s = r$ for the typical member shown in Fig. 93, however, a distinction must be drawn between the 'fixed' suffix r and the 'free' suffix s which appear in our general treatment. Hence, if we exclude the values $r = s$ in the summations indicated by the

symbol Σ', then after collecting the A_r- and B_r-terms the last equations can be rearranged in the form

$$A_r = \frac{M_r}{g}\alpha_{rr}A_r p^2 + \frac{1}{g}\sideset{}{'}\sum_{s=1}^{n} M_s\alpha_{rs}A_s p^2 + \frac{1}{g}\sum_{s=1}^{n} M_s\gamma_{rs}B_s p^2,$$

$$B_r = \frac{M_r}{g}\delta_{rr}B_r p^2 + \frac{1}{g}\sum_{s=1}^{n} M_s\beta_{rs}A_s p^2 + \frac{1}{g}\sideset{}{'}\sum_{s=1}^{n} M_s\delta_{rs}B_s p^2,$$

i.e. $$A_r\left(1 - \frac{M_r}{g}\alpha_{rr}p^2\right) = \frac{1}{g}\sideset{}{'}\sum_{s=1}^{n} M_s\alpha_{rs}A_s p^2 + \frac{1}{g}\sum_{s=1}^{n} M_s\gamma_{rs}B_s p^2,$$

$$B_r\left(1 - \frac{M_r}{g}\delta_{rr}p^2\right) = \frac{1}{g}\sum_{s=1}^{n} M_s\beta_{rs}A_s p^2 + \frac{1}{g}\sideset{}{'}\sum_{s=1}^{n} M_s\delta_{rs}B_s p^2,$$

or $$\left.\begin{aligned} \sideset{}{'}\sum_{s=1}^{n} M_s\alpha_{rs}A_s + \sum_{s=1}^{n} M_s\gamma_{rs}B_s + A_r\left(M_r\alpha_{rr} - \frac{g}{p^2}\right) &= 0,\\ \sum_{s=1}^{n} M_s\beta_{rs}A_s + \sideset{}{'}\sum_{s=1}^{n} M_s\delta_{rs}B_s + B_r\left(M_r\delta_{rr} - \frac{g}{p^2}\right) &= 0.\end{aligned}\right\} \quad (54.16)$$

These equations for the joint J_r supply, with the n values assigned to r, the $2n$ expressions necessary for eliminating the ratios $A_1 : A_2 : \ldots : A_n : B_1 : B_2 : \ldots : B_n$. The result, with λ^2 written for $\frac{g}{p^2}$, takes the form of a symmetrical determinant of degree $2n$ in λ^2, namely

$$\begin{vmatrix}
M_1\alpha_{11}-\lambda^2, & M_2\alpha_{12} & ,\ldots, & M_n\alpha_{1n} & , M_1\gamma_{11} & , M_2\gamma_{12} & ,\ldots, & M_n\gamma_{1n}\\
M_1\alpha_{21} & , M_2\alpha_{22}-\lambda^2, & \ldots, & M_n\alpha_{2n} & , M_1\gamma_{21} & , M_2\gamma_{22} & ,\ldots, & M_n\gamma_{2n}\\
--- & , --- & ,\ldots, & --- & , --- & , --- & ,\ldots, & ---\\
--- & , --- & ,\ldots, & --- & , --- & , --- & ,\ldots, & ---\\
M_1\alpha_{n1} & , M_2\alpha_{n2} & ,\ldots, & M_n\alpha_{nn}-\lambda^2, & M_1\gamma_{n1} & , M_2\gamma_{n2} & ,\ldots, & M_n\gamma_{nn}\\
M_1\beta_{11} & , M_2\beta_{12} & ,\ldots, & M_n\beta_{1n} & , M_1\delta_{11}-\lambda^2, & M_2\delta_{12} & ,\ldots, & M_n\delta_{1n}\\
M_1\beta_{21} & , M_2\beta_{22} & ,\ldots, & M_n\beta_{2n} & , M_1\delta_{21} & , M_2\delta_{22}-\lambda^2, & \ldots, & M_n\delta_{2n}\\
--- & , --- & ,\ldots, & --- & , --- & , --- & ,\ldots, & ---\\
--- & , --- & ,\ldots, & --- & , --- & , --- & ,\ldots, & ---\\
M_1\beta_{n1} & , M_2\beta_{n2} & ,\ldots, & M_n\beta_{nn} & , M_1\delta_{n1} & , M_2\delta_{n2} & ,\ldots, & M_n\delta_{nn}-\lambda^2
\end{vmatrix} = 0 \quad (54.17)$$

This obviously refers to oscillations in two dimensions, and corresponds to equation (52.7) in the general theory.

In view of the previous example we see without much difficulty that lengthy calculations might well be involved in the work of solving the last equation even when n is not large in value. But the foregoing analysis shows that the degree of the determinantal expression will be halved if the displacement parallel to one of the axes is negligibly small compared with that parallel to the other; also, if only k of the joints be loaded, where $k < n$, it is readily seen that the final equation reduces to one of degree $2k$ in λ^2. Moreover, in problems involving a symmetrical arrangement of the struc-

tural and loading systems, as is the case here, we have demonstrated that the equations to be solved are simplified if the symmetrical and antisymmetrical modes of vibration are treated separately.

Considering these points in order with reference to the specified system, in the above determinant we can omit the terms which contain α_{rs} and β_{rs}, because there are no X-forces present. Further reduction follows if, for reasons already explained, we neglect the γ_{rs}-coefficients in comparison with the corresponding δ_{rs}-terms. To this degree of approximation equation (54.17) reduces to

$$\begin{vmatrix} M_1\delta_{11}-\lambda^2, & M_2\delta_{12} & , M_3\delta_{13} & ,\ldots, & M_n\delta_{1n} \\ M_1\delta_{21} & , M_2\delta_{22}-\lambda^2, & M_3\delta_{23} & ,\ldots, & M_n\delta_{2n} \\ M_1\delta_{31} & , M_2\delta_{32} & , M_3\delta_{33}-\lambda^2, & \ldots, & M_n\delta_{3n} \\ --- & , --- & , --- & ,\ldots, & --- \\ --- & , --- & , --- & ,\ldots, & --- \\ --- & , --- & , --- & ,\ldots, & --- \\ M_1\delta_{n1} & , M_2\delta_{n2} & , M_3\delta_{n3} & ,\ldots, & M_n\delta_{nn}-\lambda^2 \end{vmatrix} = 0 \quad . \quad (54.18)$$

In the problem stated above, $n = 5$, and $M_1 = M_2 = M_3 = M_4 = M_5 = 4{\cdot}75$ tons, which may be denoted briefly by M. On making these substitutions in the last expression, and writing μ^2 for $\dfrac{\lambda^2}{M}$, we thus obtain

$$\begin{vmatrix} \delta_{11}-\mu^2, & \delta_{12} & , \delta_{13} & , \delta_{14} & , \delta_{15} \\ \delta_{21} & , \delta_{22}-\mu^2, & \delta_{23} & , \delta_{24} & , \delta_{25} \\ \delta_{31} & , \delta_{32} & , \delta_{33}-\mu^2, & \delta_{34} & , \delta_{35} \\ \delta_{41} & , \delta_{42} & , \delta_{43} & , \delta_{44}-\mu^2, & \delta_{45} \\ \delta_{51} & , \delta_{52} & , \delta_{53} & , \delta_{54} & , \delta_{55}-\mu^2 \end{vmatrix} = 0 \quad . \quad (54.19)$$

If we now consider the symmetrical modes of oscillation, given, as previously, by writing

$$\ddot{y}_1 = \ddot{y}_5,\ \ddot{y}_2 = \ddot{y}_4$$

in the second of equations (54.6), it is readily shown that in consequence of these relations equation (54.19) becomes

$$\begin{vmatrix} (\delta_{11}+\delta_{15})-\mu^2, & (\delta_{12}+\delta_{14}) & , \delta_{13} \\ (\delta_{21}+\delta_{25}) & , (\delta_{22}+\delta_{25})-\mu^2, & \delta_{23} \\ (\delta_{31}+\delta_{35}) & , (\delta_{32}+\delta_{34}) & , \delta_{33}-\mu^2 \end{vmatrix} = 0.$$

Similarly, the antisymmetrical modes are defined by

$$\ddot{y}_1 = -\ddot{y}_5,\ \ddot{y}_2 = -\ddot{y}_4,\ \ddot{y}_3 = 0,$$

by reason of which equation (54.19) reduces to

$$\begin{vmatrix} (\delta_{11}-\delta_{15})-\mu^2, & (\delta_{12}-\delta_{14}) \\ (\delta_{21}-\delta_{25}), & (\delta_{22}-\delta_{24})-\mu^2 \end{vmatrix} = 0.$$

In this way, reverting to Table 3 for the purpose of assigning the

numerical values to the coefficients δ_{rs} in the last two expressions, we conclude that

$$\begin{vmatrix} 9{\cdot}945 - \rho^2, & 13{\cdot}64, & 7{\cdot}435 \\ 13{\cdot}64, & 24{\cdot}81 - \rho^2, & 13{\cdot}64 \\ 14{\cdot}87, & 27{\cdot}28, & 17{\cdot}38 - \rho^2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 3{\cdot}901 - \rho^2, & 2{\cdot}359 \\ 2{\cdot}359, & 3{\cdot}901 - \rho^2 \end{vmatrix} = 0,$$

where $\rho^2 = 1{,}000\mu^2$. Hence, on expanding,

$$(\rho^2)^3 - 52{\cdot}16(\rho^2)^2 + 182{\cdot}4\rho^2 - 135{\cdot}1 = 0,$$
$$(\rho^2)^2 - 7{\cdot}802\rho^2 + 9{\cdot}645 = 0.$$

The roots of these equations in ρ^2 will be found, when arranged in decreasing order of magnitude, to be

$$48{\cdot}40,\ 6{\cdot}250,\ 2{\cdot}520,\ 1{\cdot}540,\ 1{\cdot}190,$$

approximately. If these roots be inserted in the relation for the frequency of vibration ν, namely

$$\nu = \frac{p}{2\pi} = \frac{1}{2\pi\rho}\sqrt{\frac{1{,}000g}{M}},$$

with $M = 4{\cdot}75$ tons, it thus appears that

$$\begin{aligned} \nu_{48\cdot40} &= 6{\cdot}54, \\ \nu_{6\cdot250} &= 18{\cdot}16, \\ \nu_{2\cdot520} &= 28{\cdot}62, \\ \nu_{1\cdot540} &= 36{\cdot}68, \\ \nu_{1\cdot190} &= 41{\cdot}60 \end{aligned}$$

are the corresponding frequencies in cycles per second for the normal modes of oscillation of the given system.

To calculate the related deflections of the rth joint when the structure is executing small vibrations with a given period $\frac{2\pi}{p}$, in the general case it is merely necessary to substitute the particular value of p in equations (54.16), and thus evaluate the corresponding A_r- and B_r-coefficients. The deflections are then given by inserting the known values of A_r, B_r in the equations

$$x_r = A_r \cos(pt + \varepsilon),$$
$$y_r = B_r \cos(pt + \varepsilon).$$

In this manner it is possible, with the aid of the theory of structures, to determine the inertia-forces for the several members of the system under consideration.

At this point it is instructive to compare the frequencies of oscillation obtained in the above examples, since they refer to different distributions of the same total load over similar structures. The comparison shows that for like modes the frequency for the

distributed loading is invariably greater than that for the concentrated loading, and notably so in the higher modes. The reason lies in the fact that under these conditions, with a given *total* load, constraints must be imposed on the system with concentrated masses in order to produce the same *deflection* when the load is evenly distributed. This is a further exemplification of a matter which will be treated more fully in Art. 55, where it is demonstrated that the effect of an additional constraint is to raise the frequency. Hence, it is not generally practicable to assume that a set of concentrated masses is dynamically equivalent to the same total load distributed evenly over a specified structure. It is in this way that the arrangement of a given total load influences the natural period of vibration of a particular system, and the result suggests a useful method of modifying this period, by effecting suitable changes in the distribution of a specified load.

55. Stationary Property of the Normal Modes. Owing to the analytical difficulties which attend the determination of the free modes of oscillation of structures in general, and the continuous type formed by riveted joints in particular, we proceed to a method which greatly facilitates the work of finding the gravest frequency. Remembering that the corresponding mode of vibration is by far the most important in many engineering problems, a practical significance is attached to the matter.

Limiting our considerations to the case of *complete stability* in normal modes, let frictionless constraints be imposed on the system referred to in Art. 53 until it retains only one degree of freedom. If, then, the normal co-ordinate ϕ specify the configuration, and $\mu_1, \mu_2, \ldots, \mu_n$ represent constants, the relations between the two sets of co-ordinates, corresponding to the original and modified systems, may be written in the form

$$\theta_1 = \mu_1\phi,\ \theta_2 = \mu_2\phi,\ \ldots,\ \theta_n = \mu_n\phi \quad . \quad . \quad . \quad (55.1)$$

With these relations inserted in equations (53.3), the motion of the modified system is given by introducing the expressions

$$\left.\begin{aligned} 2T &= (\alpha_{11}\mu_1^2 + \alpha_{22}\mu_2^2 + \ldots + \alpha_{nn}\mu_n^2)\dot{\phi}^2, \\ 2V &= (\gamma_{11}\mu_1^2 + \gamma_{22}\mu_2^2 + \ldots + \gamma_{nn}\mu_n^2)\phi^2 \end{aligned}\right\} \quad . \quad (55.2)$$

into the formula

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) + \frac{\partial V}{\partial \phi} = 0.$$

As the resulting equation has a solution

$$\phi = A \cos (pt + \varepsilon),$$

where
$$p^2 = \frac{\gamma_{11}\mu_1^2 + \gamma_{22}\mu_2^2 + \ldots + \gamma_{nn}\mu_n^2}{\alpha_{11}\mu_1^2 + \alpha_{22}\mu_2^2 + \ldots + \alpha_{nn}\mu_n^2}, \quad . \quad . \quad (55.3)$$

it is seen that $\frac{p}{2\pi}$ determines the frequency of the modified system.

Here p^2 is intermediate in value between the greatest and least of the quantities $\frac{\gamma_{11}}{\alpha_{11}}, \frac{\gamma_{22}}{\alpha_{22}}, \ldots, \frac{\gamma_{nn}}{\alpha_{nn}}$ which are proportional to the squares of the frequencies in the several normal modes. If the modified mode differs but slightly from a normal mode, implying that $\mu_2, \mu_3, \ldots, \mu_n$ are small compared with μ_1, our result shows further that the effect of the constraints on the frequency is of the second order of small quantities. To this degree of approximation the frequency in a normal mode is, therefore, stationary for small variations in the values of the ratios $A_1 : A_2 : \ldots : A_n$.

This stationary property yields an extremely valuable method of estimating the gravest frequency in instances where the normal modes cannot be accurately determined, by assuming at the outset an approximate type of vibration which does not greatly differ from the actual mode. Since we must usually impose constraints on the actual system in order to obtain the assumed shape or configuration, the frequency derived in this manner is commonly somewhat higher than the value given by more exact analysis. The theorem on which this is based is due to Lord Rayleigh,[1] and it has been elaborated by W. Ritz.[2]

To investigate the effect of a partial constraint, suppose that its application results in the disappearance of one of the co-ordinates, say q_1, from equations (52.1) and (52.2). In the modes thus modified the frequencies are determined by the roots of the equation obtained by deleting the first row and column of (52.7). If Δ_n and Δ_{n-1} denote respectively the original and reduced determinants, it can be shown[3] that the roots of Δ_{n-1} separate those of Δ_n, hence the lowest frequency is raised. In a like manner, if $q_2 = 0$ results from an additional constraint, the frequencies are then determined by the roots of the above-mentioned determinant with its first two rows and columns omitted. It appears that every additional constraint raises the gravest frequency, and that the introduction of fresh constraints increases the 'stiffness' defined by $\gamma_{11}, \gamma_{22}, \ldots$ in equation (55.3) when fixed values are associated with the coefficients of inertia $\alpha_{11}, \alpha_{22}, \ldots$

Ex. Evaluate by Rayleigh's method the longest period in a normal mode of free vibration for a slender wire of length L, and weight ρ per unit length, when subjected to a tension P. Any frictional forces which act on the wire may be neglected.

Let the x-axis be the equilibrium-position of the wire, having

[1] *Theory of Sound*, Vol. I, Chap. IV.

[2] *Gesammelte Werke*, pages 192 and 265 (Paris, 1911).

[3] H. Lamb, *Higher Mechanics*, page 220, second edition.

its ends at $x = 0$, $x = L$, and y be the deflection at the point x. In small oscillations, when P may be treated as a constant quantity, the expressions for the kinetic and potential energies T and V are

$$2T = \frac{\rho}{g}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx,$$

$$2V = P\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx,$$

as is shown in Art. 68.

Now assume the shape of the disturbed wire to be

$$y = Ax(L - x)\cos pt,$$

where A is an arbitrary constant. It is permissible to suppose also that the maximum values of T and V are equal, because the system is of the conservative type implied in equation (38.5).

With the assumed relation for y we have

$$\frac{\partial y}{\partial t} = -Apx(L - x)\sin pt,$$

$$\frac{\partial y}{\partial x} = A(L - 2x)\cos pt,$$

so that

$$\left(\frac{\partial y}{\partial t}\right)^2 = A^2p^2x^2(L - x)^2\sin^2 pt,$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = A^2(L - 2x)^2\cos^2 pt$$

in the above expressions for T and V. The maximum values of the trigonometrical terms being unity, it follows that

$$2T_{\text{max.}} = \frac{\rho A^2p^2}{g}\int_0^L x^2(L - x)^2\,dx$$

$$= \frac{\rho A^2p^2L^5}{30g},$$

$$2V_{\text{max.}} = A^2P\int_0^L (L - 2x)^2\,dx$$

$$= \frac{A^2PL^3}{3}.$$

Hence, on equating,

$$\frac{\rho p^2L^2}{30g} = \frac{P}{3},$$

whence the period $\frac{2\pi}{p}$ is

$$1{\cdot}987L\sqrt{\frac{\rho}{Pg}},$$

which is sensibly in agreement with the value given by the more exact treatment of Art. 68.

56. In applications of the approximate method it is advisable to consider each case on its merits, and to infer from the nature of the problem what are the most suitable assumptions to make. The following examples will serve as an explanation of the general procedure.

Ex. 1. Estimate the natural frequency of small vibrations in a normal mode for the *pin-jointed* frame examined in Ex. 2 of Art. 54, *with the mass of the structure itself taken into account,* when the weight of the material is 489 lb. per cubic foot. For convenience of reference it may be repeated that: the span is 55 ft.; the sectional area of each boom-member is 9 sq. in., and that of each web-member 5·3 sq. in.; the direct modulus of elasticity is 12,000 tons per square inch.

In this example of *concentrated loading* let

M_u = weight of a member on the upper boom,
M = weight of a web-member,
M_l = weight of a member on the lower boom *plus* the load imposed on it,
w = load applied at each of the five joints in Fig. 96.

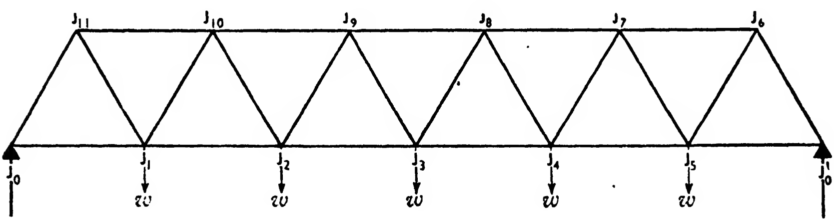


FIG. 96.

The kinetic energy T_{rs} of any member having an effective weight M_{rs}, and associated with the joints J_r, J_s, is defined by equation (54.1), namely

$$2T_{rs} = \frac{M_{rs}}{g}\left[\left(\frac{\dot{x}_r + \dot{x}_s}{2}\right)^2 + \left(\frac{\dot{y}_r + \dot{y}_s}{2}\right)^2 + \tfrac{1}{12}\{(\dot{y}_s - \dot{y}_r)\cos\theta - (\dot{x}_s - \dot{x}_r)\sin\theta\}^2\right]$$

But in the prescribed system there are no X-forces, and it has been remarked that the numerical values of the coefficients γ_{rs} are of the second order of small quantities. To this degree of approximation, in which a practical significance is therefore attached only to the

terms containing the coefficients δ_{rs}, we have, on writing $\theta = 0$ in the relation for T_{rs},

$$2T_{rs} = \frac{M_{rs}}{3g}(\dot{y}_r^2 + \dot{y}_s^2 + \dot{y}_r\dot{y}_s) \quad . \quad . \quad . \quad . \quad (56.1)$$

for any boom-member; and, with $\theta = 60$ deg.,

$$2T_{rs} = \frac{M_{rs}}{48g}(13\dot{y}_r^2 + 13\dot{y}_s^2 + 22\dot{y}_r\dot{y}_s) \quad . \quad (56.2)$$

for any web-member. Bearing in mind the previous treatment of the suffixes s, r in equation (54.9) and the end conditions stated in Ex. 1of Art. 54, we can similarly express the kinetic energy T of the complete system in the form

$$\begin{aligned} 2T = {} & \frac{M_u}{3g}\left\{\dot{y}_6^2 + \dot{y}_{11}^2 + 2\sum_{r=7}^{10}\dot{y}_r^2 + \sum_{r=6}^{10}\dot{y}_r\dot{y}_{r+1}\right\} \\ & + \frac{M}{48g}\left\{26\sum_{r=1}^{11}\dot{y}_r^2 + 22\sum_{r=1}^{5}\dot{y}_r(\dot{y}_{r+5} + \dot{y}_{r+6})\right\} \\ & + \frac{M}{3g}\left\{2\sum_{r=1}^{5}\dot{y}_r^2 + \sum_{r=1}^{4}\dot{y}_r\dot{y}_{r+1}\right\} \quad , \quad . \quad . \quad . \quad . \quad . \quad (56.3) \end{aligned}$$

being the sum obtained by assigning the proper values to the masses as well as the suffixes r, s in equations (56.1) and (56.2).

Were we employing a method other than that of Rayleigh, we should next derive an expression for the potential energy. To avoid the considerable amount of work thus involved, we might here suppose the mode of oscillation to be such that the maximum value of the actual displacement y is proportional to the 'static' deflection under the complete load, including the weight of the structure. Seeing that the mass of the present frame, being approximately 10 per cent. that of the extraneous load, causes a deflection which is only about 1 per cent. of the total, we shall, however, assume the maximum value of y to be proportional to the displacement induced by the static effect of the applied load alone.

If, on this hypothesis, the deflection at the joint J^r is σ times that due to the static load applied at the joint J_s, where σ represents a constant for the given frame, a simple substitution in equation (51.2) yields

$$(y_r)_{\text{max.}} = \sigma\sum_{s=1}^{n}\delta_{rs} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56.4)$$

for the maximum value of the displacement caused at the rth joint by a *unit* load at the sth joint.

Making our usual assumption that

$$y_r = (y_r)_{\text{max.}}\cos(pt + \epsilon),$$

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where $\frac{2\pi}{p}$ denotes the period in a normal mode, the velocity is accordingly

$$\dot{y}_r = -p(y_r)_{\text{max.}} \sin(pt + \varepsilon).$$

Hence, with maximum values ascribed to the variables,

$$(\dot{y}_r)_{\text{max.}} = -p(y_r)_{\text{max.}}$$

$$= -\sigma p \sum_{s=1}^{n} \delta_{rs}, \quad . \quad . \quad . \quad . \quad . \quad (56.5)$$

by equation (56.4), whence, after squaring both sides,

$$(\dot{y}_r^2)_{\text{max.}} = \sigma^2 p^2 \left(\sum_{s=1}^{n} \delta_{rs} \right)^2 \quad . \quad . \quad . \quad (56.6)$$

We are now in a position to write down the corresponding relations for the kinetic and potential energies, with the ultimate aim of equating the maximum values of these quantities in accordance with the supposition that the effect of friction on the motion may be neglected.

Thus, as a result of substituting in equation (56.3) from (56.5) and (56.6) and assigning the proper values to the limits of summation, it will be found that the maximum value, $T_{\text{max.}}$, of the kinetic energy is given by an expression containing three members, the first of which may be exhibited symbolically in the form

$$\frac{\sigma^2 p^2 M_u}{3g} \left\{ (\delta_{66})^2 + (\delta_{11})^2 + 2 \sum_{r=7}^{10} \left(\sum_{s=7}^{10} \delta_{rs} \right)^2 + \sum_{r=6}^{10} \sum_{s=6}^{10} \delta_{rs} \times \sum_{r=7}^{11} \sum_{s=7}^{11} \delta_{rs} \right\} \quad . \quad . \quad . \quad (56.7)$$

The second and third members, referring in succession to M and M_l, are of the same type. It is to be noticed that in the process of expanding these expressions the double summations must be taken in the order indicated.

Turning next to the maximum value $V_{\text{max.}}$ of the potential energy, according to equation (48.4) it is equal to the work done by the applied (static) loads, so that

$$2V_{\text{max.}} = \sigma \sum_{r=1}^{5} (y_r)_{\text{max.}}$$

whence, by equation (56.4),

$$2V_{\text{max.}} = \sigma^2 \sum_{r=1}^{5} \sum_{s=1}^{5} \delta_{rs} \quad . \quad . \quad . \quad . \quad . \quad (56.8)$$

Here our notation indicates that the values $r = 1, 2, 3, 4, 5 = s$ are to be given in turn to the coefficients δ_{rs}.

Reverting to the stated dimensions of the structure, we have, in terms of ton-inch units,

$$\frac{M_u}{g} = \frac{0\cdot1496}{32\cdot2 \times 12},$$

$$\frac{M}{g} = \frac{0\cdot0897}{32\cdot2 \times 12},$$

$$\frac{M_l}{g} = \frac{0\cdot1496 + w}{32\cdot2 \times 12}.$$

Taking account of the numerical values of δ_{rs} (Table 3), we find

$$2T_{\text{max.}} = \frac{\sigma^2 p^2}{10^3}(4\cdot130 + 10\cdot61w),$$

$$2V_{\text{max.}} = \frac{89{,}590\ \sigma^2}{10^3}.$$

The final operation in the present method is that of equating these values, whence the frequency ν of the gravest mode is determined by

$$\nu^2 = \frac{p^2}{4\pi^2}$$

$$= \frac{1}{4\pi^2}\cdot\frac{89{,}590}{4\cdot130 + 10\cdot61w}.$$

The fundamental frequency of vibration, in cycles per second, for the *unloaded* structure, corresponding to $w = 0$, is accordingly 23·44; and that for the *loaded* frame, with $w = 4\cdot75$ tons, is 6·45, which agrees sensibly with the figure of 6·54 given by the more elaborate analysis in Ex. 2 of Art. 54.

Had we decided, at the outset of the foregoing problem, to omit the inertia of the frame, a solution might have been arrived at by a correspondingly shorter way of approach. As the procedure is of general application, it merits due consideration at this stage of the treatment.

Ex. 2. Find, *with the weight of the frame neglected*, the gravest frequency in a normal mode of oscillation for the system considered in the previous example.

Assuming, as before, that the actual deflection is σ times that due to the static loads concerned, it follows from equation (54.7) that a load M_s applied at the joint J_s will produce at J_r component deflections having the maximum values

$$\left.\begin{aligned}(x_r)_{\text{max.}} &= \sigma\sum_{s=1}^{n} M_s\gamma_{rs},\\ (y_r)_{\text{max.}} &= \sigma\sum_{s=1}^{n} M_s\delta_{rs},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (56.9)$$

since the X-forces are absent.

If, as we shall further suppose, x and y vary as e^{ipt}, then

$$(x_r) = (x_r)_{\max.} \cos (pt + \varepsilon),$$
$$(y_r) = (y_r)_{\max.} \cos (pt + \varepsilon)$$

in equations (56·9), by reason of which

$$\left.\begin{aligned}(\dot{x}_r^2)_{\max.} &= p^2(x^2_r)_{\max.},\\ (\dot{y}_r^2)_{\max.} &= p^2(y^2_r)_{\max.},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (56.10)$$

as in the preceding examples.

It has already been observed that in a normal mode of vibration the maximum kinetic energy, $T_{\max.}$, occurs when the frame is passing through its equilibrium-position, that is when the generalized co-ordinate $q_r = 0$. From definition we have also

$$2T_{\max.} = \frac{1}{g}\sum_{r=1}^{n} M_r(\dot{x}^2_r + \dot{y}^2_r)_{\max.}$$

In view of these facts and equations (56.9) and (56.10), it is readily seen that

$$2T_{\max.} = \frac{\sigma^2 p^2}{g}\sum_{r=1}^{n} M_r\left\{\left(\sum_{s=1}^{n} M_s\gamma_{rs}\right)^2 + \left(\sum_{s=1}^{n} M_s\delta_{rs}\right)^2\right\} . \quad (56.11)$$

Moreover, since the present method implies that the maximum value, $V_{\max.}$, of the potential energy is equivalent to the work which would be done if the forces were applied slowly,

$$\begin{aligned}2V_{\max.} &= \sigma\sum_{r=1}^{n} M_r\{(x_r)_{\max.} + (y_r)_{\max.}\}\\ &= \sigma^2\sum_{r=1}^{n} M_r\left\{\sum_{s=1}^{n} M_s\gamma_{rs} + \sum_{s=1}^{n} M_s\delta_{rs}\right\}, \quad . \quad (56.12)\end{aligned}$$

from equations (56.9).

In the final process of equating $T_{\max.}$ and $V_{\max.}$ we shall treat terms in γ_{rs} as negligibly small compared with the other quantities, in accordance with foregoing conclusions. This simplification leads to

$$\frac{\sigma^2 p^2}{g}\sum_{r=1}^{n} M_r\left(\sum_{s=1}^{n} M_s\delta_{rs}\right)^2 = \sigma^2\sum_{r=1}^{n} M_r\sum_{s=1}^{n} M_s\delta_{rs},$$

or

$$p^2 = \frac{g\sum_{r=1}^{n} M_r\sum_{s=1}^{n} M_s\delta_{rs}}{\sum_{r=1}^{n} M_r\left(\sum_{s=1}^{n} M_s\delta_{rs}\right)^2} \quad . \quad . \quad . \quad . \quad (56.13)$$

The numerical data and Table 3, when expressed in ton-inch-

second units, show that the quantities in the last equation amount to

$$M_r = 4{\cdot}75 \text{ when } r = 1,\ 2,\ 3,\ 4,\ 5 = s,$$
$$M_r = 0 \quad \text{ when } r = 6,\ 7,\ 8,\ 9,\ 10,\ 11 = s,$$
$$\sum_{r=1}^{5}\sum_{s=1}^{5}\delta_{rs} = 232{\cdot}1 \times 10^{-3},$$
$$\sum_{r=1}^{5}\left(\sum_{s=1}^{5}\delta_{rs}\right)^2 = 11{\cdot}53 \times 10^{-3}.$$

Therefore
$$\nu = \frac{p}{2\pi}$$
$$= \frac{1}{2\pi}\sqrt{\frac{232{\cdot}1 \times 32{\cdot}2 \times 12}{4{\cdot}75 \times 11{\cdot}53}}$$
$$= 6{\cdot}56$$

cycles per second is the gravest frequency; and it will be noticed that this compares favourably with the figure of 6·54 in Ex. 2 of Art. 54.

The significance of the approximate method is best exemplified by applications to the 'continuous' type of structure involving joints which are more or less rigid, considering that it would otherwise be difficult to determine the constraints. This point may be illustrated in connection with a problem in the design of buildings for regions subject to earthquakes.

Ex. 3. Investigate, with special reference to the *shearing forces* associated with earthquakes, the fundamental frequency in a normal mode of small oscillations for the 3-storey building represented in Fig. 97, on the supposition that the foundations and joints are perfectly rigid. As indicated in the figure, l is the height of each storey, and M the weight of each set of floor-beams together with the imposed load; also w is the weight per unit distance, measured *vertically*, of the external walls taken as a whole. It is permissible to neglect the effect of the roof and the internal walls, as well as that of the frictional forces which usually operate in these circumstances.

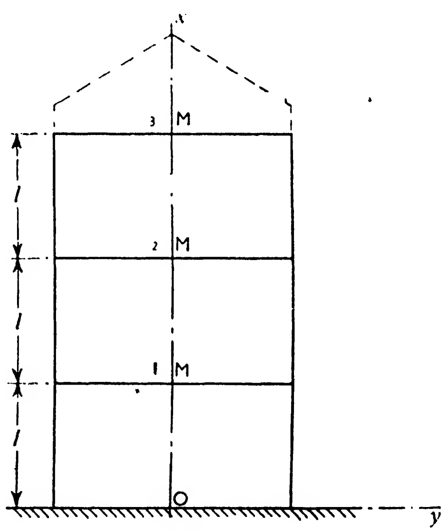


FIG. 97.

Imagine, for simplicity, the structure to be rotated through a right-angle, so that the y-axis becomes vertical, and let the x-axis define the equilibrium-position. Then the system is, to the first approximation, equivalent to a cantilever subjected to a uniformly distributed load of w per unit length together with three concentrated loads M, all acting vertically.

Writing y for the deflection at the point x, with the origin at O, the end conditions are

$$y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ when } x = 0,$$

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = 0 \text{ when } x = 3l,$$

by reason of which we make the rational assumption that the actual deflection due to the shearing force is proportional to the deflection which would be produced if that force were applied slowly. Hence we suppose that in the free motion described parallel to the y-axis the building moves under the influence of the gravitational forces acting on the 'rotated' system.

Let, then, y_1, y_2, y_3 denote the deflections produced at the floors numbered 1, 2, 3 in the figure by the associated load M, and y the *additional* deflection caused at the point x by the weight of the external wall alone.

On repeating the procedure followed in Ex. 2, with the same significance attached to the symbol σ, it will be seen that

$$2T_{\text{max.}} = \frac{\sigma^2 p^2}{g}\left\{M(y^2_1 + y^2_2 + y^2_3) + \int_0^{3l} wy^2\, dx\right\} \quad . \quad (56.14)$$

represents the maximum value of the kinetic energy of the disturbed building, where $\frac{2\pi}{p}$ refers to the period in a normal mode of vibration. Also, to the same approximation,

$$2V_{\text{max.}} = \sigma^2\left\{M(y_1 + y_2 + y_3) + \int_0^{3l} wy dx\right\} \quad . \quad . \quad (56.15)$$

is the maximum value of the potential energy. The symbol of integration indicates that the external walls are here treated as an elastic beam, which is probably a reasonable assumption to make in the case of small vibrations about equilibrium.

If $T_{\text{max.}}$ and $V_{\text{max.}}$ be equated, to effect the next operation, the frequency $\frac{p}{2\pi}$ under consideration is determined by

$$p^2 = \frac{g\left\{M(y_1 + y_2 + y_3) + w\int_0^{3l} y\, dx\right\}}{M(y^2_1 + y^2_2 + y^2_3) + w\int_0^{3l} y^2\, dx}, \quad . \quad . \quad (56.16)$$

where w is taken as a constant quantity.

The theory of structures now provides a means of evaluating the unknown quantities in the expression for p. Suppose, by way of illustration, that the shearing force is carried by the webs of the structural members and, for the moment, that the shearing stress is uniformly distributed over the relevant section. Let A denote the *total* sectional area of the webs, S_x the shearing force at the point x, and N the modulus of rigidity for the material, then, with the stated law of distribution for the stress,

$$\frac{dy}{dx} = \frac{S_x}{AN},$$

whence

$$y = \frac{xS_x}{AN} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56.17)$$

It is also evident that the shearing force, being due to gravitational effects, amounts to

$\{3(M + wl) - \frac{1}{2}wx\}$ . . . over the range $x = 0$ and $x = l$,

$\left\{\frac{Ml}{x} + (2M + 3wl) - \frac{1}{2}wx\right\}$ . . . over the range $x = l$ and $x = 2l$,

$\left\{\frac{3Ml}{x} + (M + 3wl) - \frac{1}{2}wx\right\}$ . . . over the range $x = 2l$ and $x = 3l$.

Successive substitution of these expressions in equation (56.17) thus shows that the values of y_1, y_2, y_3, y in equation (56.16) are determined by the relations

$y = \frac{1}{AN}\{3(M + wl)x - \frac{1}{2}wx^2)$ . . . over the range $x = 0$ and $x = l$,

$y = \frac{1}{AN}\{Ml + (2M + 3wl)x - \frac{1}{2}wx^2\}$ . . . over the range $x = l$ and $x = 2l$,

$y = \frac{1}{AN}\{3Ml + (M + 3wl)x - \frac{1}{2}wx^2\}$ . . . over the range $x = 2l$ and $x = 3l$.

Thus, after inserting the appropriate values for the y-terms in equation (56.16), we ascertain

$$\nu = \frac{p}{2\pi}$$

$$= \frac{1}{2\pi l}\left\{\frac{30gAN(22Mwl^2 + 14M^2l + 9w^2l^3)}{2{,}100M^3 + 4{,}760M^2wl + 3{,}680Mw^2l^2 + 972w^3l^3}\right\}^{\frac{1}{2}} \quad . \quad (56.20)$$

to be the expression for the fundamental frequency in a normal mode of vibration.

To improve upon this result and so make it agree more closely

with the fact that the shearing stress is not evenly distributed over the web-section, we might introduce the factor $\frac{9}{8}$ on the right of equation (56.17) and proceed to the corresponding solution.

57. Effect of Centrifugal Force. A noteworthy example of free oscillations is presented by a system which is constrained to rotate uniformly round a fixed axis, as is to be inferred from the discussion in Art. 19(*d*). The conclusions then arrived at, however, require modification, in so far as they relate to a system of rigid bodies which does not represent that formed by the elastic materials used in the construction of machinery, such as, for example, turbine-wheels and airscrews.

It will make for simplicity of treatment if we limit the examination to the 'conservative' type of system and consider the natural vibrations for the structure shown in Fig. 98, consisting of a bar or

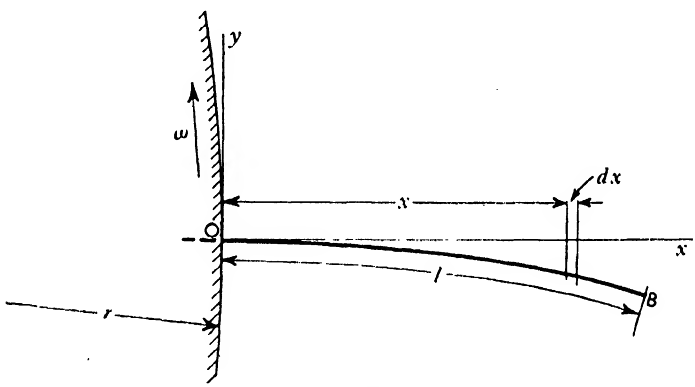


FIG. 98.

blade rigidly attached to a rotor of radius r, and revolving with constant angular velocity ω about a fixed axis. The blade may be specified by its length l, cross-sectional area A at a point distant x from the fixed root or end, and density ρ of the material.

If the x-axis be the equilibrium-position of the blade indicated by OB, and y the vertical displacement of the element dx, in the case where the oscillatory motion is confined to the plane of the paper and ρ is a constant, we have

$$2T = \frac{\rho}{g}\int_0^l A\dot{y}^2\,dx \quad . \quad . \quad . \quad . \quad . \quad (57.1)$$

for the kinetic energy T of the bar, the sectional area A being expressed as a function of x.

To apply the method of Art. 55 on the supposition that the vibration is of the 'normal' type, assume

$$y = X\cos pt$$

approximately represents the shape of the disturbed blade, where X is a function of x. Then, on taking maximum values in the manner explained above,

$$y_{\text{max.}} = X$$

and
$$\dot{y}_{\text{max.}} = -pX,$$

which, combined with equation (57.1), enable us to write

$$2T_{\text{max.}} = \frac{\rho p^2}{g}\int_0^l AX^2\,dx \quad . \quad . \quad . \quad . \quad (57.2)$$

The related expression for the potential energy $V'_{\text{max.}}$ of the *non-rotating* bar is readily proved to be

$$V'_{\text{max.}} = \frac{\rho}{2}\int_0^l Ay_{\text{max.}}\,dx$$

$$= \frac{\rho}{2}\int_0^l AX\,dx \quad . \quad . \quad . \quad . \quad . \quad (57.3)$$

To this quantity we must add the potential energy involved in the term $\omega^2 W$ of equation (19.11), which is clearly equal to the work done by the centrifugal force

$$\frac{\rho\omega^2}{g}\int_0^l A(r+x)\,dx$$

in displacing the blade radially through the distance

$$-\tfrac{1}{2}\int_0^x\left(\frac{dX}{dx}\right)^2 dx,$$

with the outward direction taken as positive. Hence, in the notation of Art. 19(*d*),

$$\omega^2 W = -\frac{\rho\omega^2}{2g}\int_0^l A(r+x)\,dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx \quad . \quad . \quad (57.4)$$

The maximum potential energy of the *rotating* blade, being $(V'_{\text{max.}} - \omega^2 W)$ according to equation (19.11), is therefore

$$2V_{\text{max.}} = \rho\int_0^l AX\,dx + \frac{\rho\omega^2}{g}\int_0^l A(r+x)\,dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx \quad . \quad (57.5)$$

Thus, after equating $T_{\text{max.}}$ and $V_{\text{max.}}$, it follows that

$$p^2 = \frac{g\int_0^l AX\,dx + \omega^2\int_0^l A(r+x)\,dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx}{\int_0^l AX\,dx} \quad . \quad . \quad (57.6)$$

determines the period $\frac{2\pi}{p}$ of free vibration in the assumed normal mode for the *rotating* bar. The corresponding frequency

$$\begin{aligned}\nu &= \frac{p}{2\pi} \\ &= (\nu_0^2 + \nu_1^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57.7)\end{aligned}$$

where
$$\nu_0^2 = \frac{1}{4\pi^2}\frac{g\int_0^l AX\,dx}{\int_0^l AX^2\,dx},$$

$$\nu_1^2 = \frac{1}{4\pi^2}\frac{\omega^2\int_0^l A(r+x)\,dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx}{\int_0^l AX^2\,dx}.$$

Seeing that ν_0 is the frequency of the *non-rotating* blade, ν_0 is the frequency due to the stiffness which is furnished by the elastic properties of the material. In the same sense, ν_1 is the frequency due to the stiffness induced by the centrifugal effect alone, the elastic properties being then neglected.

The last equation shows that the effect of centrifugal force is to raise the natural frequency when referred to the non-rotating system, which is equivalent to a partial constraint, by the treatment of Art. 55. Put into other words, the vibratory motion of the rotating blade is the same as if the prescribed angular velocity were zero, and constraints were imposed so as to secure an additional term $-\omega^2 W$ in the potential energy. But we have just identified $\omega^2 W$ with ν_1 of equation (57.7), and pointed out in Art. 19(*d*) that W is a function of the generalized co-ordinates $q_1, q_2, q_3, \ldots$, whence ν is a function of the same co-ordinates. It therefore appears that, in addition to being a function of the *speed* ω, the frequency ν will vary with the *direction of vibration*, when account is taken of the non-symmetrical form and cross-section of the blades on air-screws, and turbine-wheels.

While it is not usually practicable to determine completely the complex group of factors which accordingly influence ν_1, it is possible, as in our expansion of the potential energy in the general theory, to expand equation (57.7) and write

$$\nu^2 = \nu_0^2 + \alpha_1\omega^2 + \alpha_2(\omega^2)^2 + \alpha_3(\omega^2)^3 + \ldots \quad . \quad (57.8)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ denote constants for a specified blade.

These results imply that in general: (i) the α-terms depend on the mode of vibration, as well as on the size and shape of the blade; (ii) the oscillations of the blade comprise a fundamental and a number of higher harmonic components or 'partials'.

By the aid of this formula we can estimate the natural frequency of a rotating blade, provided sufficient data are available for the purpose. If, for example, $\nu_0 = 500$ cycles per second and $\alpha_1 = 10$ in the case of a turbine-blade designed to rotate at 3,000 r.p.m., its natural frequency at that speed will amount to 524·9 cycles per second.

In the general problem of airscrews the effect of the term $\omega^2 W$ as defined above is of practical importance, since the tip-speed of a blade may approach the velocity of sound in air. A considerable amount of labour has been devoted to experimental and analytical investigations into the matter, as is made manifest in such publications as the *Technical Reports and Memoranda* of the Aeronautical Research Committee (*R. & M.*), and the *Technical Reports and Notes* of the U.S. National Advisory Committee for Aeronautics (*N.A.C.A.*). With regard to bars of simple shape, A Berry [1] made the first exact determination of the influence of centrifugal force on the vibrations of a uniform rod fixed at its root; he found $\alpha_1 = 1{\cdot}19$ for small values of the ratio $\frac{\omega}{\nu_0}$, and $\alpha_1 = 1{\cdot}22$ when that ratio was approximately equal to unity.

When the sectional area A and, consequently, the moment of inertia for the section vary from point to point on the blade of an airscrew, we may, on expressing those variables as functions of the ratio $\frac{x}{l}$ in Fig. 98, proceed by the method of Art. 95. As considerable difficulty is encountered at the outset of an analytical determination of the constraint due to the hub on an airscrew, its law of variation within a given range of speed cannot well be obtained without recourse to tests on models. Various mathematical methods have, nevertheless, been used for the purpose of evaluating the coefficients $\alpha_1, \alpha_2, \alpha_3, \ldots$ in equation (57.8) for prescribed shapes. H. A. Webb and L. M. Swain [2] determined the frequency for a wedge-shaped rod, in which the cross-section varied linearly and the moment of inertia as the cube of the dimension x in Fig. 98, and found a mean value of 1·3 for α_1. The gravest mode of oscillation for a thin tapered rod rotating about its root has been examined by L. M. Swain,[3] who arrived at values within the range of 1·17 and 1·19 for α_1. Moreover, a graphical method was utilized by R. V. Southwell and H. J. Gough,[4] to find the upper limit of the centrifugal effect in this sense, and their result corresponds with $\alpha_1 = 1{\cdot}52$. To facilitate a comparison with the data derived from tests of actual airscrews, reference may be made to the work of A. Fage,[5] who

[1] *R. & M.* No. 488 (1918–1919). [2] *R. & M.* No. 626 (1918–1919).
[3] *Phil. Mag.*, vol. 41, page 259 (1921).
[4] *R. & M.* No. 766 (1921–1922). [5] *R. & M.* No. 967 (1925–1926).

obtained mean values of $\alpha_1 = 1{\cdot}54$ for narrow and $\alpha_1 = 1{\cdot}04$ for broad propellers.

It is to be observed that the present examination refers, strictly speaking, to aircraft making a straight and level course, for the disturbed motion of the airscrew-blades would become still more complicated if these conditions were not fulfilled, owing to the forces which we shall investigate in Art. 98 and exemplify in Art. 124. Mainly on account of those forces the transverse oscillations, described approximately at right-angles to the direction under consideration, are usually accompanied by torsional vibrations about the longitudinal axis of a blade.

Under working conditions the unbalanced effect of aero-engines is naturally transmitted to the airscrew by way of the crankshaft, so that an engine and its propeller should be treated as a unit in a general solution of the present problem, in a manner which will be understood from Arts. 60 and 61. Several instruments and types of apparatus have been devised for the purpose, descriptive accounts of which are given in publications devoted to this sphere of investigation.[1]

58. Forced Vibrations without Friction. If we impose a set of n external forces, $Q_1, Q_2, \ldots, Q_n$ on the system referred to in Art. 52, equation (52.3) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial V}{\partial q_r} = Q_r, \quad . \quad . \quad . \quad . \quad . \quad (58.1)$$

whence the resulting motion is defined by the n relations

$$(a_{1r}\ddot{q}_1 + c_{1r}q_1) + (a_{2r}\ddot{q}_2 + c_{2r}q_2) + \ldots + (a_{nr}\ddot{q}_n + c_{nr}q_n) = Q_r, \quad . \quad (58.2)$$

where $r = 1, 2, \ldots, n$.

In systems formed by actual machines, it has been noticed that the generalized component of force Q_r commonly involves a number of harmonic components which, in virtue of the principle of superposition, can be expressed by a number of terms of the type $\cos(\omega t + \varepsilon)$. For the general purpose of analysis, it is, however, most convenient to assume that the terms Q_r vary as $e^{i\omega t}$, with a complex coefficient. On this assumption equations (58.2) yield,

[1] J. Morris, *Jour. Roy. Aero. Soc.*, vol. 26, page 472 (1922); *ibid.*, vol. 27, page 182 (1923); *ibid.*, vol. 40, page 311 (1936). F. Liebers, *N.A.C.A. Tech. Memo.* No. 568 (1930); *ibid.*, No. 683 (1932). F. Seewald, *N.A.C.A. Tech. Memo.* No. 642 (1931). F. L. Prescott, *Air Corps Inform. Circ.* No. 664 (1932). T. Theodorsen, *N.A.C.A. Tech. Note* No. 516 (1935). M. Hansen and G. Mesmer, *Aircraft Engineering*, vol. 7, page 65 (1935). H. H. Crouch, *Mech. Engineering*, vol. 58, page 215 (1936). K. Lürenbaum, *S.A.E. Jour.*, vol. 39, page 469 (1936). B. C. Carter, *R. & M.* No. 1758 (1937). B. C. Carter, *Jour. Roy. Aero. Soc.*, vol. 41, page 749 (1937).

after cancelling the common factor $e^{i\omega t}$, n linear equations of the form

$$(c_{1r}-a_{1r}\omega^2)q_1+(c_{2r}-a_{2r}\omega^2)q_2+\ldots+(c_{nr}-a_{nr}\omega^2)q_n=Q_r \quad . \quad (58.3)$$

The 'particular integral' of equations (58.2) may accordingly be written as

$$\Delta(\omega^2)q_r = \alpha_{1r}Q_1 + \alpha_{2r}Q_2 + \ldots + \alpha_{nr}Q_n, \quad . \quad (58.4)$$

where $\Delta(\omega^2)$ is the determinant formed by the coefficients in equations (58·3), and $\alpha_{1r}, \alpha_{2r}, \ldots, \alpha_{nr}$ are the minors of its rth row. This solution, as previously remarked, represents the *forced* motion, in which every point on the system describes in general a simple oscillation of the imposed period $\frac{2\pi}{\omega}$, and all the particles pass simultaneously through their equilibrium-positions. It shows further that, in the absence of friction, the amplitude tends to infinitely large values when ω^2 approximates to a root of the determinantal equation

$$\Delta(\omega^2) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (58.5)$$

that is when the imposed frequency practically coincides with one of the natural frequencies of the structural system concerned. Moreover, since the determinant $\Delta(\omega^2)$ is symmetrical, $\alpha_{rs} = \alpha_{sr}$, so that the coefficient of Q_r in the expression for q_s is the same as the coefficient of Q_s in the expression for q_r, according to Art. 49.

The complete solution is obtained by superposing on this forced motion the *free* oscillations which were determined in Art. 52, when the $2n$ arbitrary constants of the previous analysis enable us to adapt the solution to $2n$ arbitrary initial conditions of displacement and velocity.

If the system be now referred to its normal co-ordinates, equations (58·3) reduce, in the notation of Art. 53, to n relations of the type

$$Q_1'=(\gamma_1-\delta_1\omega^2)\theta_1,\ Q_2'=(\gamma_2-\delta_2\omega^2)\theta_2,\ \ldots,\ Q_n'=(\gamma_n-\delta_n\omega^2)\theta_n, \quad (58.6)$$

where the accented symbols denote the forces when expressed in terms of the normal co-ordinates. Consequently, if $\omega_1, \omega_2, \ldots, \omega_n$ be the consecutive values of ω in the n normal modes, the related frequencies are given by

$$\omega_1^2 = \frac{\gamma_1}{\delta_1},\ \omega_2^2 = \frac{\gamma_2}{\delta_2},\ \ldots,\ \omega_n^2 = \frac{\gamma_n}{\delta_n}, \quad . \quad . \quad . \quad (58.7)$$

and the corresponding displacements by

$$\theta_1 = \frac{Q_1'}{\gamma_1\left(1-\frac{\omega^2}{\omega^2_1}\right)},\ \theta_2 = \frac{Q_2'}{\gamma_2\left(1-\frac{\omega^2}{\omega^2_2}\right)},\ \ldots,\ \theta_n = \frac{Q_n'}{\gamma_n\left(1-\frac{\omega^2}{\omega^2_n}\right)} \quad . \quad (58.8)$$

Hence the displacement θ_r in a normal mode becomes very great when the ratio $\frac{\omega^2}{\omega_r^2}$ approaches unity, as in the general motion.

It is of practical interest to observe here that the quantities $\frac{Q_1'}{\gamma_1}, \frac{Q_2'}{\gamma_2}, \ldots, \frac{Q'_n}{\gamma_n}$ in equations (58.8) are the *statical* displacements which would be produced by *constant* forces of the proper normal types, equal to the instantaneous values of the forces in the actual system. On account of this fact, if the imposed frequency $\frac{\omega}{2\pi}$ is small compared with those in the normal modes of free vibration, the system is, at any instant, practically in the equilibrium-configuration corresponding to the disturbing forces. To this degree of approximation the equilibrium-position of a *railway bridge*, for example, is the same as the curve of the deflections produced by a locomotive moving at 'crawling' speed across the structure, so that in experimental work on the subject the 'crawl' record may be taken as the datum line upon which the effects of hammer-blow are superposed.

59. Damped Vibrations. The simple problems examined in Arts. 41 and 42 show that the oscillations of actual structures and mechanisms are modified by the inherent frictional forces of such systems. The dissipative forces may arise from a number of sources and vary according to different laws, those of dry- and fluid-friction representing the limits.

The general effect of damping will, however, be sufficiently explained if we suppose the frictional forces to be proportional to the generalized velocities and, for brevity, consider a system having two degrees of freedom. We may conveniently take as co-ordinates those variables which would be normal co-ordinates if there were no friction.

Let $2F$ be the rate at which the mechanical energy is dissipated on this account, then the expressions for the kinetic energy T, the *dissipation-function* F and the potential energy V are of the form

$$\left.\begin{aligned} 2T &= \dot{q}_1^2 + \dot{q}_2^2, \\ 2F &= b_{11}\dot{q}_1^2 + 2b_{12}\dot{q}_1\dot{q}_2 + b_{22}\dot{q}_2^2 \\ 2V &= c_1 q_1^2 + c_2 q_2^2, \end{aligned}\right\} \quad . \quad . \quad (59.1)$$

where the frictional coefficients b_{11}, b_{12}, b_{22} are constants for given conditions. To secure stability in the absence of friction, it will be supposed further that c_1 and c_2 are positive.

If the free oscillations given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial V}{\partial q_r} = 0$$

are subjected to this type of friction, the equations of motion are seen to be

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial F}{\partial \dot{q}_r} + \frac{\partial V}{\partial q_r} = 0,$$

where $r = 1, 2$, or, on applying the formula to equations (59.1),

$$\left.\begin{aligned}\ddot{q}_1 + b_{11}\dot{q}_1 + b_{12}\dot{q}_2 + c_1 q_1 &= 0,\\ \ddot{q}_2 + b_{12}\dot{q}_1 + b_{22}\dot{q}_2 + c_2 q_2 &= 0.\end{aligned}\right\} \quad . \quad . \quad . \quad (59.3)$$

Now assume that the relations

$$q_1 = A_1 e^{pt} \text{ and } q_2 = A_2 e^{pt} \quad . \quad . \quad . \quad (59.4)$$

hold in the above equations, then substitution leads to

$$\left.\begin{aligned}A_1(p^2 + b_{11}p + c_1) + A_2 b_{12} p &= 0,\\ A_1 b_{12} p + A_2(p^2 + b_{22}p + c_2) &= 0,\end{aligned}\right\} \quad . \quad . \quad . \quad (59.5)$$

whence, on eliminating the ratio $A_1 : A_2$,

$$(p^2 + b_{11}p + c_1)(p^2 + b_{22}p + c_2) - b^2_{12}p^2 = 0 \quad . \quad (59.6)$$

This quadratic in p^2 yields two roots, which will be referred to as p^2_1 and p^2_2.

In the case where the frictional forces are comparatively small, we may neglect all but the first powers of the quantities b_{11}, b_{12}, b_{22}, and so obtain

$$p_1 = i\sqrt{c_1} - \tfrac{1}{2}b_{11},\ p_2 = i\sqrt{c_2} - \tfrac{1}{2}b_{22} \quad . \quad . \quad (59.7)$$

as the roots of the last equation, where $i = \sqrt{-1}$.

To determine the amplitudes of vibration, we may first substitute the root p_1 in the second of equations (59.5), and thus derive

$$\frac{A_2}{A_1} = \frac{ib_{12}\sqrt{c_1}}{c_1 - c_2},$$

if only the first powers of b_{11}, b_{12}, b_{22} are retained. Therefore, as can readily be verified,

$$q_1 = (c_1 - c_2)e^{-\frac{1}{2}b_{11}t}(\cos\sqrt{c_1}t + i\sin\sqrt{c_1}t),$$

and $$q_2 = b_{12}\sqrt{c_1}e^{-\frac{1}{2}b_{11}t}(i\cos\sqrt{c_1}t - \sin\sqrt{c_1}t)$$

form a particular solution of equations (59.3). A second particular solution is, moreover, given by changing i to $-i$ in these expressions. These two independent particular solutions, when taken together, enable us to write in terms of 'real' quantities

$$q_1 = (c_1 - c_2)e^{-\frac{1}{2}b_{11}t}\cos\sqrt{c_1}t,\quad q_2 = -b_{12}\sqrt{c_1}e^{-\frac{1}{2}b_{11}t}\sin\sqrt{c_1}t,$$

$$q_1 = (c_1 - c_2)e^{-\frac{1}{2}b_{11}t}\sin\sqrt{c_1}t,\quad q_2 = b_{12}\sqrt{c_1}e^{-\frac{1}{2}b_{11}t}\cos\sqrt{c_1}t.$$

Hence, on combining, the most general real solution involving e^{p_1t} is

$$q_1 = (c_1 - c_2)A_1 e^{-\frac{1}{2}b_{11}t}\sin(\sqrt{c_1}t + \varepsilon_1),$$

$$q_2 = b_{12}\sqrt{c_1}A_1 e^{-\frac{1}{2}b_{11}t}\sin\left(\sqrt{c_1}t + \frac{\pi}{2} + \varepsilon_1\right),$$

where A_1 and ε_1 are arbitrary constants. These expressions represent one of the normal modes of oscillation. When to this result

is added the solution corresponding to the root p_2, which is derived in a like manner, it will be found that the general expressions for the motion are

$$\left.\begin{aligned} q_1 &= (c_1 - c_2)A_1 e^{-\frac{1}{2}b_{11}t} \sin(\sqrt{c_1}t + \varepsilon_1) \\ &\qquad + b_{12}\sqrt{c_2}A_2 e^{-\frac{1}{2}b_{22}t} \sin\left(\sqrt{c_2}t + \frac{\pi}{2} + \varepsilon_2\right), \\ q_2 &= b_{12}\sqrt{c_1}A_1 e^{-\frac{1}{2}b_{11}t} \sin\left(\sqrt{c_1}t + \frac{\pi}{2} + \varepsilon_1\right) \\ &\qquad + (c_2 - c_1)A_2 e^{-\frac{1}{2}b_{22}t} \sin(\sqrt{c_2}t + \varepsilon_2), \end{aligned}\right\} \quad . \qquad (59.8)$$

where the constants A_1, A_2, ε_1, ε_2 depend on the initial circumstances of the motion.

It is clear that in engineering problems the symbol F relates to a positive quantity, since the dissipative forces are such that mechanical energy is being continually lost to the system under consideration, by reason of which the b-terms are positive. The last set of equations plainly shows that the vibrations gradually die away, owing to the presence of the exponential terms; also, to the present order of approximation, that in the normal modes the *periods* of oscillation are the same as if the frictional forces were absent; and the *amplitude* of one of the co-ordinates is small compared with that of the other co-ordinate, while at any instant the *phases* of the vibrations in the two co-ordinates differ by 90 deg.

Our results obviously apply to a system having any number of degrees of freedom in instances where the dissipative forces are small. In general, then, with the potential energy and the dissipation function both positive, the effect of slight friction is mainly on the *amplitude* of oscillation, the *natural periods* being practically unaltered; and if $(q_1, q_2, \ldots, q_n)$ denote the normal co-ordinate of the system with the frictional forces absent, when those forces are present there is a normal vibration of the system in which the *amplitude* of the oscillations in $q_2, q_3, \ldots, q_n$ is small compared with the amplitude in q_1, and the *phase* of the vibrations in $q_2, q_3, \ldots, q_n$ differs by 90 deg. from the phase of the vibration in q_1.

Remembering the conclusions arrived at in Art. 39, equations (59.8) also disclose the fact that the oscillatory motion is generally elliptic-harmonic, the amplitude of a point on the system contracting according to the law $e^{-\alpha t}$. In the special case of slight friction, the elliptic path of a point on the system will be very flat, except when approximate equality exists between the periods of two fundamental modes.

If the frictional coefficients b_{rs}, b_{sr} be comparatively large in a system having n degrees of freedom, the inertia of the system will become ineffective, when the most convenient set of co-ordinates

to use is that which reduces the general expressions for F and V to the sums of squares. Taking these to be

$$\left.\begin{aligned} 2F &= b_1\dot{q}_1^2 + b_2\dot{q}_2^2 + \ldots + b_n\dot{q}_n^2, \\ 2V &= c_1 q_1^2 + c_2 q_2^2 + \ldots + c_n q_n^2, \end{aligned}\right\} \quad . \quad . \quad (59{\cdot}9)$$

the free motion is determined by equations of the type

$$b_r\dot{q}_r + c_r q_r = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (59{\cdot}10)$$

so that the displacement $\qquad q_r = Ce^{-\alpha t}, \quad . \quad . \quad . \quad . \quad (59{\cdot}11)$

where $\alpha = \dfrac{c_r}{b_r}$, and $r = 1, 2, \ldots, n$. Here the displacements die away without oscillation, and the motion is said to be 'aperiodic', such as is commonly secured by means of the dash-pots used in connection with valve-gears, and governors.

The treatment of *forced* vibrations with friction will now readily be understood from the method of Art. 58, the formula being

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial F}{\partial \dot{q}_r} + \frac{\partial V}{\partial q_r} = Q_r \quad . \quad . \quad . \quad . \quad (59{\cdot}12)$$

It is plain that the foregoing results would be but slightly modified in these circumstances if the frictional coefficients b_{rs}, b_{sr} were comparatively small, except in instances where either approximate or exact equality between a natural and an imposed frequency leads to the condition of resonance investigated in Art. 45.

Due consideration should be given to all the forces in a specified problem of vibration, arranged in their relative order of magnitude, as will be explained in what remains of this chapter.

60. Engine-Driven Alternators. Trouble sometimes arises from the passage of interchange currents between a set of alternators connected in parallel, in systems where some of the machines are driven by reciprocating engines and some by turbines. This undesirable state of working is, from the mechanical point of view, caused by the cyclic variation in the speed of the engines, assuming the turbines to rotate with uniform angular velocity. Since the results of Chapter I demonstrate that an increase in the flywheel-effect of the engines offers a practical means of mitigating the disturbance, we shall in the main confine our attention to this aspect of the problem.

Consider, for brevity of treatment, the system formed by a synchronous machine direct-coupled to an engine having one cylinder. According to equation (21.4) the motion is defined by

$$\frac{1}{g}\{(I_c + M_c\gamma r a) + M_c\gamma^2(k^2_A - al)\cos^2\theta + Mr^2(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta)^2\}\ddot{\theta}$$

$$+ \frac{1}{g}\{Mr^2(\cos\theta + \gamma\cos 2\theta)(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \tfrac{1}{2}M_c\gamma^2(k^2_A - al)\sin 2\theta\}\dot{\theta}^2$$
$$= Pr(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) + \mathfrak{T}, \quad . \quad . \quad (60.1)$$

where:

I_c = moment of inertia of the crankshaft, flywheel and direct-coupled machinery;

M_c = weight of the connecting rod shown in Fig. 40;

$\gamma = \dfrac{\text{throw of the crank}}{\text{length of the connecting rod}} = \dfrac{r}{l}$;

k_A = radius of gyration of the connecting rod about the gudgeon-pin;

$M = (M_A + M_Q)$ as defined in Art. 18;

P = thrust exerted at time t by the working fluid on the piston;

$\mathfrak{T}$ = torque *reaction* of the machine at time t;

θ = crank-angle at time t;

a = dimension indicated in Fig. 40.

With this notation the position of the mechanism at any instant t is given by $\theta = \omega t + \varepsilon$, where ω is the synchronous or mean angular rotation and ε the phase-difference of the alternator. The torque reaction of the machine will be a function of ε, and it can be specified by writing

$$\mathfrak{T} = C(\omega t - \theta), \quad \ldots \ldots \quad (60.2)$$

C being a constant for the given system.

In the steady state the machine will run with a constant phase-difference ε, so that the corresponding equation of motion follows by substituting $C(\omega t - \theta)$ for $\mathfrak{T}$ in (60.1).

To investigate the effect of a slight deviation from the mean angular velocity, and thus study the disturbance initiated by the unbalanced inertia forces of the complete system, let the machine be disturbed so as to increase ε by a small amount ξ. Under these conditions we have, in equation (60.1),

$$\theta = (\omega t + \xi + \varepsilon) \quad \ldots \quad (60.3)$$

and, by successive differentiation with regard to the time,

$$\dot{\theta} = \omega + \dot{\xi},\ \ddot{\theta} = \ddot{\xi},\ \dot{\theta}^2 = \omega^2 + 2\omega\dot{\xi} + \dot{\xi}^2 \quad \ldots \quad (60.4)$$

The equation of the disturbed motion follows on introducing these new values after the method explained in Art. 22, in which we cancel all terms that refer to the steady motion and neglect those of the second degree in ξ and its derivatives. The consequent increment in the armature-torque also may be included by adding a term $-B\dot{\xi}$ to the right-hand side of the resulting expression, where B relates to a constant for the machine under consideration. This term is accounted for by the fact that at the instant in question the machine, in passing through the 'equilibrium' state associated with the applied torque, is running slightly faster than the synchronous speed at which the unbalanced armature-torque is completely represented by the term $-B\dot{\xi}$.

On effecting these operations with the aid of formula (20.4), remembering that ξ is to be treated as a small quantity, in consequence of which $\cos(\omega t + \xi + \varepsilon) \longrightarrow \cos(\omega t + \varepsilon)$ and $\sin(\omega t + \xi + \varepsilon) \longrightarrow \sin(\omega t + \varepsilon)$ in the limit, it is not difficult to prove that the disturbed motion is determined by

$$\begin{aligned}
&\frac{1}{g}\{(I_c + M_c\gamma ra) + \tfrac{1}{2}M_c\gamma^2(k^2{}_A - al) + \tfrac{1}{2}Mr^2(1 + \tfrac{1}{2}\gamma^2)\}\ddot{\xi} \\
&+ \frac{1}{g}\{Mr^2(\tfrac{1}{2}\cos 2\theta - \gamma\sin\theta\sin 2\theta + \tfrac{1}{8}\gamma^2\cos 4\theta) \\
&\qquad\qquad - \tfrac{1}{2}M_c\gamma^2(k^2{}_A - al)\cos 2\theta\}\ddot{\xi} \\
&+ \frac{1}{g}\{Mr^2(\cos\theta + \gamma\cos 2\theta)(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) \\
&\qquad\qquad - \tfrac{1}{2}M_c\gamma^2(k^2{}_A - al)\sin 2\theta\}(2\omega\dot{\xi} + \omega^2) \\
&+ B\dot{\xi} + C\xi = Pr(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \mathfrak{T}_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (60.5)
\end{aligned}$$

where $\mathfrak{T}_0$ signifies the mean load torque of the synchronous machine. Here the coefficients are supposed to have the values corresponding to $\dot{\theta} = \omega$ and are, therefore, constants.

Before proceeding, a few remarks might be made on the significance of certain quantities in the last equation. It is clear from the treatment of Chapter I that the *first* coefficient of $\ddot{\xi}$ is a *constant* quantity which denotes the equivalent weight of the moving parts, while the product of ω^2 and its coefficient is the torque associated with the unbalanced effects of the mechanism. In a first approximation we may neglect the *variable* coefficient of $\ddot{\xi}$ and the terms which involve $2\omega\dot{\xi}$, since both refer to quantities which affect chiefly the higher harmonic components identified with 2θ, 3θ, 4θ, . . . The symbol C is the 'overall' coefficient of stiffness for the mechanism; if it were equivalent to that of a shaft of length L and uniform diameter d, then $C = \dfrac{\pi d^4 N}{32L}$ for a material with a modulus of rigidity N.

At this stage of the work we can superpose the graph of

$$\begin{aligned}
&-\frac{1}{g}\{Mr^2(\cos\theta + \gamma\cos 2\theta)(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) \\
&\qquad\qquad - \tfrac{1}{2}M_c\gamma^2(k^2{}_A - al)\sin 2\theta\}\omega^2
\end{aligned}$$

on that of

$$Pr(\sin\theta + \tfrac{1}{2}\gamma\sin 2\theta) - \mathfrak{T}_0,$$

and express the result in the form of a Fourier series, a complete working cycle being taken in both instances. We require, to the present degree of approximation, only the first harmonic term of that series, which will here be signified by $A\cos\omega t$.

So far no account has been taken of the inevitable friction between the several parts of the mechanism, but this resistance

may be represented by inserting a term $-K\dot{\xi}$ on the right-hand side of equation (60.5), on the assumption that the dissipative forces are proportional to the corresponding velocity. It is here assumed that the frictional coefficient K is known, from experimental or other data. For analytical purposes it is convenient to combine $B\dot{\xi}$ and $K\dot{\xi}$, the sum of which will now be indicated by $D\dot{\xi}$ on the left-hand side of equation (60.5).

Writing, for brevity,

$$\begin{aligned} h &= \{(I_c + M_c\gamma ra) + \tfrac{1}{2}M_c\gamma^2(k^2{}_A - al) + \tfrac{1}{2}Mr^2(1 + \tfrac{1}{2}\gamma^2)\}, \\ b &= gD, \\ c &= gC, \\ f &= gA \end{aligned}$$

in equation (60.5), to the first approximation the disturbed motion of the system is accordingly determined by

$$h\ddot{\xi} + b\dot{\xi} + c\xi = f\cos\omega t, \quad . \quad . \quad . \quad . \quad (60.6)$$

where h, b, c, f represent constants. This equation is of the type examined in Art. 43, whence the angular displacement

$$\xi = \xi_0 e^{-\mu t}\cos(pt + \alpha) + \frac{f\cos(\omega t + \beta)}{\left\{\left(\frac{c}{h} - \omega^2\right)^2 + 4\mu\omega^2\right\}^{\frac{1}{2}}}, \quad . \quad (60.7)$$

where ξ_0 and α are arbitrary constants, and $\mu = \frac{1}{2}\frac{b}{h}$, $p^2 = \frac{c}{h} - \frac{1}{4}\frac{b^2}{h^2}$, $\tan\beta = \dfrac{b\omega}{h\omega^2 - c}$.

In the ordinary damped vibrations of such a system the coefficient b in the expression for ξ is always positive, as is also h. The presence of the 'time factor' $e^{-\mu t}$ consequently indicates that the free motion dies away, leaving only the forced oscillation represented by the second member of the solution (60.7). Hence the phenomenon of resonance is likely to occur if the impressed frequency $\dfrac{\omega}{2\pi}$ approximates to the natural frequency $\dfrac{p}{2\pi}$, being conditioned by the relative magnitude of the term b, as shown in Art. 45. The synchronous vibration produced in this way will lead to corresponding fluctuations in the energy delivered by the machine to the bus-bars, which may become so great as to blow the fuses. Trouble, as already remarked, has occasionally been experienced from this cause, and it is usually overcome with the help of flywheels heavy enough to result in the natural period being greater than the imposed period of the engine. It is readily seen that this procedure also prevents resonance with the higher harmonic components of the turning moment.

To determine the size of the flywheel by means of equation (60.7), it is accordingly necessary to arrange that the ratio

angular displacement of the machine in a revolution when running disconnected from the bus-bars : mean angular rotation

shall not exceed a specified fraction of the angular pitch of the poles. This information supplies the ratio $\xi : \omega$ in the last equation, in which the only unknown is then the flywheel-effect of the moving parts implied in the term h. If, for example, the equation be solved for I_c, the result gives the weight of the flywheel required to secure a prescribed value of ξ with fixed values assigned to the weights of the other moving parts in the system.

When the electrical characteristics of the machine are included in the analysis, however, b may become negative, because its sign then depends chiefly on the flow of electrical energy at the instant when ξ passes through the zero value. The index of the time-factor in equation (60.7) is then positive, and the motion may be unstable even when the turning moment is quite uniform and b is very small. This particular type of instability can be prevented by fitting 'amortisseurs' to the pole faces of the armature, the analytical effect of which is to add a negative term to b. The copper grids used for the purpose should, therefore, be of such dimensions as to introduce a negative term large enough to counteract the forces acting in the opposite sense.

Reverting to the mechanical considerations, our result shows in connection with the torsional oscillation of the crankshaft that if its natural period $2\pi\sqrt{\frac{h}{c}}$ in this mode approximately coincides with the imposed period, the state of resonance will induce appreciable amplitudes of vibration which will be limited only by the inherent frictional agencies in the system.

While this form of solution is sufficiently accurate for most practical purposes, a more exact determination can be made by taking the value of ξ thus obtained and substituting it in equation (60.5) together with the second order terms. The improved expression for ξ follows by solving the resulting equation.

It will be understood that the above method is of general application, with engines having any number of cylinders, for the extension involves nothing more than a superposition of the indicator-diagrams and the related load-graphs of the various cylinders.

In the analogous case of engines fitted to ships and aircraft, the direct-coupled 'machinery' should include the weight of the propeller and any entrained fluid; the complete shaft and its attachments must be treated as a structural unit, involving the proper number of coefficients of stiffness for the crankshaft,

propeller-blades, and so on. It is supposed in such applications that the ship or aeroplane is, strictly speaking, neither 'rolling' nor 'pitching', nor deviating from a straight course, for reasons that will be explained in Chapters V and VI.

61. Nose-Suspended Motor Drives. The essential parts of this mechanism, as indicated in Fig. 99, consist of a driving axle and direct-coupled wheels of radius r_1 connected to a motor, through a gear-wheel of radius r_2 and a pinion of radius r_3. Moreover, the

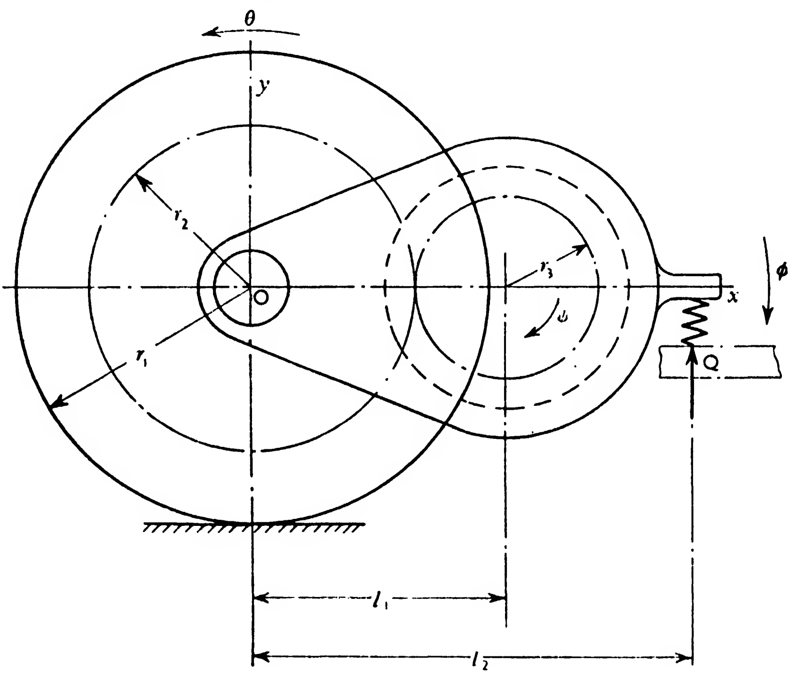


FIG. 99.

'nose' of the motor casing is spring-borne, and is therefore capable of executing an independent movement ϕ about the axis through O. Consequently, the given system has two degrees of freedom, associated in turn with the co-ordinates θ and ϕ in the figure.

A general treatment of this type of problem will be sufficiently elucidated if we omit the coefficients of stiffness for the shafts, teeth on the gears, and the spring mechanism that usually supports the main axle in such systems. These coefficients may be introduced without difficulty into the following analysis, for they are evaluated separately at various places in this work.

In these circumstances it is possible to refer the motion of the specified mechanism to either θ or ϕ; the former of these coordinates should be chosen in instances where the torsional oscillation of the various shafts is under consideration, and the latter where

the vibratory motion of the motor casing is the object of study. Here we shall confine our attention to the motion of the 'nose' of the casing, as being a matter of primary importance in finding the stress on the spring under working conditions. Our ultimate aim is therefore that of expressing the equations in terms of the co-ordinate ϕ.

To facilitate reference in this process, let:

M_1 = weight of the driving wheels, axle and gear-wheel of radius r_2;
M_3 = weight of the armature;
M_4 = weight of the motor casing, the centre of gravity of which is supposed to coincide with the axis of the driving shaft;
I_1 = moment of inertia of M_1 about the axis through O;
I_3 = moment of inertia M_3 about the axis of the driving shaft;
I_4 = moment of inertia of M_4 about the axis through O;
n = gear ratio $\frac{r_2}{r_3}$;
ψ = absolute angular displacement of the driving shaft, i.e. $\psi = n(\theta + \phi) + \phi$.

Seeing that the kinetic energy T of the given system is, by definition, equal to half the sum of the product of each mass and the square of its proper velocity, we can at once write down from an inspection of the figure

$$2T = \frac{1}{g}\{(M_1 + M_3 + M_4)\dot{x}^2 + M_3\dot{y}^2 + I_1\dot{\theta}^2 + I_3\dot{\psi}^2 + I_4\dot{\phi}^2\}, \quad (61.1)$$

with the x- and y- axes as exhibited.

In order to state T in terms of the variables θ, ϕ and their derivatives, use must be made of the geometrical relations for the mechanism, which are easily seen to be

$$x = r_1\theta,\ y = l_1\phi,\ \psi = n\theta + (1 + n)\phi.$$

Therefore, differentiating with respect to the time,

$$\dot{x} = r_1\dot{\theta},\ \dot{y} = l_1\dot{\phi},\ \dot{\psi} = n\dot{\theta} + (1 + n)\dot{\phi}$$

in the above equation, whence

$$2T = \frac{1}{g}[\{(M_1 + M_3 + M_4)r^2_1 + I_1 + n^2I_3\}\dot{\theta}^2 + 2n(1 + n)I_3\dot{\theta}\dot{\phi} + \{M_3l^2_1 + (1 + n)^2I_3 + I_4\}\dot{\phi}^2] \quad . \quad . \quad (61.2)$$

If $\mathfrak{T}_\theta$, $\mathfrak{T}_\phi$ denote in succession the effective torques referred to the co-ordinates θ, ϕ, it is clear that these quantities can be found with the help of the Lagrangian formulae

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \mathfrak{T}_\theta, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \mathfrak{T}_\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (61.3)$$

By the first of these formulae we have, from equation (61.2),

$$\frac{\partial T}{\partial \dot\theta} = \frac{1}{g}[\{(M_1 + M_3 + M_4)r_1^2 + I_1 + n^2 I_3\}\dot\theta + n(1 + n)I_3\dot\phi],$$

so that, differentiating with respect to the time,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = \frac{1}{g}[\{(M_1 + M_3 + M_4)r_1^2 + I_1 + n^2 I_3\}\ddot\theta + n(1 + n)I_3\ddot\phi]$$
$$= \mathfrak{T}_\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.4)$$

since $\dfrac{\partial T}{\partial \theta} = 0$. Differentiation with regard to the velocity $\dot\phi$ further leads to

$$\frac{\partial T}{\partial \dot\phi} = \frac{1}{g}[n(1 + n)I_3\dot\theta + \{M_3 l_1^2 + (1 + n)^2 I_3 + I_4\}\dot\phi],$$

whence

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) = \frac{1}{g}[n(1 + n)I_3\ddot\theta + \{M_3 l_1^2 + (1 + n)^2 I_3 + I_4\}\ddot\phi]$$
$$= \mathfrak{T}_\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.5)$$

because $\dfrac{\partial T}{\partial \phi} = 0$.

There remains to be found the corresponding expression for the potential energy, which may be conveniently investigated by writing :—

C = force necessary to compress the spring through unit distance, which will be assumed to remain constant ;

Q = reaction of the spring for an angular displacement ϕ about the equilibrium-configuration, i.e. $Q = Cl_2\phi$;

$\mathfrak{T}$ = driving torque of the motor ;

ε = relative angular displacement between the shaft and casing of the motor, i.e. $\varepsilon = n(\theta + \phi)$;

H = horizontal component of the reaction on the main bearings of the locomotive ;

R = tractive force at the tread of the wheels, i.e.

$$R = (M_1 + M_3 + M_4 + M)r_1\ddot\theta + H ;$$

M = weight of the complete locomotive or vehicle with the exception of the system shown in Fig. 99 ;

F = track-resistance to the motion of the locomotive.

Thus, on equating the effective force to the product of mass and acceleration of the vehicle concerned, we have

$$H - F = \frac{1}{g}Mr_1\ddot\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.6)$$

Moreover, an application of the principle of virtual work yields the relations

$$
\begin{aligned}
\mathfrak{T}_\theta &= \mathfrak{T}\frac{\partial \varepsilon}{\partial \theta} - H\frac{\partial x}{\partial \theta} \\
&= n\mathfrak{T} - r_1 H \\
&= n\mathfrak{T} - r_1\left(\frac{1}{g}Mr_1\ddot{\theta} + F\right), \qquad (61.7)
\end{aligned}
$$

by equation (61.6), and

$$
\begin{aligned}
\mathfrak{T}_\phi &= \mathfrak{T}\frac{\partial \varepsilon}{\partial \phi} - Ql_2 + (M_3 + M_4)l_1 \\
&= n\mathfrak{T} - Ql_2 + (M_3 + M_4)l_1 \\
&= n\mathfrak{T} - Cl_2{}^2\phi + (M_3 + M_4)l_1 \qquad (61.8)
\end{aligned}
$$

Inserting these values for $\mathfrak{T}_\theta$, $\mathfrak{T}_\phi$ in equations (61.4) and (61.5) we obtain the simultaneous equations

$$
\begin{aligned}
\frac{1}{g}[\{(M_1 + M_3 + M_4 + M)r_1{}^2 + I_1 + n^2I_3\}\ddot{\theta} \\
+ n(1 + n)I_3\ddot{\phi}] = n\mathfrak{T} - r_1F, \qquad (61.9)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{g}[n(1 + n)I_3\ddot{\theta} + \{M_3l_1{}^2 + (1 + n)^2I_3 + I_4\}\ddot{\phi}] + Cl_2{}^2\phi \\
= n\mathfrak{T} + (M_3 + M_4)l_1 \qquad (61.10)
\end{aligned}
$$

To combine these and thus state the result in terms of the co-ordinate ϕ and its derivatives, write A for the coefficient of inertia $\{(M_1 + M_3 + M_4 + M)r_1{}^2 + I_1 + n^2I_3\}$ in equation (61.9), then

$$
\ddot{\theta} = \frac{ng\mathfrak{T} - r_1gF - n(1 + n)I_3\ddot{\phi}}{A}
$$

in equation (61.10), whence

$$
\begin{aligned}
\frac{1}{g}\left\{M_3l_1{}^2 + (1+n)^2I_3 + I_4 - \frac{n^2(1+n)^2I_3}{A}\right\}\ddot{\phi} + Cl_2{}^2\phi \\
= \left\{n - \frac{n^2(1+n)I_3}{A}\right\}\mathfrak{T} + \frac{n(1+n)I_3r_1}{A}F + (M_3+M_4)l_1 \qquad (61.11)
\end{aligned}
$$

determines the general motion of the prescribed system.

With ϕ_0, $\mathfrak{T}_0$ representing the 'equilibrium' values of ϕ, $\mathfrak{T}$, respectively, we can derive the related expression for the steady motion by inserting ϕ_0, $\mathfrak{T}_0$ in the last equation. If now, while so running, the system be slightly disturbed so as to increase ϕ by ξ, and $\mathfrak{T}$ by $\Delta\mathfrak{T}$, both increments being small, then

$$
\phi = \phi_0 + \xi, \quad \mathfrak{T} = \mathfrak{T}_0 + \Delta\mathfrak{T},
$$

by reason of which the relations

$$
\dot{\phi} = \dot{\xi}, \quad \ddot{\phi} = \ddot{\xi}, \quad n\mathfrak{T}_0 = Cl_2{}^2\phi_0 - (M_3 + M_4)l_1
$$

hold in the disturbed state. Making these substitutions in equation

(61.11) and cancelling the terms which relate to the steady motion, in the manner already described, we have

$$\frac{1}{g}\left\{M_3l_1^2 + (1+n)^2I_3 + I_4 - \frac{n^2(1+n)^2I_3}{A}\right\}\ddot{\xi} + Cl_2^2\xi$$
$$= \left\{n\Delta\mathfrak{T} - \frac{n^2(1+n)I_3}{A}\mathfrak{T}\right\} + \frac{n(1+n)I_3r_1}{A}F, \quad . \quad . \quad . \quad (61.12)$$

on the assumption that F remains constant throughout the disturbance.

Hence

$$\frac{1}{g}\left\{M_3l_1^2 + (1+n)^2I_3 + I_4 - \frac{n^2(1+n)^2I_3}{A}\right\}\ddot{\xi} + Cl_2^2\xi = 0 \quad (61.13)$$

represents the equation for the free motion, in which the displacement

$$\xi = B\cos(pt + \alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.14)$$

B, α being arbitrary constants, and

$$p^2 = \frac{ACl_2^2g}{A\{M_3l_1^2 + (1+n)^2I_3 + I_4\} - n^2(1+n)^2I_3}.$$

The mechanism therefore executes, to the implied degree of approximation, free vibrations with a period $\frac{2\pi}{p}$ about the position corresponding to the steady conditions, and the instability associated with resonance will occur if $\Delta\mathfrak{T}$ varies with a periodicity that approximates to $\frac{2\pi}{p}$.

In actual machines, however, this instability is checked to some extent by the inherent frictional effects, but in order to prevent it a viscous resistance is usually introduced, to oppose variations of ϕ. This modification may be represented, as previously, by inserting a term $K\dot{\xi}$ on the left-hand side of equation (61.13). In consequence that expression becomes

$$\frac{1}{g}\left\{M_3l_1^2+(1+n)^2I_3+I_4-\frac{n^2(1+n)^2I_3}{A}\right\}\ddot{\xi}+K\dot{\xi}+Cl_2^2\xi = 0 \quad (61.15)$$

or, briefly,

$$\ddot{\xi} + k\dot{\xi} + c\xi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.16)$$

where the constants

$$k = \frac{AKg}{A\{M_3l_1^2 + (1+n)^2I_3 + I_4\} - n^2(1+n)^2I_3},$$

$$c = \frac{ACl_2^2g}{A\{M_3l_1^2 + (1+n)^2I_3 + I_4\} - n^2(1+n)^2I_3}.$$

The results obtained in Art. 42 show that here oscillations will be prevented if the viscous resistance is great enough to result in either $c < \frac{1}{4}k^2$ or $c = \frac{1}{4}k^2$. But in instances where $c > \frac{1}{4}k^2$ we

have damped harmonic vibrations represented by

$$\xi = De^{-\frac{1}{2}kt}\cos(p_1 t + \beta), \quad . \quad . \quad . \quad . \quad (61.17)$$

where the constants D, β depend on the initial circumstances of the motion, and $p_1{}^2 = c - \frac{1}{4}k^2$. In troublesome cases, then, the remedy lies in increasing the viscous resistance to such an extent that $c \ngtr \frac{1}{4}k^2$.

It is, of course, practicable to change the periodic time of oscillation by effecting appropriate modifications in the magnitude of the rotating parts of the system specified above, in accordance with equation (61.16).

The disturbed motion in question occasionally arises from want of accuracy or excessive wear on the teeth of the gearing, as is to be inferred from the treatment of the matter in Art. 116, and irregularities on the track sometimes form a contributory factor.

62. We here take leave of this introduction to the general theory of vibrations, where an attempt has been made to put the matter on as definite a basis as possible with special reference to systems having a finite number of degrees of freedom. To avoid, as far as may be, the suspicion of vagueness which sometimes attaches to the use of 'generalized co-ordinates' in engineering practice, the theory has frequently been explained by the aid of particular problems, with the result that a degree of prolixity may appear in the methods used for the purpose. It is, nevertheless, to be observed that the same procedure is involved in all cases, and once this is understood, Lagrange's method can be utilized without difficulty in questions of the type considered in the present work. The successive steps in a given problem of small oscillations about the equilibrium-configuration consist, briefly, in finding expressions for the kinetic and potential energies of the specified system, after which we proceed to the equation of motion for the variable q_r by way of the formula

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial F}{\partial \dot{q}_r} + \frac{\partial V}{\partial q_r} = Q_r,$$

in the notation of Art. 59. It has been shown that, with extraneous forces absent, this expression reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial F}{\partial \dot{q}_r} + \frac{\partial V}{\partial q_r} = 0$$

in the case of damped harmonic vibrations, and to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial V}{\partial q_r} = 0$$

in the simple case of free vibrations without friction. We have also seen that the consequences of resonance are to be expected if any one of the generalized components of force Q_r acts with a periodicity that approximates to the natural period of oscillation for any one of the principal parts of the structural system concerned.

CHAPTER IV

PROPAGATION OF STRESS IN ELASTIC MATERIALS

63. Plane Waves. An examination of the manner in which stress is transmitted through elastic materials will throw light not only on the significance of the phenomena to be considered in this chapter, but also on certain aspects of the problem of the 'continuous' type of structural system which is discussed in Chapters V and VI. The stress may be associated with various causes, such as the impact of the hammer on the head of a pile, earthquakes, or the unbalanced effects of reciprocating engines. It thus appears that we may have either discontinuous or continuous applications of a disturbing force on the material of piles, buildings and structures in general, or on any one of the several parts of an engine.

Let us first investigate the matter with reference to the bar of

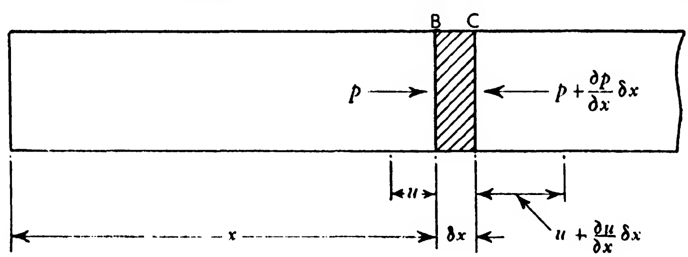


FIG. 100.

isotropic elastic material represented by Fig. 100, having a constant cross-sectional area A arranged symmetrically about the x-axis, and the ends plane and perpendicular to that axis. The left-hand end of the bar will be taken as the origin, with the positive direction to the right.

Now suppose a disturbing force of any kind to be applied uniformly over the plane $x = 0$ and of a magnitude such as to result in a stress p on the material. Since the value of p must not, for reasons already mentioned, violate Hooke's law of elasticity, it is implied in what follows that the consequent displacement remains small throughout a given disturbance. We shall therefore neglect, as of the second order of small quantities, the transverse strain associated with Poisson's ratio. No practical significance is attached

to the slight error thus introduced into the work, as is to be inferred from discussions on the subject.[1] It will make for further simplicity if, for the present, we assume that both the gravitational and the dissipative forces are negligibly small.

The prescribed disturbance will produce a uniform stress on the particles that lie in the plane $x = 0$, and considerations of continuity show that this state is transmitted to the adjacent particles. We can in this way imagine the stressed material as forming, at time t, the element shown hatched in Fig. 100, bounded by the normal planes at x and $x + \delta x$. These boundaries will, on the present assumptions, remain plane and perpendicular to the x-axis, because the dissipative forces are supposed to be of no account. If, then, the plane B be subjected to a stress p, it is evident that $-\left(p + \frac{\partial p}{\partial x}\delta x\right)$ represents the related stress on the plane C, whence the force on the element of cross-sectional area A is $A\left\{p - \left(p + \frac{\partial p}{\partial x}\delta x\right)\right\}$, i.e. $- A\frac{\partial p}{\partial x}\partial x$. This force naturally displaces the particles from their position of equilibrium, in the direction of x-positive, by an amount which will be referred to as the *shift*. If u denote the shift of the particles in the plane B, $-\left(u + \frac{\partial u}{\partial x}\delta x\right)$ is accordingly the shift of the particles in the plane C. It may be remarked that the symbol for 'partial' differentiation is required because the quantities p and u are both functions of x and t.

To proceed, write ρ for the density of the material, then $\frac{A\rho}{g}\delta x$ is the mass of the element, and $\frac{\partial^2 u}{\partial t^2}$ its acceleration. The dynamical equation is given by equating the product of mass and acceleration to the disturbing force, whence

$$-\frac{A\rho}{g}\delta x\frac{\partial^2 u}{\partial t^2} = A\frac{\partial p}{\partial x}\delta x,$$

or

$$-\frac{\rho}{g}\frac{\partial^2 u}{\partial t^2} = \frac{\partial p}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (63.1)$$

Turning next to the elastic properties of the material, it is clear that the difference between the shifts for the planes B and C amounts to $\left\{u - \left(u + \frac{\partial u}{\partial x}\delta x\right)\right\}$, i.e. $-\frac{\partial u}{\partial x}\delta x$. Hence, with e written for the

[1] Lord Rayleigh, *Theory of Sound*, vol. 1, Art. 157; L. Pochhammer, *J. für Math.* (*Crelle*), vol. 81, page 324 (1876); C. Chree, *Quart. J. Math.*, vol. 21, page 287 (1886), and vol. 24, page 340 (1890).

corresponding stretch of the material,

$$e = -\frac{\partial u}{\partial x}\frac{\delta x}{\delta x}.$$

But by definition we have also

$$p = Ee,$$

where E signifies the direct modulus of elasticity concerned, so that

$$p = -E\frac{\partial u}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (63.2)$$

in virtue of the previous equation. The presence of the negative sign indicates that the element is under compression as a result of the disturbance.

Consequently, on differentiating this expression with respect to x,

$$\frac{\partial p}{\partial x} = -E\frac{\partial^2 u}{\partial x^2},$$

which we may now insert in equation (63·1) and so derive

$$\frac{\partial^2 u}{\partial t^2} = \frac{Eg}{\rho}\frac{\partial^2 u}{\partial x^2}$$

$$= a^2\frac{\partial^2 u}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63.3)$$

where $a^2 = \frac{Eg}{\rho}$.

The general solution of this equation of motion is of the form

$$u = f(x - at) + F(x + at), \quad . \quad . \quad . \quad . \quad (63.4)$$

where f and F are arbitrary functions. To verify this, we notice

$$\frac{\partial u}{\partial t} = -af'(x - at) + aF'(x + at),$$

with $\quad f'(x - at) = \frac{\partial f(x - at)}{\partial(x - at)}$, etc.,

by reason of which

$$\frac{\partial^2 u}{\partial t^2} = a^2 f''(x - at) + a^2 F''(x + at);$$

and, in the same notation,

$$\frac{\partial u}{\partial x} = f'(x - at) + F'(x + at),$$

whence $\quad \frac{\partial^2 u}{\partial x^2} = f''(x - at) + F''(x + at).$

Therefore $\quad \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2},$

as was to be proved.

The interpretation of these results can be simply explained by considering first the case where F is zero, when

$$u = f(x - at),$$

by equation (63.4). Taking, as represented in Fig. 101, the values

$$u = s \text{ when } t = t_1 \text{ and } x = x_1,$$
$$u = s \text{ when } t = t_2 \text{ and } x = x_2,$$

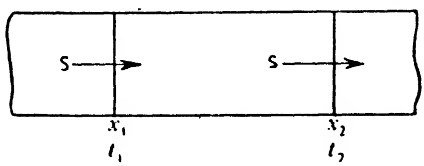


FIG. 101.

the rate of propagation of the disturbance is obviously given by

$$\frac{x_2 - x_1}{t_2 - t_1}.$$

In this connection we have also the relations

$$s = f(x_1 - at_1) \text{ and } s = f(x_2 - at_2),$$

which yield the information

$$x_1 - at_1 = x_2 - at_2,$$

so that

$$\begin{aligned} a &= \frac{x_2 - x_1}{t_2 - t_1} \\ &= \text{rate of propagation of the disturbance} \\ &= \sqrt{\frac{Eg}{\rho}}, \qquad (63.5) \end{aligned}$$

by equation (63.4).

Thus it appears that the disturbance advances unchanged with a constant velocity a in space, showing that $u = f(x - at)$ is a 'progressive wave' travelling with velocity a in the direction of x-positive. Our equation therefore describes a *plane wave,* since the shift u has the same value at all points in any plane x = const. The surface containing all the points for which u has the same value at a given instant is called the *wave-front,* so that our expressions relate to a wave having a plane front. Similarly, the equation

$$u = F(x + at)$$

refers to a progressive plane-wave travelling with velocity a in the direction of x-negative.

Since (63.4) is a *complete* solution of equation (63.3), it follows that any displacement or shift whatever, subject to the implied assumptions, may be regarded as made up of waves of these two

kinds. If, for example, the given bar were subjected to a periodic disturbance, a series of waves forming a *wave-train* would be initiated.

Seeing that the presence of these waves connotes a state of stress in the material concerned, we shall refer to them as *stress-waves*, to emphasize their significance for engineers.

It will be understood that the velocity of propagation a is independent of the *velocity of shift* $\left(\frac{\partial u}{\partial t}\right)$ of the particles, and that the arbitrary functions f and F in equation (63.4) depend on the boundary and initial conditions. The former of these velocities is fixed by the elastic properties of the material, whereas the latter is determined by the nature of the disturbing force.

In steel, for example, with $E = 30{,}000{,}000$ lb. per square inch and $\rho = 480$ lb. per cubic foot, equation (63.5) gives

$$a = \sqrt{\frac{30 \times 10^6 \times 144 \times 32{\cdot}2}{480}} \text{ ft. per sec.,}$$

or approximately 17,000 ft. per sec. Hence in the case of an engine operating at 2,400 r.p.m., with a connecting rod $10\frac{1}{2}$ in. long, the time required for a plane wave to travel, in the absence of dissipative agencies, from the gudgeon- to the crank-pin amounts to

$$\frac{10{\cdot}5}{12 \times 17{,}000} \text{ sec.,}$$

and this corresponds to an angular rotation

$$\frac{40 \times 10{\cdot}5 \times 360}{12 \times 17{,}000} \text{ deg.}$$

of the crank, say 0·74 deg.

64. As the quantity denoted above by a is sensibly the velocity with which sound is propagated in a specified material, its magnitude can be found by acoustical methods, and the following Table exhibits the mean values for various substances which have been thus obtained by different investigators.

Apart from its intrinsic value, this tabular matter makes manifest the fact that the velocity a is influenced by those factors which affect the modulus of elasticity of a material. With ordinary metals the value of the modulus of elasticity E undergoes sensible changes at the high temperatures now used in practice,[1] in consequence of which the velocity of propagation also exhibits corresponding variations. The modulus of rigidity is, on the contrary, practically constant for metals in normal conditions, as is demonstrated by the work of P. W. Bridgeman,[2] who found an increase of less than 6 per cent. in this modulus as a result of raising the pressure to 10,000 atmospheres.

[1] H. J. Tapsell, *Creep of Metals*.

[2] *Proc. Amer. Acad. Arts and Science*, vol. 63, page 401 (1929).

VELOCITY OF SOUND IN VARIOUS SOLIDS

Material.	Temp. in deg. Cent.	Velocity in ft. per sec.
Aluminium	—	16,740
Brass	—	11,480
Copper	20	11,670
,,	100	10,800
,,	200	9,690
Steel, soft	—	16,410
,, cast	20	16,360
,, ,,	200	15,710
Tin	—	8,200
Zinc	—	12,140
Brick	—	11,980
Cork	—	1,640
Glass	—	16,500–19,600
Granite	—	12,960
Marble	—	12,500
Rubber, vulcanized (black)	0	177
,, ,, ,,	50	103
Slate	—	14,800
Wax	17	2,890
,,	28	1,450
Ash, with the grain	—	15,310
,, against the rings	—	4,570
,, with the rings	—	4,140
Beech, with the grain	—	10,960
,, against the rings	—	6,030
,, with the rings	—	4,640
Elm, with the grain	—	13,516
,, against the rings	—	4,665
,, with the rings	—	3,324
Fir, with the grain	—	15,220
Oak, ,, ,, ,,	—	12,620
Pine, ,, ,, ,,	—	10,900

As regards wood, the above Table indicates that the magnitude of the velocity a varies with the relative direction of the grain; and this is true also of the shear modulus N which we have seen, in Ex. 3 of Art. 56, to be of primary importance in connection with the effect of earthquakes on buildings.

In the special case of such materials as concrete, masonry, and wood, tests show that the direct modulus of elasticity in tension (E_t) differs from that in compression (E_c). The point may be illustrated by reference to the experiments of A. N. Johnson,[1] who

[1] *Public Roads*, vol. 9, page 237 (1929).

obtained $E_t = 3{,}410{,}000$ lb. per square inch and $E_c = 3{,}840{,}000$ lb. per square inch for concrete; and $E_t = 2{,}920{,}000$ lb. per square inch and $E_c = 3{,}200{,}000$ lb. per square inch for mortar. These differences in the values of the direct modulus of elasticity, although comparatively small, may, according to this treatment, give rise to a phenomenon of interest to engineers, as will be pointed out in Chapter VII.

65. To deduce a convenient expression for the stress p, consider the disturbance defined by

$$u = f(x - at).$$

Here, by differentiation,

$$\frac{\partial u}{\partial x} = f'(x - at), \quad \frac{\partial u}{\partial t} = -af'(x - at)$$

in the previous notation, whence

$$\frac{\partial u}{\partial x} = -\frac{1}{a}\frac{\partial u}{\partial t}.$$

By the aid of equation (63.2) we can now write

$$\begin{aligned} p &= -E\frac{\partial u}{\partial x} \\ &= \frac{E}{a}\frac{\partial u}{\partial t} \\ &= \sqrt{\frac{E\rho}{g}}\frac{\partial u}{\partial t}, \qquad \qquad (65.1) \end{aligned}$$

from equation (63.5).

The stress thus brought about at any point in a given material consequently depends on the velocity of shift $\frac{\partial u}{\partial t}$, and is, of course, superposed on the stress due to static loads.

It is apparently possible for the value of p to be such as will induce a velocity of shift which exceeds the corresponding velocity of propagation. This condition would violate our assumptions, since it makes for *rupture* of the material in consequence of the fact that the energy of the wave is then momentarily concentrated within a very thin layer of the constituent particles. Notwithstanding this limitation to the theory, a practical acquaintance with the subject shows that the propagation of stress would then depend on the *plastic* properties of a given material, because the phenomenon of *flow* is the only means of relieving the stress thus produced. Therefore a material having physical properties comparable in this respect with those of copper, for example, may, without breaking, execute vibrations that involve an appreciable amount of plastic deformation. No practical purpose would be

served by generalizing on this point in the present work, for some materials exhibit remarkable plastic characteristics when subjected to fluctuating stresses while behaving as brittle substances under a constant load.

66. Pile Driving. An instructive application of the foregoing results is to be found in pile driving, provided that our suppositions remain valid throughout the operation. In short, we assume that an evenly distributed blow is given to the head of a pile made of isotropic elastic material, and that the motion is sensibly free from the effects of frictional forces.

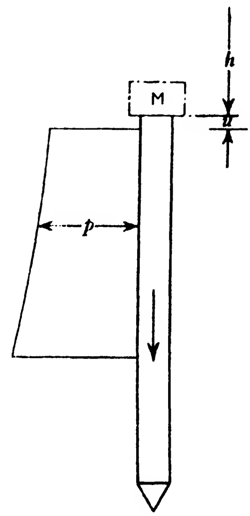


FIG. 102.

To fix ideas, let M be the weight of the hammer, falling freely through a vertical distance h on to the head of the pile indicated in Fig. 102. We shall suppose that the hammer comes into direct contact with the pile, and that the impact occurs at the instant $t = 0$.

If V denote the velocity of the hammer just before impact, then

$$V^2 = 2gh.$$

The particles in the uppermost layer of the material will be displaced at the instant of impact by an amount u which is now known to be related to the resulting stress p by the expression

$$p = \sqrt{\frac{E\rho}{g}}\frac{\partial u}{\partial t},$$

according to equations (65.1). The material of the pile is here specified by its direct modulus of elasticity E, and density ρ.

Writing A for the cross-sectional area of the pile, we have further, by Newton's second law, the equation of motion

$$M - pA = \frac{M}{g}\frac{\partial^2 u}{\partial t^2},$$

or

$$M - A\sqrt{\frac{E\rho}{g}}\frac{\partial u}{\partial t} = \frac{M}{g}\frac{\partial^2 u}{\partial t^2}.$$

For conciseness put $\dfrac{\partial u}{\partial t} = v$ and $\dfrac{A\sqrt{Eg\rho}}{M} = \beta$ in this expression, then it becomes

$$\frac{dv}{dt} = g - \beta v,$$

i.e.

$$\frac{dv}{g - \beta v} = dt,$$

the solution of which is readily shown to be

$$-\frac{1}{\beta}\log_e(g-\beta v)=t+b,$$

or
$$g-\beta v=C_1e^{-\beta t}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66.1)$$

where the constant C_1 depends on the initial circumstances of the motion.

Hence, reverting to the original notation, we obtain

$$\frac{\partial u}{\partial t}=\frac{g}{\beta}-\frac{1}{\beta}C_1e^{-\beta t},$$

which has the solution

$$u=\frac{g}{\beta}t+\frac{C_1}{\beta^2}e^{-\beta t}+C_2, \quad . \quad . \quad . \qquad (66.2)$$

with C_2 denoting a second arbitrary constant. If the initial conditions, namely

$$\frac{\partial u}{\partial t}=V \text{ and } u=0 \text{ when } t=0,$$

be substituted in equations (66.1) and (66.2), it will be found that

$$C_1=g-\beta V,\ C_2=-\frac{1}{\beta^2}C_1.$$

From these results we deduce

$$\begin{aligned} p&=\sqrt{\frac{E\rho}{g}}\frac{\partial u}{\partial t}\\ &=\frac{M}{A}\left(1-e^{-\frac{A\sqrt{Eg\rho}}{M}t}\right)+\sqrt{\frac{E\rho}{g}}Ve^{-\frac{A\sqrt{Eg\rho}}{M}t}, \quad . \quad . \quad . \qquad (66.3)\end{aligned}$$

for the stress at time t, so long as the pile goes freely through the soil.

In order to avoid excessive damage to the head of such a pile during the process of driving, the maximum value of p should not, of course, exceed the crushing strength of the material. Now

$$\frac{dp}{dt}=\sqrt{Eg\rho}\,e^{-\frac{A\sqrt{Eg\rho}}{M}t}-\frac{AE\rho}{M}Ve^{-\frac{A\sqrt{Eg\rho}}{M}t},$$

and this becomes negative if

$$\frac{AE\rho}{M}V>\sqrt{Eg\rho},$$

i.e. if
$$V>\frac{M\sqrt{Eg\rho}}{AE\rho},$$

so that the head of the pile will be unnecessarily damaged if

$$V>\frac{M}{A}\sqrt{\frac{g}{E\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66.4)$$

From this point of view the drop h is therefore conditioned by the last relation.

It is of interest to observe that if s denote the distance travelled

by the stress-wave in a given interval of time t, then equation (63.5) yields

$$s = \sqrt{\frac{Eg}{\rho}}\,t,$$

which can be introduced into equation (66.3) by way of expressing p as a function of s. The resulting expression affords a ready means of tracing the pressure-distance graph for a specified pile, in the form shown to the left in Fig. 102.

Ex. Find the greatest height from which a hammer of 2,000 lb. weight can be dropped without causing excessive damage to the head of a timber pile with a cross-sectional area of 1 square foot. The material has a crushing strength of 5,000 lb. per square inch, density of 40 lb. per cubic foot, and direct modulus of elasticity of 2,000,000 lb. per square inch.

According to equation (66.4) the maximum value of the stress p will occur at the start of the driving operation provided that at impact the velocity of the hammer

$$V > \frac{2{,}000\sqrt{32\cdot 2}}{\sqrt{2 \times 10^6 \times 144 \times 40}} \text{ ft. per sec.}$$

i.e. if $\quad V > 0\cdot 106$ ft. per sec.,

nearly.

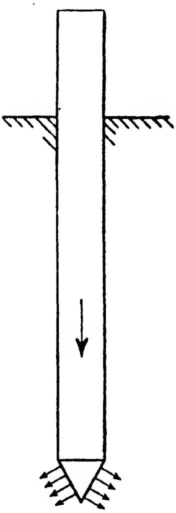


Fig. 103.

There are two possible conditions of motion which deserve notice here.

(i) *When the pile goes freely through the soil,* as indicated by Fig. 103, the maximum value of p takes place at the head of the pile when $t = 0$. But if crushing of the pile is to be avoided

$$p \not> 5{,}000 \times 144 \text{ lb. per square foot,}$$

hence, imposing this restriction on equation (65.1), namely

$$p = V\sqrt{\frac{E\rho}{g}},$$

it follows that the limiting value of

$$V = \frac{5{,}000 \times 144\sqrt{32\cdot 2}}{\sqrt{2 \times 10^6 \times 144 \times 40}} \text{ ft. per sec.}$$
$$= 38\cdot 0 \text{ ft. per sec.,}$$

in round numbers. To this degree of approximation the required drop

$$h = \frac{38^2}{64\cdot 4}$$
$$= 22\cdot 5 \text{ ft.,}$$

and this figure must not be exceeded if the damage to the pile is to be minimized.

(ii) *When the resistance to motion is such that the pile does not penetrate,* the stress-wave will be reflected back at the foot of the pile, as is shown in Art. 69. Assuming no dissipation of the energy, this reflection makes for a twofold increase in the initial value of p, when the maximum pressure on the material will amount to

$$2V\sqrt{\frac{E\rho}{g}}.$$

In these conditions the drop of the hammer should not exceed one-quarter of the previous value for h, say 5·6 ft.

It must not be supposed that these results are actually attained, or are even attainable. Many causes conspire to prevent this, and some of the causes of imperfection cannot be removed. The energy of stress-waves in general is, in fact, dispersed during the process of transmission, as will be explained later in this chapter.

67. Vibrations of a Light Wire. If Fig. 100 be taken to represent part of a wire-rope, it is now evident that any departure from the equilibrium condition or state of steady motion will initiate a stress-wave, having an intensity which is approximately expressed by equation (65.1). Besides such longitudinal disturbances, however, we may have to do with the transverse vibrations of wires. This problem deserves mention here because the longitudinal and transverse vibrations of a light wire are mathematically the same, so that the following results should help us to visualize the characteristics of waves in the motion of the particles involved in the previous work.

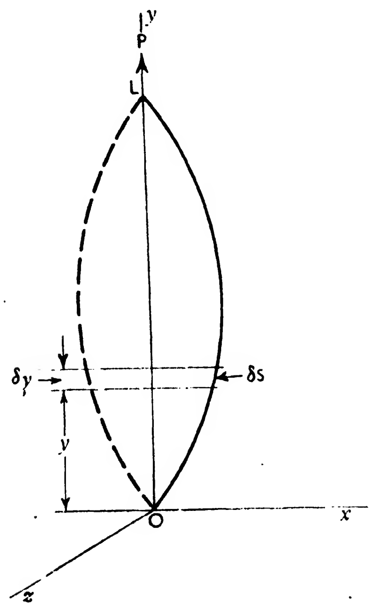


FIG. 104.

In Fig. 104 let the curve shown full indicate the disturbed configuration of a slender wire, having a line-density ρ, when executing small oscillations about the position of rest which we take to coincide with the y-axis. It will be supposed further that rigid end-attachments at $y = 0$ and $y = L$ in the figure exert a tension P when the wire is at rest, and that the disturbed wire always lies in the plane of the paper.

With reference to the element of length δs, at the point (x, y, z), the wire forms at any instant a continuous curve in space which is defined by

$$\frac{ds}{dy} = \left\{1 + \left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial y}\right)^2\right\}^{\frac{1}{2}},$$

where z and x both are functions of y and t. In the prescribed motion we may neglect, as of the second order of small quantities, the squares of the inclinations $\frac{\partial z}{\partial y}$ and $\frac{\partial x}{\partial y}$. To this degree of approximation $\frac{ds}{dy} = 1$; and the tension P will therefore remain sensibly constant throughout slight displacements about the y-axis.

The pull on the given element, the ends of which may be defined by y and $y + \delta y$ with respect to the y-axis, clearly consists of the tensions together with any extraneous forces which operate. Let the z- and x- components of these forces be $\frac{Z\rho}{g}\delta s$ and $\frac{X\rho}{g}\delta s$ in engineers' units, respectively. At the y-end of the element the corresponding components of the tension are $-P\frac{\partial z}{\partial s}$ and $-P\frac{\partial x}{\partial s}$, or $-P\frac{\partial z}{\partial y}$ and $-P\frac{\partial x}{\partial y}$ if the squares of the inclinations are neglected. Likewise, at the other end of the element the components of the tensions are

$$P\left\{\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2}\delta y\right\} \text{ and } P\left\{\frac{\partial x}{\partial y} + \frac{\partial^2 x}{\partial y^2}\delta y\right\},$$

taken in the same order. Thus it follows that the z- and x- components of the tension on the element are respectively $P\frac{\partial^2 z}{\partial y^2}\delta y$ and $P\frac{\partial^2 x}{\partial y^2}\delta y$. By equating the effective force to the product of mass and acceleration we may now write the equations of motion in the form

$$P\frac{\partial^2 z}{\partial y^2}\delta y + \frac{Z\rho}{g}\delta y = \frac{\rho}{g}\frac{\partial^2 z}{\partial t^2}\delta y$$

$$P\frac{\partial^2 x}{\partial y^2}\delta y + \frac{X\rho}{g}\delta y = \frac{\rho}{g}\frac{\partial^2 x}{\partial t^2}\delta y,$$

i.e.

$$\left.\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= a^2\frac{\partial^2 z}{\partial y^2} + Z,\\ \frac{\partial^2 x}{\partial t^2} &= a^2\frac{\partial^2 x}{\partial y^2} + X,\end{aligned}\right\} \quad . \quad . \quad (67.1)$$

where $a = \sqrt{\frac{Pg}{\rho}}$. It is easily inferred from the previous argument that the symbol a here defines the velocity with which the disturbance is propagated in the wire.

These equations are independent, and evidently refer to the *forced* oscillations which will be described under the influence of the impressed components of force Z and X. Hence the free vibrations are given by

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= a^2\frac{\partial^2 z}{\partial y^2}, \\ \frac{\partial^2 x}{\partial t^2} &= a^2\frac{\partial^2 x}{\partial y^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (67.2)$$

Since these are of the same type as equation (63.3), the solution obtained in Art. 63 consequently applies here. We may in this way imagine the full and dotted curves in Fig. 104 as representing the disturbed configuration of the wire.

Moreover, with the ends of the wire fixed at the points $y = 0$ and $y = L$ in Fig. 104, for all values of the time t

$$x = 0 \text{ at } y = 0, \; x = 0 \text{ at } y = L.$$

In view of these conditions the solution (63.4) shows that the relation

$$0 = f(L - at) - f(L + at)$$

holds throughout the vibratory motion. Hence if θ be written for $L - at$,

$$f(\theta) = f(2L - \theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (67.3)$$

showing that $f(\theta)$ is a periodic function which can, consequently, be expanded in a Fourier series. The period of the motion is therefore $2L$, this being the time a progressive wave would take to travel twice the length of the wire.

68. There is an alternative form of solution for this important type of equation which may be discussed with advantage at this stage of the treatment.

If for this purpose we take the second of the expressions (67.2) and, for convenience, write $x = \tau Y$, where τ and Y are separately functions of t and y alone, then the equation may be arranged in the form

$$\frac{\dfrac{d^2\tau}{dt^2}}{a^2\tau} = \frac{\dfrac{d^2Y}{dy^2}}{Y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68.1)$$

The left-hand side of this relation is now independent of y, whence the right-hand side also must be independent of that variable. If its constant value be represented by C, then we have

$$\frac{d^2Y}{dy^2} = CY, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68.2)$$

and, as can easily be verified, the solution

$$Y = Ae^{y\sqrt{C}} + Be^{-y\sqrt{C}}, \quad . \quad . \quad (68.3)$$

where the constants A, B depend on the boundary conditions.

For example, at the point $y = 0$ in Fig. 104 the displacement x is zero for all values of the time t, so that Y must vanish in similar circumstances. Therefore $A + B = 0$, whence

$$Y = A(e^{y\sqrt{C}} - e^{-y\sqrt{C}}).$$

Further, $Y = 0$ when $y = L$, so that $\sqrt{C}$ must be imaginary and equal to $\frac{in\pi}{L}$, where n is an integer and $i = \sqrt{-1}$; hence

$$Y = 2iA \sin \frac{n\pi y}{L}.$$

With this information we can determine τ by inserting the known value of C in equations (68.1) and (68.2); thus

$$\frac{d^2\tau}{dt^2} = -\frac{n^2\pi^2a^2}{L^2}\tau,$$

for which the solution is

$$\tau = F'_n \cos \frac{n\pi a}{L}t + G'_n \sin \frac{n\pi a}{L}t,$$

where the arbitrary constants F'_n, G'_n depend on the initial conditions of motion.

One solution of

$$\frac{\partial^2 x}{\partial t^2} = a^2\frac{\partial^2 x}{\partial y^2}$$

is, therefore,

$$x = \left(F_n \cos \frac{n\pi a}{L}t + G_n \sin \frac{n\pi a}{L}t\right) \sin \frac{n\pi y}{L}, \quad . \quad . \quad (68.4)$$

where $F_n = 2iAF'_n$, $G_n = 2iAG'_n$.

Now, bearing in mind the identities

$$2 \cos \frac{n\pi a}{L}t \sin \frac{n\pi}{L}y = \sin \frac{n\pi}{L}(at + y) - \sin \frac{n\pi}{L}(at - y),$$

$$2 \sin \frac{n\pi a}{L}t \sin \frac{n\pi}{L}y = \cos \frac{n\pi}{L}(at - y) - \cos \frac{n\pi}{L}(at + y),$$

we see, as is to be expected, that the solution (68.4) satisfies (63.4).

It is sometimes more convenient to arrange the solution (68.4) in the form

$$x = H_n \sin \frac{n\pi}{L}y \cos \left(\frac{n\pi a}{L}t + K_n\right), \quad . \quad . \quad . \quad (68.5)$$

where $H^2_n = F^2_n + G^2_n$, and $\tan K_n = -\frac{G_n}{F_n}$. In both cases x varies as a simple-harmonic function of the time, signifying that our results relate to a *normal mode* of vibration.

Returning to the wire indicated in Fig. 104, it can be shown,

on expanding the solution (67.3) in a Fourier sine-series, that here

$$x = \sum_{n=1}^{\infty}\left(F_n \cos \frac{n\pi a}{L}t + G_n \sin \frac{n\pi a}{L}t\right) \sin \frac{n\pi}{L}y \quad . \quad (68.6)$$

is the most general form of the solution given by equation (68.4). If, on the one hand, the wire starts from rest at time $t = 0$, every G_n-term must vanish by reason of the condition $x = 0$ initially; and if, on the other hand, the wire starts from its equilibrium-position with specified velocities at time $t = 0$, then every F_n-term must vanish.

For the normal mode implied in equation (68.4) it is evident that the kinetic energy

$$T = \tfrac{1}{2}\frac{\rho}{g}\int_0^L \left(\frac{\partial x}{\partial t}\right)^2 dy$$

$$= \frac{n^2\pi^2a^2}{4L}\frac{\rho}{g}\left(-F_n \sin \frac{n\pi a}{L}t + G_n \cos \frac{n\pi a}{L}t\right)^2 \quad . \quad (68.7)$$

To obtain an expression for the corresponding potential energy V, on the assumption that there is no dissipation of energy, we equate V to the work done in stretching the wire. As the work done on the element of infinitesimal length δs in the prescribed motion is

$$P\left(\frac{ds}{dy} - 1\right)\delta y,$$

which is equal to $P\left[\left\{1 + \left(\frac{\partial x}{\partial y}\right)^2\right\}^{\frac{1}{2}} - 1\right]\delta y,$

or

$$\tfrac{1}{2}P\left(\frac{\partial x}{\partial y}\right)^2 \delta y,$$

to the present degree of approximation we have

$$V = \tfrac{1}{2}P\int_0^L \left(\frac{\partial x}{\partial y}\right)^2 dy$$

$$= \frac{n^2\pi^2}{4L}P\left(F_n \cos \frac{n\pi a}{L}t + G_n \sin \frac{n\pi a}{L}t\right)^2, \quad . \quad (68.8)$$

from equation (68.4). Consequently the total energy

$$T + V = \frac{n^2\pi^2}{4L}P(F^2_n + G^2_n),$$

since $a = \sqrt{\dfrac{Pg}{\rho}}$.

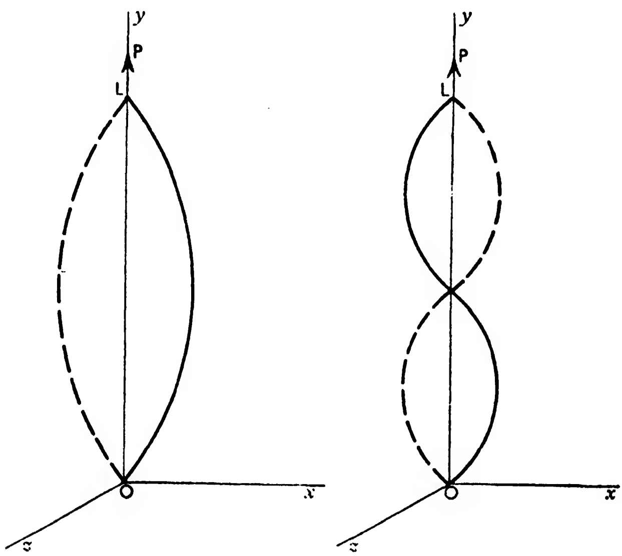
Similarly, in the general equation (68.6), the total energy of the wire is given by

$$T + V = \frac{\pi^2}{4L}P\sum_{n=1}^{\infty} n^2(F^2_n + G^2_n). \quad . \quad . \quad . \quad . \quad . \quad (68.9)$$

When a system of this type is executing *forced* oscillations, the

condition to be avoided is that of approximate equality between the frequency of the impressed force and the natural frequency of the wire, for a state of resonance will bring about large displacements if the frictional forces are comparatively small in magnitude. In this connection it is worth noting that a periodic force applied at any point on a stretched wire will not excite any fundamental mode which has a *node* at that point, even in a state of synchronism.

Ex. 1. Examine, on the foregoing assumptions, the transverse oscillations in a normal mode for the slender wire shown in Fig. 105,



(*a*) (*b*)

FIG. 105.

fixed at its ends $y = 0$, $y = L$, and starting initially from rest, under a tension P.

Assuming the small vibrations to take place in the xOy plane, we have seen that the G_n-terms in equation (68.6) must vanish in the given conditions, whence the displacement of the wire

$$x = \sum_{n=1}^{\infty} F_n \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} y,$$

where the velocity of propagation $a = \sqrt{\frac{Pg}{\rho}}$ for a wire with a line-density ρ.

In the fundamental mode, corresponding to the value $n = 1$, we thus obtain

$$x = F_1 \cos \frac{\pi a}{L} t \sin \frac{\pi}{L} y,$$

indicating that in this mode the graph of x is a half-period sine curve, as represented by Fig. 105(a). The wire may be imagined as then vibrating between the limits described by the full and dotted curves, with a frequency

$$\frac{a}{2L}, \text{ i.e. } \frac{1}{2L}\sqrt{\frac{Pg}{\rho}}.$$

It is here worth drawing attention to the result that the frequency of oscillation for a wire of specified material varies directly as the square root of the tension (P) and inversely as the length (L). We can therefore alter the frequency by effecting suitable changes in these variables, and so prevent excessive vibratory movement of the wires and cables used in telegraph and power-transmission systems. In such cases, however, the weight of the wire modifies the disturbance, to an extent which can be estimated by the method of the following example.

To determine the next mode we write $n = 2$ in the above expression for x, and so deduce that in this case

$$x = F_2 \cos\frac{2\pi a}{L}t \sin\frac{2\pi}{L}y,$$

whence it is clear that the graph of x is a complete sine curve. Hence Fig. 105(b) may be regarded as representing the extreme positions of the wire, where the point $x = \frac{L}{2}$, like the fixed ends, is a *node*, being always at rest; and the maximum displacement takes place at the *loop* or mid-point between a pair of nodes.

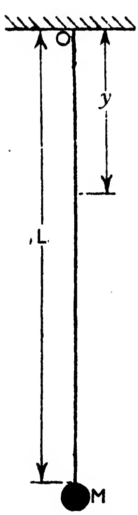


FIG. 106.

Thus it is seen that there are $n - 1$ nodes in the nth mode of vibration, when the length of wire between any two nodes oscillates in the fundamental mode associated with the length $\frac{L}{n}$. The *wave-length* in any mode is easily found with the help of the general relation

velocity of propagation = wave-length × frequency.

Finally, an application of the principle of superposition at once reveals the combined effect of any number of such modes.

Ex. 2. Investigate the longitudinal vibrations in a natural mode about the position of rest for the system represented in Fig. 106, consisting of a wire of length L, to the lower end of which is attached a mass of weight M. The wire, which is fixed at its upper end $y = 0$, has a cross-sectional area A, and is made of

material with a density ρ and a direct modulus of elasticity E. It may be assumed that the frictional forces are negligibly small.

Let us first neglect the weight of the wire, and write v for the vertical shift of a constituent particle, then the equation of motion is

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial y^2},$$

where $a = \sqrt{\frac{Eg}{\rho}}$, as in equation (63.3).

If the wire starts from rest, it follows from equation (68.4) that in the *fundamental mode* the shift

$$v = F \cos \frac{\pi a}{L} t \sin \frac{\pi}{L} y,$$

where F is an arbitrary constant. To our degree of approximation the stress-waves accordingly execute simple-harmonic motion with a period $\frac{2L}{a}$.

Further, if p denote the stress induced in the wire by the disturbed mass, this will result in a total force pA on the sectional area A. But, by equation (63.2),

$$p = -E\frac{\partial v}{\partial y},$$

which means that the disturbing force

$$\begin{aligned} pA &= -EA\frac{\partial v}{\partial y} \\ &= -\frac{EAF\nu}{a} \cos \nu t \cos \frac{\nu}{a} y, \end{aligned}$$

where the pulsatance $\nu = \frac{a}{2L}$. This relation, as remarked in Art. 66, can be used for tracing either a stress-time or a stress-distance graph of the disturbed motion.

Account may now be taken of the weight of the wire, by a method which is applicable also to the transverse oscillations of such a system, since equations of the type (67.2) hold in both kinds of motion.

The last equation shows that at the lower end of the wire, where $y = L$, the disturbing force amounts to

$$-\frac{EAF\nu}{a} \cos \nu t \cos \frac{\nu}{a} L.$$

But this quantity is obviously equal to the product of mass and acceleration of the suspended weight, so that

$$\frac{M}{g}\frac{\partial^2 v}{\partial t^2} = -\frac{M}{g}F\nu^2 \cos \nu t \sin \frac{\nu}{a}L$$

at the point $y = L$. We obtain in this way

$$\frac{AEg}{M} = \nu a \frac{\sin \frac{\nu}{a}L}{\cos \frac{\nu}{a}L},$$

or

$$\frac{AEgL}{Ma^2} = \beta \tan \beta,$$

where $\beta = \frac{\nu}{a}L$,

i.e.

$$\frac{AL\rho}{M} = \beta \tan \beta,$$

since $a^2 = \frac{Eg}{\rho}$. This relation serves to determine the pulsatance ν, and it can be solved either by graphical means or by the method of successive approximation.

To explain the latter procedure with reference to the case where $\beta^2 < \frac{\pi^2}{4}$, use may be made of the known expansion for $\tan \beta$, since it gives

$$\frac{AL\rho}{M} = \beta(\beta + \tfrac{1}{3}\beta^3 + \tfrac{2}{15}\beta^5 + \tfrac{17}{315}\beta^7 + \ldots)$$

$$= \beta^2(1 + \tfrac{1}{3}\beta^2 + \tfrac{2}{15}\beta^4 + \tfrac{17}{315}\beta^6 + \ldots).$$

Hence, by reason of the fact that

$$\frac{AL\rho}{M} = \frac{\text{total weight of the wire}}{\text{weight of the suspended mass}}$$

$$= \alpha,$$

say, we have $\quad \alpha = \beta^2(1 + \tfrac{1}{3}\beta^2 + \tfrac{2}{15}\beta^4 + \tfrac{17}{315}\beta^6 + \ldots).$

This expansion enables us to include, in instances where β is sufficiently small, the weight of the wire, by the method of successive approximation. If the weight of the wire is small compared with that of the suspended mass, we may write $\alpha = \beta^2$, and obtain a first approximation in the form

$$\frac{AL\rho}{M} = \frac{\nu^2}{a^2}L^2,$$

i.e.

$$\nu = \sqrt{\frac{AEg}{ML}}$$

$$= \sqrt{\frac{g}{\Delta}},$$

where Δ is the stretch which would be produced in the wire by a *static* load M, namely $\frac{ML}{AE}$.

To improve upon this result, we next take the first two terms of the above series, and utilize the first approximation by putting $\alpha = \beta^2$. Thus

$$\alpha = \beta^2(1 + \tfrac{1}{3}\alpha),$$

whence
$$\beta = \sqrt{\frac{\alpha}{1 + \frac{1}{3}\alpha}},$$

in consequence of which the new value is

$$\nu = \frac{a}{L}\sqrt{\frac{\alpha}{1 + \frac{1}{3}\alpha}}$$

$$= \sqrt{\frac{g}{(1 + \frac{1}{3}\alpha)\Delta}}.$$

Therefore the mass of the wire may be accounted for, to this degree of accuracy, by adding one-third of its weight to the quantity denoted here by M. It is readily verified that our result is within 1 per cent. of the true value given by the significant root of the equation $\alpha = \beta \tan \beta$ if the weight of the wire is not greater than that of the suspended mass.

A more exact approximation to the frequency of vibration can be obtained, on the present suppositions, by introducing the first three terms of the expansion for $\tan \beta$, but this extension of the method commonly involves lengthy calculations which cannot always be justified.

In instances where a system of this type is vibrating either longitudinally or transversely in the nth mode, we may use equation (68.6) in the general theory of Chapter III, since equations (68.7) and (68.8) then completely express the kinetic and potential energies, to the present degree of accuracy.

This problem illustrates the magnitude and motion of the stress-waves which are induced in *winding ropes* in service, when the disturbing force may be associated with the retarding effect of the brakes on the drum, or the impact produced by dropping the load into a skip at the bottom of a shaft. In this connection it is to be remarked that the periodic time of a winding rope is not uniform throughout a 'wind' owing to the change in the tension P which is brought about by inevitable variations in the angular velocity of the winding drum, as well as to the change in free length of the rope. A graphical method is generally most convenient to use in instances where the weight of the wire greatly exceeds that of the loaded cage or skip, signified by M in the foregoing discussion.

When a stress-wave arrives at the end of a wire, or meets with an abrupt change in the line-density of the material, the disturbance suffers partial reflection, in a manner which we are now in a position to examine.

69. Reflection. We have, so far, confined our attention to the propagation of waves in a material having constant values for the modulus of elasticity and the density. There are, however, instances where the path of such disturbances passes through materials for which the elastic properties vary from point to point, or change suddenly in the direction of propagation. These conditions are exemplified in turn by *case-hardened* metals, and the interface of the end of a bar and the surrounding air.

Now it appears from the linearity of our approximate equations of motion that, in the case of sufficiently small amplitudes of oscillation, any number of independent solutions may be superposed. For example, in Fig. 100 with a wave of given form travelling in one direction, if we superpose its *image* in the plane $x = 0$, travelling in the opposite direction, it is evident that in the resulting motion the horizontal velocity will vanish at the origin, and the conditions are the same as if there were a rigid and fixed barrier at that place. We can thus understand how the original wave suffers *reflection* at the specified barrier, the horizontal velocity being reversed and associated with the superposed wave, while the 'crests' and 'troughs' are reflected unchanged.

Since a rigid barrier is an extreme kind of discontinuity, let us now consider the practical case of the propagation of stress in the bar shown in Fig. 107, of unit cross-sectional area, and composed of two materials with different elastic properties. Suppose further

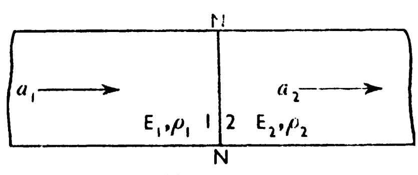


FIG. 107.

that these materials are separated by the plane interface NN, taken as perpendicular to the direction of propagation of the stress. If the suffixes 1 and 2 be assigned to the elastic properties of the materials on the left- and right-hand sides of NN, respectively, by virtue of equation (63.3) we have

$$a_1 = \sqrt{\frac{E_1 g}{\rho_1}}, \quad a_2 = \sqrt{\frac{E_2 g}{\rho_2}},$$

where, as usual, the direct modulus of elasticity is denoted by E, and the density by ρ. In what follows it will be assumed that the

materials are separately isotropic and homogeneous, and that the dissipative forces are negligibly small.

Here energy in the form of stress-waves can be transmitted across the interface NN, since it is an elastic boundary.

If, to fix ideas, we let the initial wave-front be plane and perpendicular to the axial direction of the bar and, therefore, parallel to NN, these conditions will hold throughout the slight disturbance to be examined, seeing that dissipative agencies are absent. Thus with u_A written for the shift of the particles in the plane at x, and time t, which is associated with a stress-wave advancing from the left towards NN, we have, from equation (63.4),

$$u_A = f(x - a_1 t), \quad . \quad . \quad . \quad . \quad . \quad (69.1)$$

where f is an arbitrary function.

It has been noticed that a fraction of the wave-energy will in general be reflected at the plane NN, and if the consequent shift of the particles be represented by u_B, then we may write

$$u_B = F(x + a_1 t), \quad . \quad . \quad . \quad . \quad . \quad (69.2)$$

where F is an arbitrary function. It will be understood that the phenomenon of reflection is secured by this change of sign.

The remaining part of the original energy will, on the present suppositions, be transmitted across the interface, in the direction of the initial disturbance. Hence if the related shift of the particles be u_C, it may be defined by

$$u_C = \phi(x - a_2 t), \quad . \quad . \quad . \quad . \quad . \quad (69.3)$$

where ϕ is an arbitrary function.

These phenomena will together give rise to a stress p which is determined by equation (63.2), namely

$$p = -E\frac{\partial u}{\partial x},$$

combined with the interfacial conditions:

(i) a common stress on the interface, and

(ii) a common displacement in that plane for each particle at any instant t.

In the work of differentiating the above expressions we may conveniently write

$$A = f'(x - a_1 t),$$
$$B = F'(x + a_1 t),$$
$$C = \phi'(x - a_2 t).$$

The condition (i) gives $\quad p_A + p_B = p_C,$

whence $\quad AE_1 + BE_1 = CE_2,$

i.e. $$A + B = C\frac{E_2}{E_1} \quad . \quad . \quad . \quad . \quad . \quad (69.4)$$

Similarly, the condition (ii) gives

$$u_A + u_B = u_C,$$

i.e.
$$\frac{\partial u_A}{\partial t} + \frac{\partial u_B}{\partial t} = \frac{\partial u_C}{\partial t},$$

whence $\quad - a_1 f'(x - a_1 t) + a_1 F'(x + a_1 t) = - a_2 \phi'(x - a_2 t),$
or $\quad - a_1 A + a_1 B = - a_2 C,$

and this reduces to

$$- A + B = - C\frac{a_2}{a_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (69.5)$$

Hence, from equations (69.4) and (69.5),

$$2B = C\left(\frac{E_2}{E_1} - \frac{a_2}{a_1}\right)$$

follows by addition, and

$$2A = C\left(\frac{E_2}{E_1} + \frac{a_2}{a_1}\right)$$

by subtraction, so that in terms of A

$$C = A\frac{2E_1 a_1}{E_2 a_1 + E_1 a_2},$$

$$B = A\left(\frac{E_2 a_1 - E_1 a_2}{E_2 a_1 + E_1 a_2}\right).$$

If these values be now inserted in the relations between the stresses p_A, p_B, p_C and the corresponding shifts u_A, u_B, u_C, which obviously are

$$\frac{p_B}{p_A} = \frac{B}{A}, \frac{p_C}{p_A} = \frac{C}{A}\frac{E_2}{E_1},$$

we obtain

$$\frac{p_B}{p_A} = \frac{E_2 a_1 - E_1 a_2}{E_2 a_1 + E_1 a_2}$$

$$= \frac{\dfrac{E_2}{a_2} - \dfrac{E_1}{a_1}}{\dfrac{E_2}{a_2} + \dfrac{E_1}{a_1}},$$

$$\frac{p_C}{p_A} = \frac{2E_1 a_1}{E_2 a_1 + E_1 a_2} \cdot \frac{E_2}{E_1}$$

$$= \frac{2\dfrac{E_2}{a_2}}{\dfrac{E_2}{a_2} + \dfrac{E_1}{a_1}}.$$

Therefore, remembering that $\dfrac{E_1}{a_1} = \sqrt{\dfrac{E_1 \rho_1}{g}}$ and $\dfrac{E_2}{a_2} = \sqrt{\dfrac{E_2 \rho_2}{g}}$, the

reflected stress

$$p_B = \frac{\sqrt{E_2\rho_2} - \sqrt{E_1\rho_1}}{\sqrt{E_2\rho_2} + \sqrt{E_1\rho_1}} p_A, \quad . \quad . \quad . \quad . \quad (69.6)$$

and the transmitted stress

$$p_C = \frac{2\sqrt{E_2\rho_2}}{\sqrt{E_2\rho_2} + \sqrt{E_1\rho_1}} p_A$$

$$= \frac{2}{1 + \sqrt{\dfrac{E_1\rho_1}{E_2\rho_2}}} p_A \quad . \quad . \quad . \quad . \quad . \quad (69.7)$$

These equations afford a means of calculating the approximate magnitudes of the stresses which we have in this way associated with the reflected and transmitted waves at the interface of different materials. With materials of the prescribed kind no reflection will take place if $\sqrt{E_2\rho_2} = \sqrt{E_1\rho_1}$; and only a small fraction of the initial stress p_A will be transmitted across the interface if $\sqrt{\dfrac{E_1\rho_1}{E_2\rho_2}}$ is very large in value.

Had the plane NN in Fig. 107 referred to an abrupt change in the cross-sectional area of a bar made of the same material throughout, the phenomena of reflection and transmission would likewise occur at the section NN, as is demonstrated in the example of Art 75. Thus it appears that under certain conditions the reflected wave will make for failure near one of the ends of a specimen subjected to periodic stress, owing to the effect of the 'interface' formed by the test-piece and the grips of the machine. The matter therefore has a bearing on the testing of metals for *fatigue*.

For most practical purposes the foregoing results hold in the transmission of stress through materials of construction in general, though in special cases due attention must be given to the relative values of the amplitude and the wave-length compared with the dimensions of the bar or structural member concerned.

70. Reinforced Concrete. This method of construction presents another instructive application of the preceding treatment, owing to the presence of the interface of steel and concrete. If the foregoing suffixes 1 and 2 be ascribed to the elastic properties of the steel and the concrete, taken in order, and the plane NN in Fig. 107 be regarded as a boundary between the two materials, we can at once write down the equations for the type of disturbance examined in Art. 69.

To discuss the problem on a numerical basis, let

$E_1 = 30{,}000{,}000$ lb. per square inch, $\rho_1 = 480$ lb. per cubic foot. for the steel, and

$E_2 = \frac{1}{12}E_1$, $\rho_2 = 145$ lb. per cubic foot.

for the concrete, then, in pound-foot units,

$$\sqrt{E_1 g \rho_1} = 1{,}000(30 \times 144 \times 32{\cdot}2 \times 480)^{\frac{1}{2}}$$
$$= 12K,$$
$$\sqrt{E_2 g \rho_2} = 1{,}000(\tfrac{30}{12} \times 144 \times 32{\cdot}2 \times 145)^{\frac{1}{2}}$$
$$= 1{\cdot}9K,$$

where $K = 683{,}000$ in round numbers.

It is of some interest to notice that, according to equation (63.3), these values correspond to the velocities of propagation

$a_1 = 17{,}000$ ft. per sec. in the steel, and
$a_2 = 8{,}940$ ft. per sec. in the concrete.

For a disturbance of the prescribed type, which will be taken to originate as a compressional wave, we may examine the three possible cases in the following manner, on the supposition that the frictional agencies are insignificant.

(i) *Steel to Concrete.* If p_A denote the stress of the initial wave in the steel, substitution in equation (69.6) of the above data shows that the 'reflected' stress

$$p_B = \frac{1{\cdot}9 - 12}{1{\cdot}9 + 12} p_A$$
$$= -\,0{\cdot}727 p_A,$$

approximately. The negative sign here signifies that the reflected part of the energy produces a state of *tension* in the steel.

As regards the stress carried by the transmitted part of the energy, equation (69.7) gives

$$p_C = \frac{2 \times 1{\cdot}9}{1{\cdot}9 + 12} p_A$$
$$= 0{\cdot}273 p_A,$$

whence only about one-quarter of the original stress is propagated in the concrete.

(ii) *Concrete to Steel.* The corresponding stresses are evidently given by interchanging the suffixes 1 and 2 in (i). Thus we see that a compressional wave is now reflected back into the concrete, with a stress

$$p_B = \frac{12 - 1{\cdot}9}{13{\cdot}9} p_A$$
$$= 0{\cdot}727 p_A\,;$$

and a similar wave is transmitted to the steel, with a stress

$$p_C = \frac{2 \times 12}{13{\cdot}9} p_A$$
$$= 1{\cdot}727 p_A.$$

Therefore no change occurs in the type of disturbance, but the transmitted part of the energy produces a stress at the interface which is nearly 75 per cent. greater than the initial value p_A.

We can in this way understand the reflection of stress-waves at the toe of a concrete pile when it encounters hard ground or rock in the process of driving. A comparatively high stress is then involved, according to these calculations, and this makes for failure by crushing at the point of such piles, particularly when driving short piles by the usual procedure.

(iii) *Concrete to Air.* A process of transmission under approximately adiabatic conditions takes place when a wave-front arrives at the boundary of the concrete and the surrounding air.

To distinguish the elastic properties of the two media, we may assign our suffixes 1 and 2 in succession to the concrete and the air. From the above data we have

$$\sqrt{E_1 g \rho_1} = 1{\cdot}297 \times 10^6 \text{ lb.-ft.}$$

for the concrete. With air specified by its adiabatic elasticity K, and density ρ_2, we require also the quantity

$$\sqrt{K g \rho_2},$$

which is sensibly equal to 76 lb.-ft. for a temperature of 60 deg. F.

On substituting these values in equations (69.6) and (69.7), we find

$$\begin{aligned} p_B &= \frac{76 - 1{\cdot}297 \times 10^6}{76 + 1{\cdot}297 \times 10^6} p_A \\ &= -p_A, \end{aligned}$$

nearly, and

$$p_C = \frac{2 \times 76}{76 + 1{\cdot}297 \times 10^6} p_A.$$

Hence the reflected part of the disturbance produces a state of *tension* in the concrete, with a stress that is almost equal to the incident stress; and only a very small fraction of the original stress is propagated in the surrounding air.

A practical significance is attached to the result that a state of tension here exists near the surface of the concrete, for this is liable to cause cracking of a material that is comparatively weak in tension. If a charge of powder be ignited at the left-hand end of the concrete column shown in Fig. 108(a), we should on this account expect Fig. 108(b) to represent the final condition of the column. Our treatment demonstrates that partial reflection will take place each time a wave-front reaches the right-hand end of the column, and the consequent tensile stress on the concrete is, in the absence of dissipative agencies, nearly doubled in intensity when two such waves pass each other in opposite directions. This cumulative

process may subsequently result in a stress sufficiently great to cause fracture at a place which in general depends on the nature and magnitude of the initial stress-wave, the relative degree of damping, and the length of the column. Each repetition of these circumstances thus leads to a piece of the column breaking away

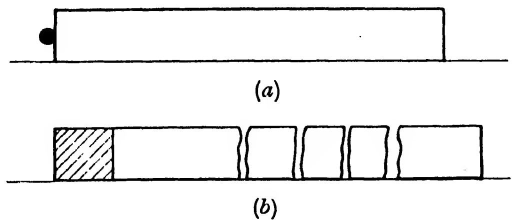


FIG. 108.

from the right-hand end, as indicated in the figure, which approximately illustrates what actually happens. The hatched part of the diagram refers to the material which is 'pulverized' by the impact.

This aspect of the matter has an obvious bearing on the design of reinforced-concrete structures subject to gunfire and bombardment. Strictly speaking, the *spherical* type of wave, to be discussed in Art. 76, will be involved in propagating the effect of a shell on the structure shown in Fig. 109. The preceding results nevertheless apply in these conditions, to the extent of explaining why the interior

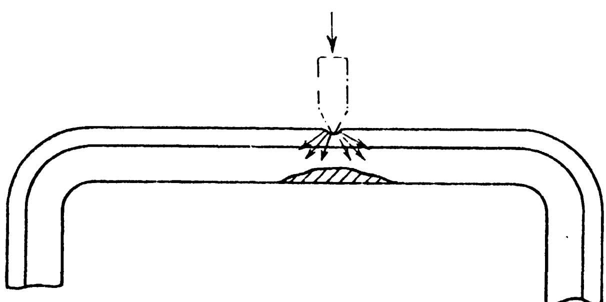


FIG. 109.

part shown hatched in the figure falls away on account of the tensile stress thus induced in the concrete. It is readily understood that in such buildings the reinforcement bars should be arranged with the metal distributed as uniformly as may be with respect to the horizontal axes of symmetry. That is to say, solid floor-slabs reinforced in both directions should be used, and full advantage taken of the 'mushroom' type of construction. Since projectiles may strike a building in various directions, all the columns should be strengthened against torsion, and all the beams against shear.

In one sense this problem of reinforcement really extends to three dimensions.

Turning, for a moment, to the interface of the concrete and steel in buildings generally, it is to be remarked that for *spherical* waves the velocities of propagation of stress are approximately 9,400 ft. per sec. in concrete, and 20,000 ft. per sec. in steel. If we take a disturbance of this type to be travelling in the concrete and steel, in a direction perpendicular to the plane of the paper with reference to Fig. 110, it is plain that the difference between these velocities will cause a corresponding *difference in phase* of the waves on each side of the interface. This gives rise to a differential movement which tends to loosen a reinforcement bar, and thereby weaken the structure as a whole.

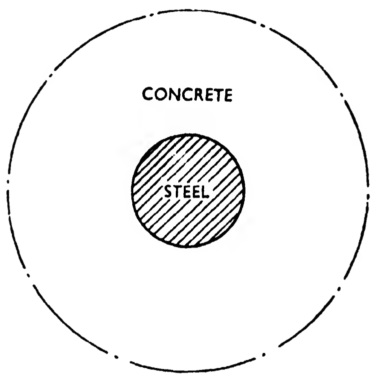


FIG. 110.

The consequences of the inherent dissipation of the wave-energy will be investigated presently, but it may be said here that in actual structures the intensity of the stress is gradually diminished by frictional agencies. This is sometimes utilized by constructing *forts* of non-reinforced concrete, and making the sections so thick that the dissipative forces shall reduce the intensity of the 'reflected' stress below the tensile strength of concrete.

71. Dispersion and Distortion. We now proceed to take account of the fact that a certain amount of the wave-energy is dissipated in the transmission of stress through actual materials, due to internal friction and other causes. Under these conditions a complex wave suffers decomposition in its course through a stiff bar or beam, by reason of which the velocity of propagation is not independent of the wave-length, or, what amounts to the same thing, the velocity is not independent of the frequency of oscillation.

The consequences of this modification in the foregoing treatment will be sufficiently explained if we imagine the disturbed element of Art. 63 to be subject to a frictional force whose magnitude is proportional to the velocity of shift of the constituent particles. Then we may write the equation of motion in the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial u}{\partial t}, \quad . \quad . \quad . \quad . \quad (71.1)$$

where b is always negative. Previous considerations show that this expression relates to a damped wave motion in which the amplitude

decreases steadily as the time increases, the positive constant $-b$ being the *damping factor.* It is convenient, in order to simplify this discussion, to change the dependent variable by means of a transformation of the type

$$u = Ue^{rt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (71.2)$$

Making this substitution in equation (71.1) and dividing throughout by e^{rt}, we obtain

$$\frac{\partial^2 U}{\partial t^2} + 2r\frac{\partial U}{\partial t} + r^2 U = a^2\frac{\partial^2 U}{\partial x^2} + 2brU + 2b\frac{\partial U}{\partial t}.$$

If we now put $r = b$, the terms involving the first derivative with respect to t cancel, and the expression reduces to

$$\frac{\partial^2 U}{\partial t^2} = a^2\frac{\partial^2 U}{\partial x^2} + b^2 U \quad . \quad . \quad . \quad . \quad . \quad (71.3)$$

A solution of this equation is, as can readily be verified,

$$U = A \sin (rx - st), \quad . \quad . \quad . \quad . \quad . \quad (71.4)$$

provided $s^2 = a^2r^2 - b^2$ holds. This proviso implies that the damping factor depends on the frequency, and involves the phenomenon known as *distortion* in which waves of different frequencies are damped unequally. The wave-front will therefore change in form or character as it advances.

It may make the matter more intelligible if we here consider a solution of the type

$$u = Ue^{-i\omega t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (71.5)$$

with U a function of x alone, and $i = \sqrt{-1}$. By equation (71.1) we must have

$$\frac{\partial^2 U}{\partial x^2} = -U\left(\frac{\omega^2 - 2i\omega b}{a^2}\right).$$

This equation of course yields two solutions, but only one of them has a practical significance, namely that which enables u to remain finite as x tends to very large values. It thus appears, with $(\alpha + i\beta)^2$ written for $\frac{\omega^2 - 2i\omega b}{a^2}$, and $\beta > 0$, that this solution is

$$U = Be^{i(\alpha + i\beta)x},$$

or, by virtue of equation (71·5),

$$u = Be^{-\beta x}e^{i(\alpha x - \omega t)} \quad . \quad . \quad . \quad . \quad . \quad (71.6)$$

In view of the fact that β now refers to the damping factor, and ω to the pulsatance of oscillation, we see that both α and β are functions of the frequency, since $\alpha^2 - \beta^2 = \frac{\omega^2}{a^2}$ and $\alpha\beta = -\frac{\omega b}{a^2}$. Moreover, the statement that α is a function of the frequency indicates the phenomenon of *dispersion*, and the statement that β is a function of the frequency indicates *distortion.* On this account

the energy is dispersed and the form of the wave varies in the process of transmission.

With the aid of these results we can trace the motion of a given stress-wave and, on the implied assumptions, visualize the changes revealed by the analysis. The presence of an interface of different materials would naturally give rise to the phenomenon of partial reflection discussed in Art. 69.

An important example of the combined effect of these phenomena is to be found in the connecting rod of, say, an internal combustion engine, where the working fluid initiates a stress-wave that is transmitted through the rod by way of the white-metal bearing and oil film to the crank-pin. Actually, as already pointed out, the frictional term does not in general vary in accordance with the simple law (71.1), but it is always feasible to use that equation if b expresses the mean value of the frictional term for a limited variation of the velocity concerned.

From a previous remark it follows that these results have reference to the propagation of waves in stiff bars and plates, as will be demonstrated in Art. 88.

72. Further light may be thrown on the matter by investigating the motion which ensues from the superposition of waves under the conditions specified in the last Article.

Let us, for definiteness, superpose two harmonic wave-trains of the same amplitude, but having different frequencies and velocities of propagation. If these be defined by

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= a_1{}^2\frac{\partial^2 u}{\partial x^2},\\ \frac{\partial^2 u}{\partial t^2} &= a^2{}_2\frac{\partial^2 u}{\partial x^2},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72.1)$$

a_1, a_2 being the velocities of propagation, we may express the two wave-trains by

$$\left.\begin{aligned}u_1 &= A\cos\left\{p_1\left(\frac{x}{a_1} - t\right) + \varepsilon_1\right\},\\ u_2 &= A\cos\left\{p_2\left(\frac{x}{a_2} - t\right) + \varepsilon_2\right\},\end{aligned}\right\} \quad . \quad . \quad . \quad (72.2)$$

where $\nu_1 = \frac{p_1}{2\pi}$, $\nu_2 = \frac{p_2}{2\pi}$ denote the corresponding frequencies. When superposed, we have a disturbance with the shift

$$\begin{aligned}u &= u_1 + u_2\\ &= 2A\cos\left\{\pi x\left(\frac{\nu_1}{a_1} + \frac{\nu_2}{a_2}\right) - \pi t(\nu_1 + \nu_2) + \frac{\varepsilon_1 + \varepsilon_2}{2}\right\}\\ &\qquad \cos\left\{\pi x\left(\frac{\nu_1}{a_1} - \frac{\nu_2}{a_2}\right) - \pi t(\nu_1 - \nu_2) + \frac{\varepsilon_1 - \varepsilon_2}{2}\right\} \quad . \quad . \quad (72.3)\end{aligned}$$

The factors on the right-hand side of this expression, taken in succession, refer to a harmonic wave-train of frequency

$$\nu = \frac{\nu_1 + \nu_2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72.4)$$

and velocity of propagation

$$a = \frac{a_1 a_2(\nu_1 + \nu_2)}{\nu_1 a_2 + \nu_2 a_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (72.5)$$

and to one of frequency

$$\nu' = \frac{\nu_1 - \nu_2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72.6)$$

and velocity of propagation

$$a' = \frac{a_1 a_2(\nu_1 - \nu_2)}{\nu_1 a_2 - \nu_2 a_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72.7)$$

Hence we see that if ν_1 be nearly equal to ν_2, then equation (72.3) represents a harmonic wave-train in which the amplitude varies slowly, the maximum value of the displacement at any point taking place at a time $\frac{x}{a'}$ later than it occurs at $x = 0$, with a periodicity $\frac{1}{\nu'}$. The noteworthy characteristics of this motion are that the wave-front changes as it advances, and the maximum amplitude reappears at intervals.

If we pass, on the same suppositions, to the consideration of harmonic wave-trains having frequencies within the narrow range of values ν and $\nu + \Delta\nu$, we again have the phenomenon of *dispersion.* The quantity a' in equation (72.7) is called the *group-velocity*, and a in equation (72.5) the *wave-* or *phase-velocity*. To determine the former we write, in equation (72.7), $a_2 = a$, $a_1 = a + \Delta a$, $\nu_2 = \nu$, $\nu_1 = \nu + \Delta\nu$, and make $\Delta\nu$ tend to zero. Thus it appears that

$$a' = \frac{a}{1 - \frac{\nu}{a}\frac{da}{d\nu}},$$

whence, on applying the binomial theorem and neglecting terms of the second order,

$$a' = a\left(1 + \frac{\nu}{a}\frac{da}{d\nu}\right), \quad . \quad . \quad . \quad . \quad . \quad (72.8)$$

provided the rate of change of velocity with respect to the frequency is small. If the given wave-trains involve the correspondingly narrow range of periods τ, $\tau + \Delta\tau$, and wave-lengths λ, $\lambda + \Delta\lambda$, it

is a simple matter to deduce from the last equation that the group-velocity

$$a' = \frac{\Delta\left(\frac{1}{\tau}\right)}{\Delta\left(\frac{1}{\lambda}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72.9)$$

The group-velocity is of importance to engineers because it defines the transmission of the *maximum* strain caused by stress-waves, as indicated by the symbol a' in Fig. 111. Taking the figure

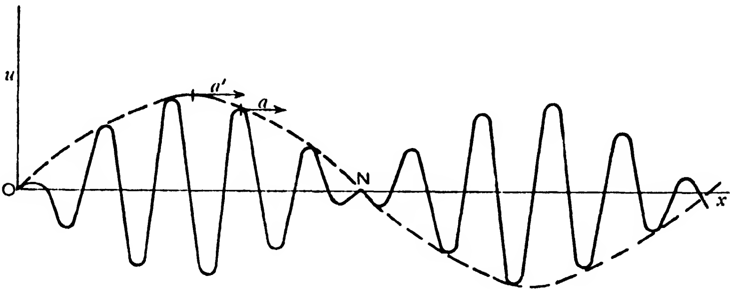


FIG. 111.

as illustrating the phenomenon, we notice that a *crest* of the 'wave profile', shown by the full line, will become a *trough* of the same profile when it passes the point N where the variable amplitudes changes sign. Such a crest will, in fact, generally travel along the curve shown by the dotted line.

An instructive illustration of the difference between group- and phase-velocities can be obtained by nearly filling a bath with water and adjusting the tap so that a drop falls once every twelve seconds or so. Each drop will initiate a group of ripples, containing about six crests and troughs, which travel appreciably faster than the group. Each crest thus advances through the group and disappears in front while new crests are formed at the rear. The experiment demonstrates that a simple group approximately maintains its shape and size like a rigid body, in the form of the sinusoidal wave represented in Fig. 111, and that the wave-length is infinitesimally short compared with the dimensions of the group.

This phenomenon is therefore to be expected in media for which the wave-velocity a varies with the frequency ν, or, with the wave-length λ. In water, for example, long waves travel faster than short, and the group-velocity is one-half the wave-velocity; but the opposite holds for capillary waves, the group-velocity being the greater. In the vibration of a stiff bar or beam of elastic material short waves, on the contrary, travel faster than long, and the group-velocity is double the wave-velocity. Hence

the group-velocity may be either less or greater than the wave-velocity, depending on the relative dimensions of the wave and on the nature of the medium. Moreover, as shown by H. Lamb,[1] there is no reason why these velocities should be in the same direction, in consequence of which we may have a series of crests moving to the right with so great a velocity that the centre of the group is continually moving to the left. This point is of special interest in connection with the optical phenomenon of *anomalous dispersion*, which has a bearing on the photo-elastic method of exploring the distribution of stress.[2]

Since the reasoning by which the foregoing results were arrived at was general, it follows that if we consider not merely two but any number of harmonic waves superposed on one another they will have a definite group-velocity, given by the equation corresponding to (72.8), provided the extreme range of frequencies is small.

Ex. Imagine the central portion of Fig. 112 to represent an

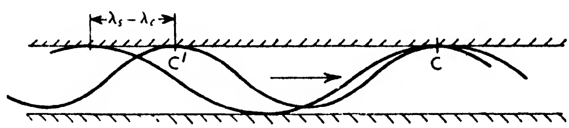


FIG. 112.

interface of steel and concrete, parallel to which are propagated two harmonic wave-trains having the different wave-lengths $\lambda_s = \frac{a_s}{\nu_s}$ and $\lambda_c = \frac{a_c}{\nu_c}$.

If it be supposed that throughout the motion one of the stress-waves is confined to the steel 'side' of the interface, and the other to the concrete 'side', then we may in this order assign the suffixes s and c to the waves. This implies that in general a process of 'selection' causes given waves to travel on a certain 'side' of such an interface, which is not an impossible condition in particular circumstances, regard being had to the related problem in seismology. Our assumption is a matter for experimental investigation, but the point will subsequently be presented in another way.

If, then, at a given instant the crests coincide at the point C in the figure, we will accordingly have this state of coincidence repeated after an interval of time $\frac{\lambda_s - \lambda_c}{a_s - a_c}$, when the faster train has gained a distance $\lambda_s - \lambda_c$ over the slower. During that interval the crest C' of the slower train will have advanced a distance

[1] *Proc. Lond. Math. Soc.*, vol. 1, page 473 (1904).

[2] E. G. Coker and L. N. G. Filon, *Treatise on Photo-Elasticity*, page 236.

$\dfrac{a_c(\lambda_s - \lambda_c)}{a_s - a_c}$, by reason of which the coincidence of crests associated with the point C and time $t = 0$ reappears at a distance

$$\frac{a_c(\lambda_s - \lambda_c)}{a_s - a_c} - \lambda_c,$$

i.e.

$$\frac{\lambda_s a_c - \lambda_c a_s}{a_s - a_c}$$

beyond C after an interval of time $\dfrac{\lambda_s - \lambda_c}{a_s - a_c}$. Thus, on dividing the related distance by the time, the expression for the group-velocity of the given wave-trains becomes

$$\frac{\lambda_s a_c - \lambda_c a_s}{\lambda_s - \lambda_c}.$$

This, as is to be expected, agrees with equation (72.7), since

$$\lambda_s = \frac{a_s}{v_s}, \quad \lambda_c = \frac{a_c}{v_c}.$$

These results illustrate, again, the fact that the group itself does not advance. On the present suppositions it appears at a certain point on the interface, fades away, reappears reversed in a new position, fades away, then reappears in its original form at another point, and so on. In actual materials the disturbance will gradually die away, due to the inherent dissipation of energy. By this means we are able to visualize the significance of the change of phase mentioned in connection with Fig. 110.

In circumstances where a state of tension is associated with each of the crests at C in Fig. 112, a corresponding increase of the resultant stress will ensue from partial reflection at any interface of different materials. In the present case the resulting stress may thus attain a value great enough to cause a localized fracture on the concrete ‘ side ’ of the boundary, in the form of a ‘ hair crack ’. A continued operation of these stresses would naturally affect the elastic properties of the material concerned. We might therefore investigate the matter by means of experiments arranged to exhibit the consequent changes in Poisson's ratio, since the distribution of stress due to unbalanced forces and couples on structural members with multiple boundaries is known to depend to some extent on the value of that ratio.

At this stage of the work it is well to distinguish the process of initiation of a crack from that of its propagation with respect to the contributory factors. For a specified structure, it will now be understood that, in the first place, the initiation of a crack depends mainly on the character and magnitude of the disturbance,

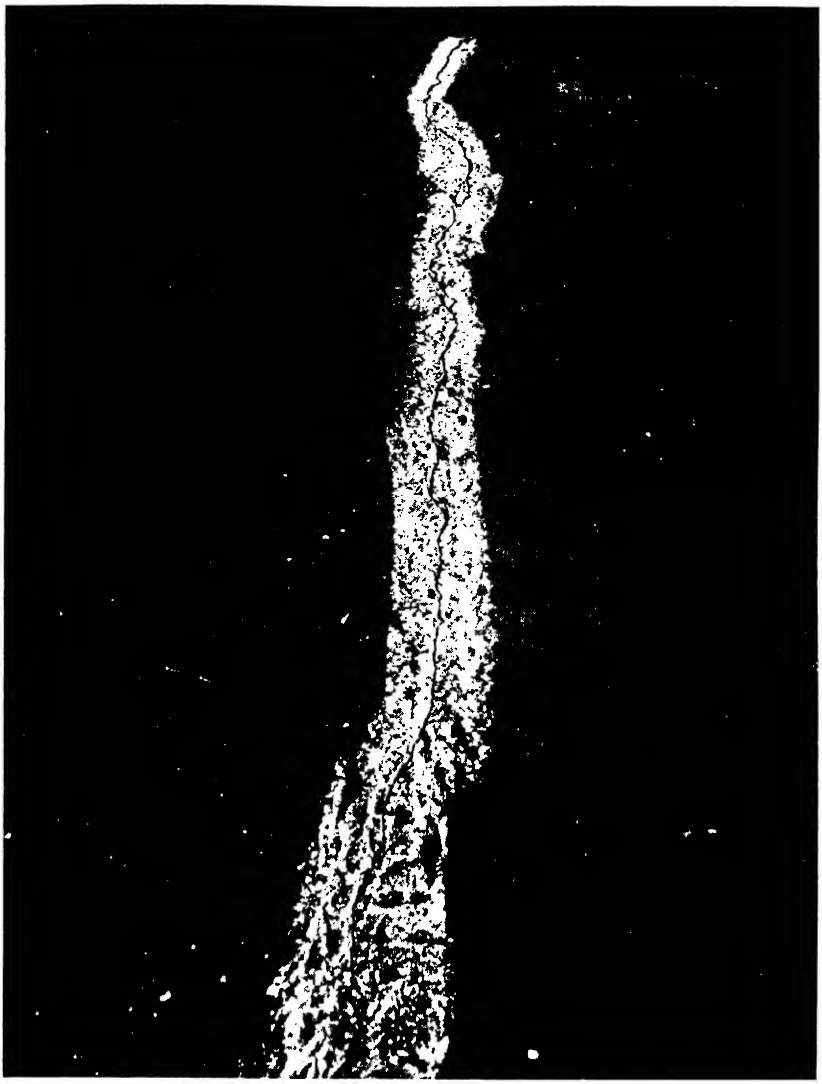
which may vary between the limits defined by the terms 'impact' and 'periodic force'. In the second place, the transmission of the crack will be influenced chiefly by the nature of the material, its history, and any changes of temperature that operate during the relevant interval of time. Due attention must, of course, be given to the shape and size of structures in the general case. This point may be illustrated by reference to the consequences of an accidental collision between a vessel of 7,000 tons and a reinforced-concrete jetty. An inspection undertaken shortly after the accident failed to reveal any serious damage to the structure, but 24 hours afterwards it was noticed that the deck of the jetty, measuring 12 ft. by 3 ft. overall in section, was completely severed by the crack shown in Fig. 113. This fracture was undoubtedly caused by the collision, and it is pertinent to remark that the crack was propagated during a period of calm weather as regards the sea. The significant thing to observe is that at least 24 hours were involved in the completion of this fracture, notwithstanding the fact that it was due to a single impact.

73. Some consideration should be given next to the case where the path of a stress-wave is intersected by a curved interface of different materials, or, what is in effect the same thing, when a train of plane waves is propagated at an oblique angle of incidence across a plane boundary of a given material. In such circumstances the transmitted part of the incident energy will suffer *refraction*, in much the same manner as a ray of light in passing from air to water. The ratio of the velocities of transmission in the two media is known as the *index of refraction*, so that a knowledge of this index enables us to predict the deviation of the refracted wave in specified conditions. With a medium having non-uniform elastic properties the refractive index is naturally variable.

It may make the discussion clear if we confine our attention to the particles which constitute one side of such an interface, and examine their motion by means of a model based on an idea which was originally put forward by Osborne Reynolds.[1]

For our present purpose imagine the two media in question to be represented in succession by the systems shown in (*a*) and (*b*) of Fig. 114. In both cases we have a light and uniform wire of elastic material under a tension P. The system (*b*) is obtained by attaching to the wire a row of similar beads, arranged at a distance l apart, and suspending from them an equal number of bobs having the same free period of oscillation. If in this model all the dimensions represent to scale very small quantities, then the bobs may be regarded as corresponding approximately to the constituent

[1] *Nature*, vol. 16, page 343 (1877); *Papers*, vol. 1, page 198.



W. T. Munns, Gravesend.

FIG. 113

[Facing p. 238

particles of matter which are capable of executing vibrations in a field of force that is exhibited by the elastic connections. In what follows we shall omit the gravitational constant g, as the problem to be solved is that of expressing the *ratio* of the velocities with

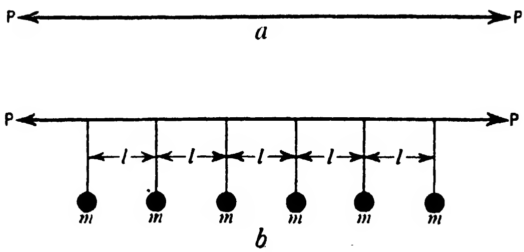


FIG. 114.

which a given wave is transmitted through the two systems thus formed.

Confining the treatment to motion in the plane of the paper, suppose that in consequence of a prescribed disturbance the *horizontal* displacement of the nth bead is y_n, and that of the nth bob is z_n. Further, let M denote the mass of each bead, m that of each bob, and c the force required to displace a bob unit distance from its position of equilibrium. Then, by equating the force to the product of mass and acceleration, we have the relations

$$\left.\begin{aligned} M\frac{d^2y_n}{dt^2} &= P\left\{\frac{(y_{n+1} - y_n)}{l} - \frac{(y_n - y_{n-1})}{l}\right\} + c(z_n - y_n), \\ m\frac{d^2z_n}{dt^2} &= c(y_n - z_n). \end{aligned}\right\} \quad (73.1)$$

On account of the fact that y_n is a function of the horizontal coordinate x we can write

$$y_{n-1} = y_n - l\frac{dy_n}{dx} + \tfrac{1}{2}l^2\frac{d^2y_n}{dx^2} - \ldots,$$

$$y_{n+1} = y_n + l\frac{dy_n}{dx} + \tfrac{1}{2}l^2\frac{d^2y_n}{dx^2} + \ldots,$$

and so, on the assumption that l is an infinitesimal, reduce equations (73.1) to

$$\left.\begin{aligned} M\frac{d^2y_n}{dt^2} &= Pl\frac{d^2y_n}{dx^2} + c(z_n - y_n), \\ m\frac{d^2z_n}{dt^2} &= c(y_n - z_n). \end{aligned}\right\} \quad . \quad . \quad . \quad (73.2)$$

If ρ be the line-density of the horizontal wire, further simpli-

fication follows from making $M = \rho l$, $m = \rho_1 l$, $c = c_1 l$ in the last equations, for then

$$\left.\begin{aligned} \rho\frac{\partial^2 y_n}{\partial t^2} &= P\frac{\partial^2 y_n}{\partial x^2} + c_1(z_n - y_n), \\ \rho_1\frac{\partial^2 z_n}{\partial t^2} &= c_1(y_n - z_n), \end{aligned}\right\} \quad . \quad . \quad . \quad (73\cdot3)$$

the symbols for partial differentiation being required because y_n is now a function of the independent variables x and t.

Hence if the wave-train $y = A_1 \cos(rx - st - \alpha)$ travels along the series of beads, we may write $z = A_2 \cos(rx - st - \beta)$ for the consequent wave-train along the row of bobs. In these conditions

$$\frac{\partial^2 y_n}{\partial t^2} = -s^2 y_n, \quad \frac{\partial^2 y_n}{\partial x^2} = -r^2 y_n, \quad \frac{\partial^2 z_n}{\partial t^2} = -s^2 z_n, \quad \frac{\partial^2 z_n}{\partial x^2} = -r^2 z_n,$$

whence, by equations (73·3),

$$-s^2 y_n = -\frac{Pr^2}{\rho}y_n + \frac{c_1}{\rho}(z_n - y_n),$$

$$-s^2 z_n = \frac{c_1}{\rho_1}(y_n - z_n).$$

Taken in order, these equations yield

$$\frac{y_n}{z_n} = \frac{c_1}{-s^2\rho + Pr^2 + c_1}$$

$$\frac{y_n}{z_n} = -\frac{s^2\rho_1 - c_1}{c_1},$$

hence, after equating, it appears that

$$\frac{1}{r^2} = \frac{P(s^2\rho_1 - c_1)}{s^2(s^2\rho\rho_1 - c_1\rho_1 - c_1\rho)} \quad . \quad . \quad . \quad (73\cdot4)$$

Denoting by a and a_1 the velocities of propagation of the given disturbance in the systems (a) and (b), respectively, we have, with the present notation,

$$a^2 = \frac{P}{\rho}$$

from equation (67.1), and

$$a_1^2 = \frac{s^2}{r^2}$$

from the specification of the wave-train in question. Thus, from equation (73·4),

$$a_1^2 = \frac{P(s^2\rho_1 - c_1)}{(s^2\rho\rho_1 - c_1\rho_1 - c_1\rho)}.$$

The index of refraction, being equal to $\frac{a}{a_1}$, is accordingly determined by

$$\frac{a^2}{a_1^2} = \frac{s^2\rho\rho_1 - c_1\rho_1 - c_1\rho}{\rho(s^2\rho_1 - c_1)} \quad . \quad . \quad . \quad . \quad (73.5)$$

To express this in more convenient terms, we notice that $2\pi\sqrt{\frac{\rho_1}{c_1}}$ denotes the period of free vibration for any one of the bobs. Signifying this period by τ, and writing $\frac{c_1}{\rho_1} = n^2$ in equation (73.5), we obtain, finally,

$$\frac{a^2}{a_1^2} = \frac{(s^2 - n^2)\rho\rho_1 - n^2\rho_1^2}{(s^2 - n^2)\rho\rho_1}$$

$$= 1 - \frac{\frac{n^2\rho_1}{\rho}}{s^2 - n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73.6)$$

Since the square root of this expression denotes the index of refraction in the stated conditions, a graphical representation of the effect of dispersion can now be traced by comparing the motions described by the beads and the bobs in Fig. 114 (*b*). Care should, however, be exercised in thus forming a picture as to what actually occurs on this account in engineering materials, by reason of the considerable difficulties encountered in the construction of a mechanical model complying with the principle of dynamical similitude in all respects.

Further information on the matter will be found in Chap. VII, where mention is made of applications in the testing of materials under periodic loads, when the resulting changes in the density give rise to characteristic spectrum phenomena which can be photographed and subjected to theoretical computation.

74. Elastic Hysteresis and Internal Friction of Metals. The phenomena investigated in Arts. 71–73 must, in the nature of things, be introduced in what is called by engineers the *elastic hysteresis* or *internal friction of metals.* It is well to emphasize, again, that a model presents only a roughly defined image of the physical mechanism behind the transmission of stress through materials, because the analogue is of a 'macroscopic' order compared with the 'microscopic' structure of crystalline metals.

With this reservation, on making the assumption that the energy dissipated in hysteresis is a function of the velocity of shift $\frac{\partial u}{\partial t}$ alone, which is in turn a function of the stress p in equation (65.1), we have the loss of energy

$$W = Kp^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74.1)$$

where K and n are constants for a fixed quantity of given material.

Several investigations into the magnitude of n for different materials have been undertaken,[1] from which it is to be inferred that $n = 2$, approximately, at the outset of a test. But it will now be apparent that this simple law of variation does not generally hold in the later stages of a test, when the phenomenon of plasticity begins to operate and makes the damping characteristic depend also on the ratio of plastic and elastic deformation of a specimen. It is to be remembered further that in certain substances the plastic deformation may be more or less localized in comparison with the elastic deformation. Moreover, in general practice the variation of the damping factor with the temperature of a metal should be borne in mind, one aspect of which has been examined by M. Guye and M. Schaffer,[2] who experimented with specimens executing small oscillations.

When regard is had to the consequences of *creep* in metals, and to the temperatures used in engineering work, it is thus seen that the behaviour of metals under fluctuating loads is influenced by a complex group of factors which do not necessarily remain constant quantities during indefinitely repeated applications of a specified load.

75. Propagation of Torque in a Shaft. We have now to consider the manner in which a fluctuating torque or turning moment is transmitted through a shaft. To simplify matters it

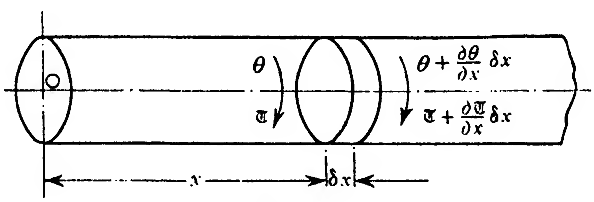


FIG. 115.

will be assumed that the dissipative forces are negligibly small and, as usual, the disturbance is slight.

In order to express this type of motion in symbols, suppose a variable torque be applied at the origin O on the uniform shaft shown in Fig. 115, where the positive direction is to the right. If, at a given instant t, $\mathfrak{T}$ be the disturbing torque and θ the angular shift or displacement of the plane section at x, then $\mathfrak{T} + \frac{\partial \mathfrak{T}}{\partial x}\delta x$ and

[1] Lord Kelvin, *Math. and Phys. Papers*, vol. 3, page 27; M. Quimby, *Phys. Review*, vol. 25, page 4 (1925); O. Föppl, *Zeits. V.d.I.*, vol. 70, page 1291 (1926); A. L. Kimball and D. E. Lovell, *Phys. Review*, vol. 30, page 6 (1927); S. F. Dorey, *Proc. I. Mech. E.*, vol. 123, page 479 (1932); R. H. Canfield, *Trans. Amer. Soc. for Steel Treating*, vol. 20, page 545 (1932).

[2] *Comptes Rendus*, vol. 15c, page 963 (1910).

$\theta + \dfrac{\partial\theta}{\partial x}\delta x$ will represent the corresponding quantities for the section at $x + \delta x$ in the figure.

The expression for the motion of the element follows by equating the disturbing couple to the product of mass and acceleration, which results in

$$-\frac{\partial \mathfrak{T}}{\partial x}\delta x = \frac{\pi\rho r^4}{2g}\frac{\partial^2\theta}{\partial t^2}\delta x$$

if ρ is the density of the material, and r is the radius of the shaft. Therefore

$$-\frac{\partial \mathfrak{T}}{\partial x} = \frac{\pi\rho r^4}{2g}\frac{\partial^2\theta}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (75.1)$$

Further, on writing L for the length of the shaft, J for the polar moment of inertia of the section, and N for the shear modulus of the material, we have from the well-known relation

$$-\frac{\mathfrak{T}}{J} = N\frac{\theta}{L}$$

the additional information

$$-\frac{\mathfrak{T}}{J} = N\frac{\partial\theta}{\partial x},$$

in view of the fact that in the disturbed state the shaft is uniformly twisted throughout its length. Hence in the case of a circular shaft, where $J = \dfrac{\pi r^4}{2}$,

$$-\mathfrak{T} = \frac{\pi r^4 N}{2}\frac{\partial\theta}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (75.2)$$

According to this relation

$$-\frac{\partial \mathfrak{T}}{\partial x} = \frac{\pi r^4 N}{2}\frac{\partial^2\theta}{\partial x^2}$$

in equation (75.1), by virtue of which we obtain, finally,

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (75.3)$$

where $a = \sqrt{\dfrac{Ng}{\rho}}$.

A comparison with equation (63.3) at once discloses a solution of the type

$$\theta = f(\theta - at) + F(\theta + at), \quad . \quad . \quad . \quad . \quad . \quad (75.4)$$

where f and F denote arbitrary functions. It is also clear that f refers to a wave travelling in the positive direction, and F to one travelling in the negative direction. A variable turning moment

on a shaft is consequently transmitted in the form of a *distortional* or *shear wave*, with a velocity equal to $\sqrt{\dfrac{Ng}{\rho}}$.

It is to be noticed that the compressional and distortional kinds of stress-wave travel at different velocities through a given material. For example, if $N = 12{,}000{,}000$ lb. per square inch and $\rho = 480$ lb. per cubic foot, then

$$a = \sqrt{\frac{12 \times 10^6 \times 144 \times 32{\cdot}2}{480}} \text{ ft. per sec.}$$

$$= 10{,}700 \text{ ft. per sec.,}$$

compared with the value 17,000 ft. per sec. for the compressional wave of Art. 63. The time involved in the transmission of a stress-wave through L ft. of this material is, therefore, $\dfrac{L}{10{,}700}$ sec. in the case of a distortional wave, and $\dfrac{L}{17{,}000}$ sec. in the case of a compressional wave. This explains why the governor of an engine cannot react instantaneously to a change of load on the flywheel, the 'lag' being equal to *twice* the time taken by a stress-wave in travelling from the flywheel, by way of the several connections, to the governor.

These results have obvious applications in the vibration of helical springs on which the stress is sensibly one of simple shear.

Ex. 1. Investigate the torsional vibration in a normal mode of a uniform shaft, fixed at one end and free at the other.

For the purpose of reference let Fig. 115 represent the shaft, with its fixed end at the origin O, and its free end at $x = L$. We may confine our attention, without in any way affecting the result, to a shaft of unit cross-sectional area, and in terms of the foregoing notation so let ρ denote the weight per unit length of the shaft.

In these circumstances we have the motion given by equation (75.3), namely

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2},$$

where the velocity of propagation $a = \sqrt{\dfrac{Ng}{\rho}}$ for a material with modulus of rigidity N. If $\dfrac{\omega}{2\pi}$ is the natural frequency of the oscillation, it is easy to confirm that this equation has the solution

$$\theta = \left(A \cos \frac{\omega}{a}x + B \sin \frac{\omega}{a}x\right) \cos (\omega t + \varepsilon), \qquad (75{\cdot}5)$$

the constants A, B depending on the end conditions. Taking account of the conditions

$$\theta = 0 \text{ at } x = 0 \text{ for all values of the time } t,$$

and, since the torque is zero at $x = L$,

$$\frac{\partial \theta}{\partial x} = 0 \text{ at } x = L \text{ for all values of the time } t,$$

it follows from equation (75.5) that

$$A = 0, \; B \cos \frac{\omega}{a} L = 0,$$

hence the series of values

$$\frac{\omega}{a} L = \frac{\pi}{2}, \; \frac{3\pi}{2}, \; \frac{5\pi}{2}, \ldots, \; \frac{(2n+1)\pi}{2}$$

define the successive modes of vibration.

The frequency $\frac{\omega_1}{2\pi}$ of the fundamental mode, for example, is therefore defined by

$$\omega_1 = \frac{\pi a}{2L}$$

$$= \frac{\pi}{2L}\sqrt{\frac{Ng}{\rho}}.$$

Let us now replace the mass of the actual shaft by an equivalent mass having a polar moment of inertia I concentrated at the free end $x = L$. If we write J for the polar moment of inertia of the circular section of the shaft, the fundamental frequency $\frac{\omega'}{2\pi}$ of the resulting system will be given by

$$\omega' = \sqrt{\frac{NJ}{IL}},$$

from equation (38.8).

Consequently, on equating the expressions for ω_1 and ω',

$$I = \frac{4JL\rho}{\pi^2 g},$$

or

$$I = \frac{4JM}{\pi^2 g}$$

if we put the weight of the shaft $L\rho = M$. Therefore

$$I = 0{\cdot}405\frac{JM}{g},$$

showing that in the fundamental mode of torsional oscillation the mass of the shaft is approximately equivalent to a mass having a polar moment of inertia $\frac{2}{5}$ that of the actual shaft and concentrated at the free end.

Ex. 2. Find the fundamental frequency in a normal mode of the shaft discussed in the last example when the free end is loaded by a disc having a polar moment of inertia I_1.

In terms of the previous notation we have here, by equation (75·5),

$$\theta = \left(A \cos \frac{\omega}{a}x + B \sin \frac{\omega}{a}x\right) \cos (\omega t + \varepsilon).$$

The condition at the fixed end $x = 0$, being

$$\theta = 0 \text{ at } x = 0 \text{ for all value of the time } t,$$

gives, as before,

$$A = 0.$$

Moreover, since the torque is $I_1\ddot{\theta}$ at the free end $x = L$, by equation (75·2) we have also

$$-I_1\ddot{\theta} = NJ\frac{\partial\theta}{\partial x}$$

in the solution when $x = L$. This leads, on effecting the operations of differentiation and then making $x = L$, to

$$I_1\omega^2 B \sin \frac{\omega}{a}L \cos (\omega t + \varepsilon) = NJ\frac{\omega}{a}B \cos \frac{\omega}{a}L \cos (\omega t + \varepsilon),$$

which reduces to

$$\frac{NJ}{I_1} = a\omega \tan \frac{\omega}{a}L,$$

i.e.

$$\frac{NJL}{I_1 a^2} = \frac{\omega}{a}L \tan \frac{\omega}{a}L,$$

or

$$\frac{JL\rho}{I_1 g} = \frac{\omega}{a}L \tan \frac{\omega}{a}L,$$

because $a^2 = \dfrac{Ng}{\rho}$. A more convenient expression ensues from writing J' for the moment of inertia of the complete shaft about its axis, so that $J' = \dfrac{JL\rho}{g}$, for then

$$\frac{J'}{I_1} = \frac{\omega}{a}L \tan \frac{\omega}{a}L;$$

or, with $\dfrac{J'}{I_1} = \alpha$, $\dfrac{\omega}{a}L = \beta$,

$$\begin{aligned}\alpha &= \beta \tan \beta \\ &= \beta(\beta + \tfrac{1}{3}\beta^3 + \tfrac{2}{15}\beta^5 + \tfrac{17}{315}\beta^7 + \dots) \\ &= \beta^2(1 + \tfrac{1}{3}\beta^2 + \tfrac{2}{15}\beta^4 + \tfrac{17}{315}\beta^6 + \dots),\end{aligned}$$

from the expansion for $\tan \beta$.

It is clearly possible to complete the solution by repeating the

method of Ex. 2 in Art. 68, but we shall here, by way of variety, follow a slightly different procedure.

A first approximation is obtained by taking only the first term of the series. To this degree of accuracy

$$\alpha = \beta^2,$$

i.e.

$$\frac{J'}{I_1} = \frac{\omega^2}{a^2}L^2$$

$$= \frac{\omega^2 \rho}{Ng}L^2,$$

whence

$$\omega = \sqrt{\frac{NgJ'}{\rho L^2 I_1}}$$

$$= \sqrt{\frac{NJ}{LI_1}},$$

as $J' = \dfrac{JL\rho}{g}$. Arranged otherwise, the frequency

$$\frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{NJ}{LI_1}},$$

which is evidently the same as the natural frequency of torsional oscillation when the weight of the shaft is negligibly small.

A second and improved approximation results from introducing the first two terms of the series, when

$$\alpha = \beta^2(1 + \tfrac{1}{3}\beta^2),$$

i.e.

$$\beta^4 + 3\beta^2 - 3\alpha = 0.$$

Now

$$\beta^2 = -\tfrac{1}{2}\{3 \pm 3(1 + \tfrac{2}{3}\alpha - \tfrac{2}{9}\alpha^2)\}$$

if terms beyond α^2 are omitted. The root

$$\beta^2 = \alpha - \tfrac{1}{3}\alpha^2$$

thus shows, on reverting to the original symbols, that

$$\omega = \sqrt{\frac{NJ}{LI_1}\left(1 - \frac{1}{3}\frac{J'}{I_1}\right)}$$

determines the corresponding frequency $\dfrac{\omega}{2\pi}$ in instances where J' is small compared with I_1.

Ex. 3. A uniform shaft of very great length has fixed to it at one point a coupling whose polar moment of inertia is equal to that of a length n of the shaft. Investigate the transmission of a shear wave through the system, on the suppositions that the coupling is very short in the axial direction, the velocity of propagation is the same for all parts of the structure thus formed, and the weight of the shaft is negligibly small compared with that of the coupling. The dissipative forces also may be neglected.

Let Fig. 116 represent the significant part of the system, and suppose the distortional wave to approach the coupling from the

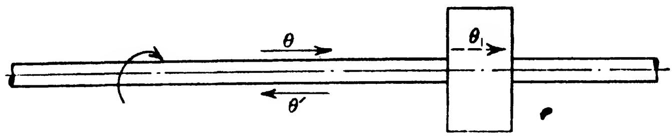


FIG. 116.

left. In the given conditions we have, in the notation of equation (75·3),

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}$$

as the relation to be satisfied by all the waves. We may, therefore, denote the incident wave by

$$\theta = A\cos(rx - st),$$

so that the disturbance has a period $\frac{2\pi}{s}$, a wave-length $\lambda = \frac{2\pi}{r}$, and a velocity of propagation $\frac{s}{r} = a = \sqrt{\frac{Ng}{\rho}}$. But, in view of the identity $\cos x + i\sin x = e^{ix}$, for analytical purposes it is better to take the wave to be the real part of $Ae^{i(rx-st)}$, where $i = \sqrt{-1}$.

The previous results indicate that this wave will suffer partial reflection and transmission at the boundary of the coupling, owing to the sudden change in the line density at that point. If, for this reason, we assume similar expressions $A'e^{i(r'x-s't)}$ and $A_1e^{i(r_1x-s_1t)}$ for the reflected and transmitted waves, respectively, considerations of continuity require that the period shall be the same for all, with the result that $s_1 = s' = s$; and, since the same velocity of propagation is involved in all three waves, $r_1 = -r' = r$. That is to say, the wave-length λ is not changed by reflection or transmission.

Hence, taking the origin at the point associated with the concentrated weight of the coupling, we may write

$$\theta = Ae^{i(rx-st)} + A'e^{-i(rx+st)} \quad . \quad . \quad . \quad (75·6)$$

for the left-hand side of the boundary involved, and

$$\theta_1 = A_1e^{i(rx-st)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (75·7)$$

for the right-hand side.

If $\mathfrak{T}$ and $\mathfrak{T}_1$ refer in turn to the torque on the corresponding sides of the boundary, the motion of the coupling is given by

$$\frac{\rho}{g}n\ddot{\theta} = \mathfrak{T}_1 - \mathfrak{T}.$$

Taking, for simplicity in working, the shaft to have a polar moment of inertia equal to unity, equation (75.2) gives

$$\mathfrak{T}_1 - \mathfrak{T} = N\left(\frac{\partial\theta_1}{\partial x} - \frac{\partial\theta}{\partial x}\right).$$

The material is, as usual, specified by its density ρ, and modulus of rigidity N. On combining these equations, it follows that at the origin $x = 0$

$$n\ddot{\theta} = \frac{Ng}{\rho}\left(\frac{\partial\theta_1}{\partial x} - \frac{\partial\theta}{\partial x}\right)$$
$$= a^2\left(\frac{\partial\theta_1}{\partial x} - \frac{\partial\theta}{\partial x}\right),$$

a being the velocity of propagation.

Thus it is seen that at the origin

$$-ns^2(A + A') = ia^2r(A_1 - A + A'),\ s^2 = a^2r^2,$$

whence $$inr(A + A') = A_1 - A + A'\,; \quad . \quad . \quad . \quad . \qquad (75.8)$$

and, as $\theta = \theta_1$ when $x = 0$ in equations (75.6) and (75.7),

$$A_1 = A + A' \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75.9)$$

Combining equations (75.8) and (75.9), we find

$$A_1 = \frac{2A}{2 - inr}.$$

Finally, by virtue of equation (75.7), we obtain

$$\theta_1 = \frac{2A}{2 - inr}e^{i(rx - st)}$$
$$= \frac{2A(2 + inr)}{4 + n^2r^2}e^{i(rx - st)}$$
$$= \frac{2A}{(4 + n^2r^2)^{\frac{1}{2}}}e^{i(rx - st + \varepsilon)},$$

where $\tan\varepsilon = \frac{1}{2}nr = \dfrac{\pi n}{\lambda}$.

This indicates that the effect of the coupling on the transmitted wave is to reduce its amplitude in the ratio $\dfrac{2}{(4 + n^2r^2)^{\frac{1}{2}}}$, and to alter its phase by the amount $\tan^{-1}\dfrac{\pi n}{\lambda}$.

Although the result exhibits the general nature of the modification brought about in distortional waves by couplings and shrunk-on flywheels, it is nevertheless only an approximate solution for actual systems of this type. This is so because we have assumed the shaft to be very long and the coupling to be very short in the axial direction, and neglected the fillets which are usually found on couplings. Nor has account been taken of the joint between the

two parts of a coupling. Moreover, it has been assumed that the inherent dissipative forces are absent, and the material is the same throughout the part of the structure that matters. The presence of a splined connection between a shaft and flywheel would also affect the result to some extent, depending on the relative dimensions of the connection. It will be understood from these remarks that the phenomena of dispersion and distortion generally enter into the problem, so that our treatment presents a correspondingly simplified account of what actually occurs on shafts fitted with couplings and flywheels.

76. Water Hammer. An instructive and important application of the theory given in Art. 63 is to be found in the case of a hydraulic main in which the flow of water or fluid is suddenly changed by closing a valve. The theory indicates that such an operation will give rise to a stress-wave in the system. In order to calculate its intensity, we shall first suppose the pipe to be inelastic, and subsequently modify the result to agree more closely with fact.

(*a*) *Inelastic Pipe.* Consider a portion of a straight pipe of uniform cross-section which is full of water flowing in the direction

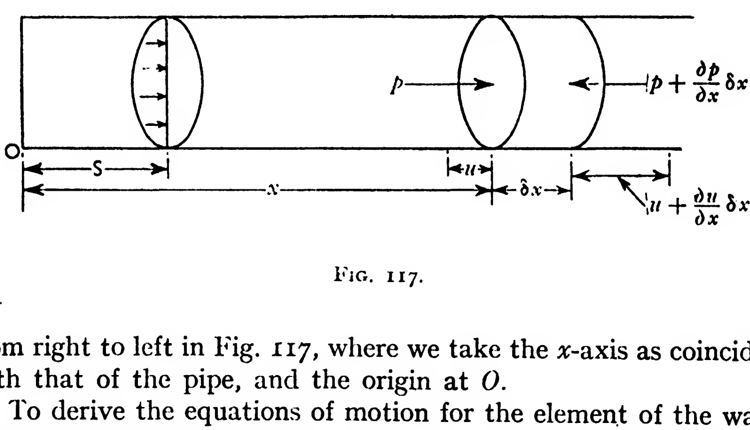


FIG. 117.

from right to left in Fig. 117, where we take the x-axis as coinciding with that of the pipe, and the origin at O.

To derive the equations of motion for the element of the water contained between the plane sections at x and $x + \delta x$, imagine it to be acted upon by a disturbing force in the direction from left to right, which will be taken as the positive direction. If that force induces a pressure p and a shift u on the fluid in the plane at x, then $p + \dfrac{\partial p}{\partial x}\delta x$ and $u + \dfrac{\partial u}{\partial x}\delta x$ will be the corresponding quantities for the plane section at $x + \delta x$ in the figure.

By the method of Art. 63 we accordingly have $-\dfrac{\partial p}{\partial x}\delta x$ for the disturbing pressure on the element of the fluid. If A denote the cross-sectional area of the water column, by equating the force to

the product of mass and acceleration we derive the dynamical equation

$$- A\frac{\partial p}{\partial x}\delta x = \frac{\rho A}{g}\frac{\partial^2 u}{\partial t^2}\delta x,$$

i.e.

$$\frac{\partial p}{\partial x} = -\frac{\rho}{g}\frac{\partial^2 u}{\partial t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (76.1)$$

where ρ is the density of the water. For most practical purposes it may be assumed that ρ remains constant throughout the motion under investigation.

Since the bulk modulus K of the fluid is defined by

$$K = -\frac{p}{\frac{\delta V}{V}},$$

where δV is the change in a volume V due to an increment of pressure p, here

$$p = -K\frac{\delta V}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (76.2)$$

Now it is plain that in this expression

$$\frac{\delta V}{V} = \frac{A\frac{\partial u}{\partial x}\delta x}{A\delta x}$$

$$= \frac{\partial u}{\partial x},$$

so that

$$p = -K\frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (76.3)$$

That is to say

$$\frac{\partial p}{\partial x} = -K\frac{\partial^2 u}{\partial x^2}$$

must hold in equation (76.1), whence

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (76.4)$$

where $a = \sqrt{\dfrac{Kg}{\rho}}$.

Our previous treatment of this type of equation shows that

$$u = f(x - at) + F(x + at)$$

is a solution, where the arbitrary functions f and F define the wave-form. Also, the negative and positive signs refer to waves travelling towards the left and the right, respectively, in Fig. 117, with the velocity

$$a = \sqrt{\frac{46 \times 10^6 \times 32{\cdot}2}{62{\cdot}5}} \text{ ft. per sec.}$$

$$= 4{,}850 \text{ ft. per sec.}$$

in water for which the bulk modulus $K = 46{,}000{,}000$ lb. per square foot and the density $\rho = 62 \cdot 5$ lb. per cubic foot.

Now suppose a valve to be situated at the origin O, and let v be the velocity of the water in the steady state, measured from right to left in the figure. If the valve be closed instantaneously, say at the time $t = 0$, the theory indicates that in the water to the right of the valve the kinetic energy will be gradually transformed into potential energy, by means of a stress-wave travelling with the velocity a of equation (76.4). Because the wave travels a distance s from the valve in the interval of time $\dfrac{s}{\sqrt{\dfrac{Kg}{\rho}}}$, it is to be deduced from the relation

$$\int p\,dt = \text{change of momentum}$$

that

$$\frac{pAs}{\sqrt{\dfrac{Kg}{\rho}}} = \frac{\rho As}{g} v,$$

i.e.

$$p = v\sqrt{\frac{K\rho}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76.5)$$

This expression for the pressure p applies until the valve is nearly closed, when our assumption as regards a plane wave-front

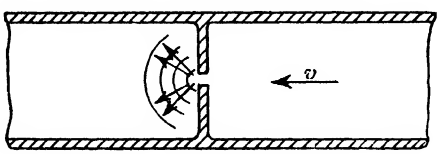


FIG. 118.

ceases to hold. The reason is, as indicated in Fig. 118, that in the later stages of the operation a spherical wave is formed at the valve, with a consequent loss of energy by dispersion and distortion.

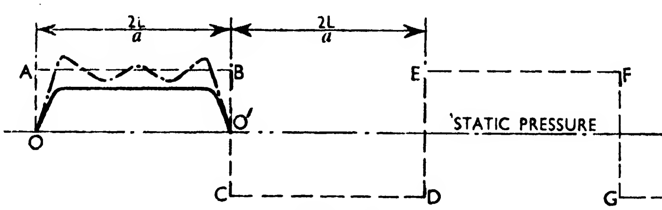


FIG. 119.

To exhibit the wave motion in the case of a main L ft. in length, take the point O as origin on the line of static pressure in Fig. 119.

Were the valve closed instantaneously without involving any dissipation of the wave-energy, the pressure on the pipe would increase by the amount p, which may be represented by the vertical line OA. The associated wave will simultaneously start from the valve and travel to the right-hand end of the pipe, in $\frac{L}{a}$ sec., where it suffers reflection and returns to the valve, in $\frac{L}{a}$ sec., as a disturbance of intensity $-p$, assuming perfect reflection. These movements are described in order by the graph ABC. The resulting wave of *negative* pressure p will, after an interval of time $\frac{2L}{a}$ sec. from the start, similarly undergo reflection at the valve, travel as such along the main, be reflected at the end of the pipe, and return to the valve as a stress-wave of intensity p, in the interval of time $\frac{2L}{a}$ sec. In the figure this second series of movements corresponds to the graph CDE. After the lapse of $\frac{4L}{a}$ sec. reckoned from the time $t = 0$ the conditions at the valve are, therefore, the same as at the beginning of the disturbance.

In the absence of frictional agencies the process would be repeated, by way of the graph EFG. . . . With actual systems, however, the inherent damping of the movement brings about the evanescence of certain harmonic components of the motion. Under these conditions the graph $OABO'$ tends to resemble that shown by the dot-dash line, where it is supposed that only the first five harmonic components of the corresponding Fourier series remain. It is to be remembered, from Art. 59, that the effect of friction is to cause the pressure p to die away without sensibly affecting the period of oscillation.

The graph is further modified by reason of imperfect reflection at both ends of an actual pipe, as well as the dispersion of the energy referred to in connection with Fig. 118. We might thus regard the full-line graph in Fig. 119 as an approximate representation of the actual motion which in ideal circumstances would correspond to the rectangular graph $OABC'$.

(*b*) *Elastic Pipe.* Account must now be taken of the fact that the pressure p will cause the pipe to stretch simultaneously in both the longitudinal and the transverse directions. The longitudinal stretch can readily be deduced from the results given in Art. 63, but it is insignificant in the usual case where the ends of a pipe are anchored.

Turning, then, to the consideration of the circumferential stretch of the pipe in Fig. 117, let r be its internal radius, and E the direct

modulus of elasticity for the material. If, as a consequence of the elastic pipe alone, the element of length δx increases an amount Δr in radius, and the longitudinal shift u is altered by an amount u_1, then we have

$$\pi r^2 u_1 = 2\pi r \delta x \Delta r,$$

i.e.

$$u_1 = \frac{2\delta x \Delta r}{r}.$$

Now the quantity $\frac{\Delta r}{r}$ defines the ratio of unit stress to the modulus E, whence, by the theory of thin cylinders

$$\frac{\Delta r}{r} = \frac{p'r}{tE},$$

where p' denotes the actual rise of pressure in a pipe with a thin wall of thickness t. Hence

$$u_1 = \frac{2p'r}{tE}\delta x.$$

In order to bring about the pressure p' in the pipe discussed above, we must shorten the element of length δx by the amount u_1, which is obviously equal to

$$\frac{p'}{p}u,$$

where $u = v\sqrt{\frac{\rho}{Kg}}\delta x$ and $p = v\sqrt{\frac{K\rho}{g}}$ in the case of water moving with a velocity v.

Therefore the shift u' for the elastic pipe is given by

$$u' = \frac{p'}{p}u + u_1.$$

Here the relation

$$\frac{p'}{p} = \frac{u}{u'}$$

holds because the pressure is inversely proportional to the shift, so that the actual pressure

$$p' = p\frac{u}{u'}$$

$$= \frac{p}{\left(1 + \frac{2r}{t}\cdot\frac{K}{E}\right)^{\frac{1}{2}}},$$

from our expressions for u and u_1.

In a thin steel pipe of internal diameter d, for example, with

$K = 46{,}000{,}000$ lb. per square foot and $E = 43{,}300{,}000$ lb. per square inch, we thus obtain the useful expression

$$p' = \frac{p}{\left(1 + \frac{d}{94t}\right)^{\frac{1}{2}}}$$

$$= \frac{65{\cdot}5v}{\left(1 + \frac{d}{94t}\right)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (76.6)$$

with v measured in feet per second.

Equation (76.6) consequently determines, for a thin-walled pipe of the specified steel, the increase of pressure brought about by the sudden closing of a valve. The phenomenon is commonly referred to as *water hammer*, and it may evidently become serious when large values of v are involved. In fact, many failures on hydraulic systems have been attributed to water hammer. If, as sometimes occurs in practice, the resulting pressure on a main falls so low as to produce a partial vacuum, the liberation of the air dissolved in the water rapidly causes the motion to break down. An appreciable increase in the pressure may take place when the two parts of the water column subsequently reunite, and this alone might well lead to the fracture of a main.

N. Joukovsky [1] conducted, in 1898, an elaborate investigation into the matter, and found that 4,200 ft. per sec. was the velocity of propagation for certain pipes. Mention should also be made of the experimental work of A. H. Gibson. [2] More recently H. K. Barrow [3] obtained comparative data from tests with pipes of different materials, and concluded that the pressure p' varied from $41{\cdot}2v$-lb. to $54{\cdot}2v$-lb. per square inch for cast iron, and from $34{\cdot}6v$-lb. to $54{\cdot}2v$-lb. per square inch for steel, in the notation of equation (76.6).

The foregoing results throw light on the principal factors for a given system. If, in the first place, the valve is completely closed before the initial wave has returned to the origin, or, what is the same thing, if the operation is carried out in an interval of time less than $\frac{2L}{a}$, the value of p will increase up to the instant of closing and so bring about a pressure which is the same as if the valve had been closed instantaneously. If, in the second place, the operation occupies an interval of time that is greater than $\frac{2L}{a}$, then the original

[1] *Stoss in Wasserleitungsröhren* (St. Petersburg, 1900), abstracted by O. Simin, *Proc. Amer. Waterworks Assoc. for* 1904, page 335.

[2] *Water Hammer in Hydraulic Pipe Lines* (1908).

[3] *Jour. New England Water Assoc.*, vol. 45, page 72 (1931).

wave will have returned to the origin as a disturbance of low pressure before the valve is completely closed, and we have a corresponding reduction in the rise of pressure during the later stages of the closing process. Put into other words, the pressure associated with the wave will be the same as for instantaneous closing if the relevant interval of time is equal to or less than $\frac{2L}{a}$; and the pressure will diminish with increase of the interval if the closing time is greater than $\frac{2L}{a}$. In the latter case the various phases of the motion are best examined by the aid of either graphical or arithmetical integration. Several methods have been evolved for this purpose, notably by N. R. Gibson,[1] and R. W. Angus.[2] Reference is made by these investigators to the instructive treatment by L. Allievi,[3] an English account of which was done by E. E. Halmos in 1925.

Before leaving this subject, it is well to notice that the phenomena of *surge* and *water hammer* together may occur when a valve is partially closed in, say, the regulating mechanism of a hydraulic turbine. This deserves mention here because the two types of disturbance differ in one important particular. The phenomenon of surging is sensibly confined to the origin, whereas water hammer involves a wave that travels with a velocity exceeding 4,200 ft. per sec. in actual pipes.

77. Spherical Waves. Proceeding to the consideration of motion in three dimensions, it will now be understood that the wave-front involves three sheets. In the special case of an isotropic elastic material, all of these sheets are spheres and two of them are coincident, with the origin of the disturbance as the centre. From the present point of view important examples are presented by the transmission of slight disturbances through water in the first place, and through a perfectly elastic solid in the second.

(*a*) *Water*. If an explosion takes place within a mass of water, say, at the time $t = 0$ and the origin O in Fig. 120, we may suppose the resulting wave to have travelled through the radial distance r during the subsequent interval of time t, the positive direction being outwards from O. In these circumstances it is possible to formulate the motion of the spherical wave by reference to the element $ABCD$ of the disturbed fluid, of radial thickness δr, subtending the angles $\delta\theta$, $\delta\theta$ at the origin O.

Writing, in accordance with our usual procedure, p and u for the pressure and the radial shift associated with the surface of

[1] *Trans. Amer. Soc. C.E.*, vol. 83, page 707 (1919).

[2] *Engineering*, vol. 140, page 154 (1935); *Proc. I. Mech. E.*, vol. 136, page 245 (1937).

[3] *Atti del Collegio degli Ingegneri ed Architteti*, Milan (1913).

radius r, we have $-\left(p + \frac{\partial p}{\partial r}\delta r\right)$ and $\left(u + \frac{\partial u}{\partial r}\delta r\right)$ for the corresponding quantities at the radius $r + \delta r$. Each of the four edges of the element will also be subjected to a tangential stress p if the water is treated as a perfect fluid. The element is consequently acted upon by the several forces indicated in the figure.

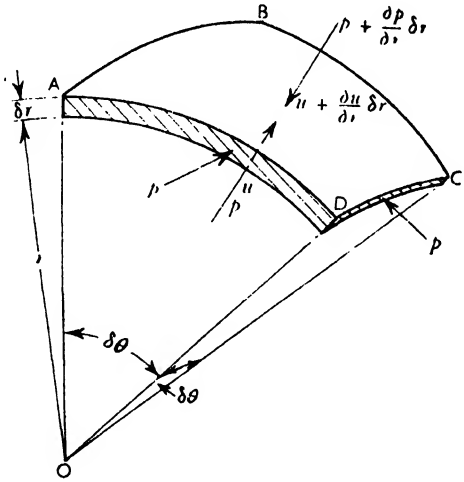


FIG. 120.

Equating the sum of these forces to the product of mass and acceleration of the element, on the assumption that the density, ρ, of the water remains constant, we thus deduce the dynamical equation

$$p(r\delta\theta)^2 - \left(p + \frac{\partial p}{\partial r}\delta r\right)(r + \delta r)^2(\delta\theta)^2 + 4pr\delta\theta\delta r\tfrac{1}{2}\delta\theta = \frac{\rho r^2}{g}\delta\theta^2\delta r\frac{\partial^2 u}{\partial t^2},$$

remembering that sin $\frac{1}{2}\delta\theta \longrightarrow \frac{1}{2}\delta\theta$ in the limit. To this degree of approximation we may neglect all but the first-order terms in δr, when the expression reduces to

$$-\frac{\partial p}{\partial r} = \frac{\rho}{g}\frac{\partial^2 u}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (77.1)$$

In this notation the related equation (76.2), namely

$$p = -K\frac{\delta V}{V},$$

becomes

$$p = -K\frac{\left\{(r + u)^2\left(\delta r + \frac{\partial u}{\partial r}\delta r\right)(\delta\theta)^2 - r^2\delta r(\delta\theta)^2\right\}}{r^2\delta r(\delta\theta)^2},$$

which means that

$$p = -K\frac{\left(r^2\frac{\partial u}{\partial r}\delta r + 2ru\delta r + 2ru\frac{\partial u}{\partial r}\delta r\right)}{r^2\delta r}$$

$$= -K\left(\frac{\partial u}{\partial r} + 2\frac{u}{r} + 2\frac{u}{r}\frac{\partial u}{\partial r}\right)$$

if terms in u^2 be omitted as of the second order. But when u itself is small, the last member can be neglected in comparison with the others, then

$$p = -K\left(\frac{\partial u}{\partial r} + 2\frac{u}{r}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (77.2)$$

and therefore

$$\frac{\partial p}{\partial r} = -K\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}u\right).$$

Hence, on combining with equation (77.1),

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}u\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (77.3)$$

where $a = \sqrt{\dfrac{Kg}{\rho}}$ for water with a bulk modulus K.

It is not difficult to prove that the solution of this equation is of the form

$$u = \frac{\partial}{\partial r}\left\{\frac{1}{r}f(r \pm at)\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (77.4)$$

f being an arbitrary function. That is to say

$$u = -\frac{1}{r^2}f(r \pm at) + \frac{1}{r}f'(r \pm at),$$

which we can differentiate with respect to r and so write

$$\frac{\partial u}{\partial r} = \frac{2}{r^3}f(r \pm at) - \frac{1}{r^2}f'(r \pm at) - \frac{1}{r^2}f'(r \pm at) + \frac{1}{r}f''(r \pm at).$$

In this way it appears, after substituting these expressions for u and $\dfrac{\partial u}{\partial r}$ in equation (77.2), that

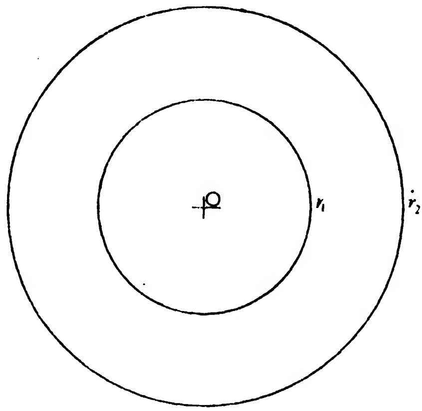
$$p = -K\left\{\frac{1}{r}f''(r \pm at)\right\}$$

$$= -K\left\{\frac{1}{r}\phi(r \pm at)\right\}, \quad . \quad . \quad . \quad . \quad . \qquad (77.5)$$

where ϕ refers to an arbitrary function.

To explain the significance of the symbol a, we shall for brevity consider only the divergent wave, which is clearly identified with the negative sign inside the brackets. Then let the radii r_1 and r_2

define the positions of the wave-front at the instants t_1 and t_2, respectively, as illustrated in Fig. 121. If p_1 and p_2 be the corre-



sponding pressures on the wave-front, in the absence of friction $p_1r_1 = p_2r_2$, by reason of which we may write

at time t_1 the pressure is p_1 at the radius r_1,

and at time t_2 the pressure is $p_2\left(=\dfrac{p_1r_1}{r_2}\right)$ at the radius r_2.

A substitution of these particular values in equation (77.5) now leads to the relations

$$p_1r_1 = -K\phi(r_1 - at_1), \quad p_2r_2 = -K\phi(r_2 - at_2),$$

since the negative sign alone is of account in the case of a divergent wave. Combining these formulae, and utilizing the condition $p_1r_1 = p_2r_2$, we find

$$r_1 - at_1 = r_2 - at_2,$$

i.e.
$$\frac{r_2 - r_1}{t_2 - t_1} = a.$$

This result clearly indicates that the symbol a in equation (77.3) expresses the velocity with which the compressional wave travels through water having a bulk modulus K and a density ρ. According to a previous calculation the numerical value of a is approximately 4,850 ft. per sec.

As the convergent wave is characterized by the positive sign in equation (77.4), that expression therefore describes the superposition of two similar waves travelling in opposite directions.

It is manifest that, on the implied assumptions, the pressure p varies inversely as the distance reckoned from the origin; and it

is a simple matter to show that the quantity of explosive required to produce a given pressure at a given distance from O varies as the cube of that distance.

(*b*) *Elastic Solid.* When the foregoing procedure is applied to the propagation of stress in an isotropic elastic solid, due attention must of course be given to the significance of Poisson's ratio $\frac{1}{m}$. To introduce this factor at the outset, let the stresses p_1, p_2, p_3 be imposed on the faces of a cube of the material as shown in Fig. 122, then by the theory of the strength of materials the stretch

$$e = \frac{p_1}{E} - \frac{p_2}{mE} - \frac{p_3}{mE}, \quad . \quad . \quad . \quad . \quad (77.6)$$

where E is the direct modulus of elasticity.

The equations of motion are obtained in much the same manner as before, by considering an element of the spherical wave which would be initiated as a result of a disturbance within the solid, at a point which we shall take as the origin. Let the element be as indicated in Fig. 123, specified by the radii r, $r + \delta r$, as well as the

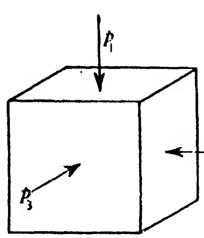


FIG. 122.

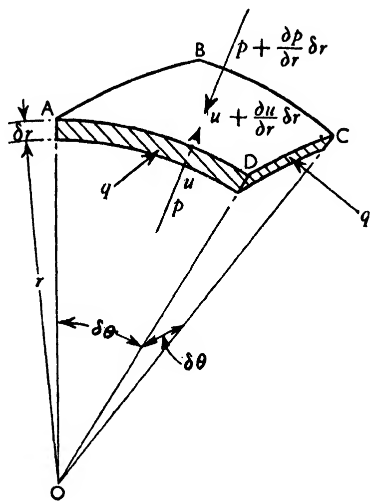


FIG. 123.

angles $\delta\theta$, $\delta\theta$. If, due to this cause alone, the surface of radius r be subjected to a pressure or stress p in the radial direction outwards, in order to secure equilibrium we have seen that a stress $p + \frac{\partial p}{\partial r}\delta r$ must act inwards on the surface of radius $r + \delta r$. The consequent shifts of the material at the radial distances r and $r + \delta r$ will, in our notation, be denoted by u and $u + \frac{\partial u}{\partial r}\delta r$, respectively. Also, write q for the tangential stress on each of the four edges of the infinitely small element $ABCD$ thus formed.

Taking the positive direction to be radially outwards, we have,

on equating the sum of the forces to the product of mass and acceleration,

$$-\left(p + \frac{\partial p}{\partial r}\delta r\right)(r + \delta r)^2(\delta\theta)^2 + pr^2(\delta\theta)^2 + 4qr\delta\theta\delta r\tfrac{1}{2}\delta\theta = \frac{\rho r^2}{g}\delta\theta^2\delta r\frac{\partial^2 u}{\partial t^2},$$

which reduces to

$$q - p - \tfrac{1}{2}r\frac{\partial p}{\partial r} = \tfrac{1}{2}\frac{\rho r}{g}\frac{\partial^2 u}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad (77.7)$$

when dimensions of the second order of small quantities are omitted.

Furthermore, from the relation (77.6),

the radial stretch $\quad \dfrac{\partial u}{\partial r} = -\dfrac{p}{E} + \dfrac{2q}{mE}$

and the tangential stretch $\quad \dfrac{u}{r} = \dfrac{p}{mE} - \dfrac{q}{E} + \dfrac{q}{mE},$

where the value $-p$ denotes a state of compression in the material. Solving these equations for p and q, we find

$$\left.\begin{aligned} p &= -\frac{mE}{m^2 - m - 2}\left\{(m-1)\frac{\partial u}{\partial r} + 2\frac{u}{r}\right\}, \\ q &= -\frac{mE}{m^2 - m - 2}\left(\frac{\partial u}{\partial r} + m\frac{u}{r}\right). \end{aligned}\right\} \quad . \quad . \quad (77.8)$$

The next operation consists in making these substitutions for p and q in equation (77.7), whence we derive the equation of motion

$$\frac{\partial^2 u}{\partial t^2} = \frac{m(m-1)Eg}{(m^2 - m - 2)\rho}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - 2\frac{u}{r^2}\right) \quad . \quad (77.9)$$

It can readily be verified that the solution is of the type

$$u = \frac{\partial}{\partial r}\left\{\frac{1}{r}f(r \pm at)\right\}, \quad . \quad . \quad . \quad . \quad . \quad (77.10)$$

f being an arbitrary function. This determines the shift u, and it is of the same form as the solution for equation (77.3). But now the velocity of propagation

$$a = \sqrt{\frac{m(m-1)Eg}{(m+1)(m-2)\rho}} \quad . \quad . \quad . \quad . \quad . \quad (77.11)$$

for a material specified by its density ρ and direct modulus of elasticity E. It is to be noticed that this wave of dilation involves no rotation of the material.

As the function f in equation (77.10) is known for a prescribed disturbance, we can, by the method already described, now substitute u in equation (77.8). In this way the stress p is determined by the expression

$$\begin{aligned} p = \frac{mE}{(m+1)(m-2)}\left\{\frac{m-1}{r}f''(r \pm at) - \frac{2(m-2)}{r^2}f'(r \pm at)\right. \\ \left. + \frac{2(m-2)}{r^3}f(r \pm at)\right\}, \quad . \quad . \quad (77.12) \end{aligned}$$

where the negative and positive signs refer, as previously, to the divergent and the convergent waves.

This phenomenon will obviously be accompanied by periodic changes in the value of the stress q, in view of the relations (77.8), whence we conclude that there is also a transverse wave in which the displacement is perpendicular to the radial direction of the element $ABCD$ in Fig. 123. The stress and the shift associated with this wave can be determined by solving for q, instead of p, in the foregoing analysis, when it will be found that the related velocity

$$a_t = \sqrt{\frac{mEg}{2(m+1)\rho}} \quad . \quad . \quad . \quad . \quad . \quad (77.13)$$

Hence the ratio

$$\frac{\text{velocity of the transverse wave}}{\text{velocity of the radial wave}} = \sqrt{\frac{m-2}{2(m-1)}} \qquad (77.14)$$

Both of these velocities will naturally contribute to the motion of a particle of the solid, and in this sense the vibration of solids is more complicated than that of perfect fluids.

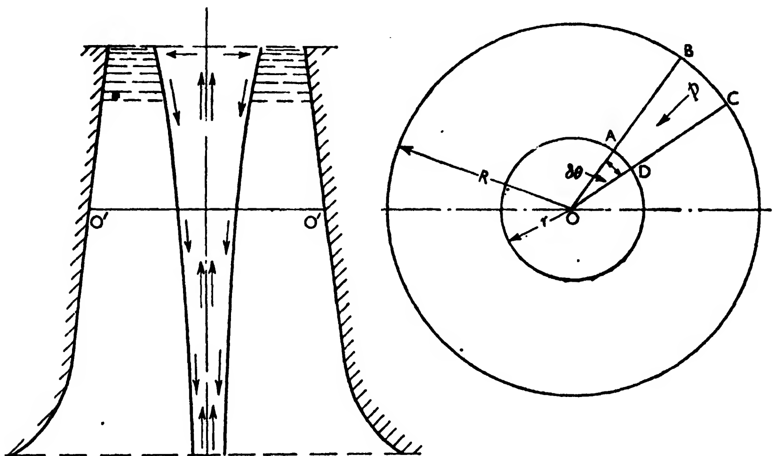
Moreover, the previous treatment suffices to indicate, when regard is had to the nature of actual materials, that a spherical wave will in general suffer reflection, distortion and dispersion at an interface of different materials. We might therefore expect, as is commonly the case, the generation of *surface waves* at the point where a disturbance encounters the boundary of an elastic solid. These remarks apply, with evident restrictions, to water, and in this connection attention may be drawn to the interface of water and vapour which surrounds a propeller operating in the condition of *cavitation*. This particular problem cannot be solved without recourse to models, but we may nevertheless throw light on the way of approach by reference to another well-known instance of instability to be found in hydraulics.

Ex. Obtain an approximate expression for the vibratory motion of the water in the draught-tube of a hydraulic turbine.

It may make the matter more intelligible if we first notice the manner in which an unstable condition may arise from a decrease in the quantity of discharge, though a full study on the point must be formulated with the aid of treatises on hydrodynamics.

For this purpose let Fig. 124(a) represent a section of a draught-tube in which the water flows parallel to the vertical axis when the machine is working steadily at its normal load, which usually corresponds to a value of from 80 per cent. to 90 per cent. of the maximum capacity for a given turbine. A slight decrease in the discharge will result in a certain amount of whirling, and then the water near the top of the tube tends to form a ' combined vortex '

of the Rankine type, consisting of a circular vortex motion about the vertical axis in the fluid moving irrotationally under the action of gravity. A further diminution of the discharge would, if continued, subsequently cause the cavity of the vortex to extend vertically from the top to the bottom of the tube, as indicated in Fig. 124(*a*). A water-particle will, on the average, then describe a spiral path about the vertical axis; and this motion will be transmitted to a vapour-particle in contact with the fluid, so that the arrows may be regarded as a rough indication of the circulation thus generated in the vapour.



(*a*) (*b*)

FIG. 124.

Now imagine this state as one of temporary equilibrium, and let Fig. 124(*b*) be a sectional view of the tube at the level $O'O'$, where R and r denote the outer and inner radii of the water column at that level.

To obtain an equation for the motion which would follow a slight disturbance about this equilibrium-position, consider the forces which then act on the fluids in OBC, subtending an angle $\delta\theta$ at the origin O, and of thickness δz in the direction at right angles to the plane of the paper. It is to be understood that these dimensions are infinitesimals, and that the arc AD relates to an interface of water and vapour. If, to this degree of approximation, a particle on the sensibly straight line AD undergoes a radial displacement δr about the position of equilibrium, it is easily proved that the consequent displacements of the centres of gravity of the water in $ABCD$ and of the vapour in OAD are $\frac{1}{2}\delta r$ and $\frac{2}{3}\delta r$, respectively.

In order to distinguish the fluids, write p and p_v for the pressures produced in this way on the water- and vapour-sides of the

interface AD, taken in succession; and let ρ, ρ_v be the densities of the corresponding media. These pressures vary according to well-known laws, but the densities remain practically constant throughout the length of an actual tube.

Taking the positive direction as radially outwards, we thus see that the disturbing force on the element amounts to $-r\delta\theta\delta z(p - p_v)$. The fact that this is equal to the product of mass and acceleration of the centres of gravity of the fluids enables us to write

$$-gr\delta\theta\delta z(p - p_v) = \tfrac{1}{2} \times \tfrac{1}{2}\rho(R^2 - r^2)\delta\theta\delta z\frac{d^2r}{dt^2} + \tfrac{1}{2} \times \tfrac{2}{3}\rho_v r^2\delta\theta\delta z\frac{d^2r}{dt^2},$$

i.e. $$-gr(p - p_v) = \tfrac{1}{4}\rho(R^2 - r^2)\frac{d^2r}{dt^2} + \tfrac{1}{3}\rho_v r^2\frac{d^2r}{dt^2}.$$

If the density of the vapour be neglected in comparison with that of the water, our result reduces to

$$-gr(p - p_v) = \tfrac{1}{4}\rho(R^2 - r^2)\frac{d^2r}{dt^2}.$$

That is to say

$$\frac{d^2r}{dt^2} = -\frac{4gr}{\rho(R^2 - r^2)}(p - p_v),$$

from which we infer that so long as $\frac{r}{R}$ remains small the relation

$$\frac{d^2r}{dt^2} = -Cr\left(1 + \frac{r^2}{R^2} + \frac{r^4}{R^4} + \frac{r^6}{R^6} + \ldots\right)(p - p_v)$$

will hold, where $C = \frac{4g}{\rho R^2}$.

A slight disturbance about the prescribed equilibrium-position would, to the first approximation, lead to the vibratory motion

$$\frac{d^2r}{dt^2} = -Cr(p - p_v),$$

given by omitting all but the first term of the series.

In specified circumstances the pressure p_v can be written down direct from tabulated data, though it is well to observe, in passing, that the value so derived may be considerably less than the vapour pressure in the actual tube, owing to the effect of impurities in the water. It is to be further noticed, from the theory of hydraulic turbines, that the pressure p is a function of the ratio $\frac{\text{partial discharge}}{\text{normal discharge}}$, for this indicates, in particular, that the rotational speed of the machine also is a factor. In other words, for a given installation the pressure p can be expressed in terms of known quantities, and the solution of the last equation inferred from (38·2). To the implied degree of accuracy the interfacial element AD therefore describes

simple-harmonic motion, and the oscillations of the complete water-column can be determined once we know the several factors which influence the pressure p in the given conditions.

The extent to which such vibrations may be sustained by water in its natural state is to be inferred from the work of B. E. Livingston and G. Lubin,[1] who demonstrated that a tensile stress amounting to hundreds of pounds per square inch can be transmitted through water containing dissolved salts and gases.

78. The Elastic Properties of the Earth. Earthquakes. Although a study of modern seismology must be sought in treatises[2] on the matter, the subject nevertheless deserves mention here for two reasons. It affords noteworthy illustrations of several phenomena which have been discussed in this chapter, and some knowledge of it naturally helps us to visualize the problems to be solved in the design of structures for regions subject to earthquakes. In order to simplify the work of reference to international tables, the various dimensions will be stated in terms of kilometres.

If, for a moment, we suppose the earth to be an elastic sphere of isotropic and homogeneous material, it is readily understood from Art. 77 (*b*) that an internal disturbance will initiate two types of waves, involving longitudinal (condensational) and transverse (distortional) displacements in the solid. In seismology these are referred to as the *primary* or *P-type* and the *secondary* or *S-type* of waves, and they are together implied in what is known as an earthquake.

But experimental and analytical investigations into the problem give strong support to the belief that the crust of the earth consists principally of three stratified layers of different composition, and that each layer by itself can be treated as isotropic. Stated briefly, the structure of the continents is approximately represented by an upper layer of granite about 12 km. in thickness with an irregular covering of sedimentary rock, upon an intermediate layer of uncertain and probably variable composition, possibly 24 km. in thickness, followed by a lower layer consisting mainly of olivine and extending without any sensible change of composition to a depth of about 480 km. At that level seismic waves would seem to undergo an appreciable change of velocity which is most easily treated by ascribing it to a sharp change in the structure, as might be associated with a corresponding variation in the density. According to R. Stoneley[3], 3 km. is the average thickness of the sedimentary layer of the continents.

[1] *Science*, vol. 65, page 376 (1927).

[2] H. Jeffreys, *The Earth*; A. E. H. Love, *Some Problems of Geodynamics*; G. W. Walker, *Modern Seismology*.

[3] *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.*, vol. 4, page 43 (1937).

Due to this lack of uniformity in composition we might expect the velocity of a given wave to vary with the depth of its path. In fact, as is shown in the following tabulated mean of the values obtained by various seismologists, the velocity of the longitudinal wave increases almost uniformly with the depth to a certain level, below which the value remains practically constant.

LONGITUDINAL OR CONDENSATIONAL WAVES IN THE EARTH

Depth in km.	Velocity in km. per sec.	Poisson's ratio.
0	7·40	0·258
100	7·68	0·272
300	8·50	0·273
600	9·62	0·274
1,000	11·16	0·270
1,400	12·23	—
2,500	12·90	—

Somewhat different values have more recently been obtained by H. Jeffreys,[1] who estimated that the velocity of the same wave increases from 7·71 km. to 9·08 km. per sec. between the levels of 40 km. and 480 km., and so up to 9·8 km. per sec. for greater depths. He also found that the velocity of the transverse wave starts from about 4·36 km. per sec. and increases nearly, but not exactly, in proportion.

It may be observed, in passing, that the manner in which the density varies will, on the average, cause the path of a wave to be concave outwards as shown in Fig. 125. Moreover, if in the work of geophysical surveying (Chapter VII) we take Poisson's ratio $\frac{1}{m}$ to be 0·26, then the expression (77.14) yields the useful relation

$$\frac{\text{velocity of the transverse wave}}{\text{velocity of the longitudinal wave}} = 0{\cdot}57.$$

To proceed with the general matter, if the depth of the origin or focus of an earthquake is such that the wave-fronts are curved and not plane, the *P*- and *S-waves* together can generate at the free surface another kind of disturbance, having an amplitude that dies away rapidly as the depth of the path increases. This is called the *Rayleigh wave,* and the particles may move both horizontally and vertically in the plane of propagation. Simultaneously, a second kind of surface movement can thus be initiated, but this involves vibrations in a plane at right angles to the direction of propagation. This is called the *Love wave,* and the particles move

[1] *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.,* vol. 4, page 55 (1937).

only horizontally and perpendicular to the direction of propagation. It follows that the combined effect of Rayleigh and Love waves is a surface movement in all three directions. Thus the theory indicates, and experience confirms, that a disturbance in the earth may produce two types of body-waves and two kinds of surface waves.

At the interfaces of the several layers the waves will, as already pointed out, be modified by such phenomena as reflection and refraction. On this account a transverse wave with vertical displacement can initiate either a reflected or a refracted disturbance

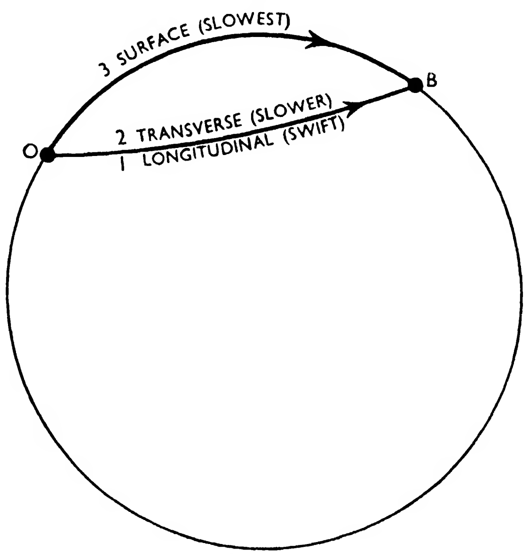


FIG. 125.

of the longitudinal type, but a transverse wave with horizontal displacement can induce a reflected or a refracted disturbance of the transverse type alone.

In the modern study of the subject it is customary to draw a distinction between ' near ' and ' distant ' earthquakes, though for engineers the former of these main classes is by far the most important, because considerable damage to buildings is commonly associated with the surface movement due to ' near ' earthquakes.

It is thus seen that if an earthquake originates in the upper layer of the crust, the resulting wave may, on encountering or traversing the various layers, undergo a modification in type, and in this way generate a number of waves identified with the reflection and refraction of a particular disturbance. On some seismograms for near earthquakes the *P*- and *S*-types are therefore accompanied by a second pair of waves which have lower velocities than those

referred to above; and a third pair appears on certain records. We have thus accounted for three condensational and three distortional forms of disturbance, though it is to be pointed out that this number by no means exhausts the series mentioned in the literature on the subject. It is usual to assign the symbols P, Pg, P^x in turn to the three condensational waves, and S, Sg, S^x to the related distortional waves. According to H. Jeffreys the mean values of the velocities, expressed in kilometres per second and arranged in decreasing order of magnitude, are

P	P^x	Pg	S	S^x	Sg.
7·73	6·25	5·50	4·28	3·73	3·23

The variations to be found in these velocities when referred to certain continents may be subsequently explained by the presence of corresponding changes in the formation of the earth at the places concerned.

In the case of an earthquake due to a fracture under shear of the granite in the upper layer of the crust, the sequence of the waves is, by Jeffreys' theory, as follows. Such a fracture would probably initiate an Sg-wave, together with a P-wave that starts from the same focus after an interval of time which may be as great as two seconds. The paths of these waves will, of course, extend in the upward and downward directions.

Taking these in succession, in travelling upwards the energy of the P-wave will be partly transmitted as a Pg-wave along the interface formed by either the surface of the earth or the base of the sedimentary layer, and partly reflected downwards as a P-wave in the upper layer. In travelling downwards, the latter will ultimately reach the boundary between the upper and intermediate layers, where part of its energy is refracted into the intermediate layer as a P^x-wave that travels along or near the interface. This refraction closely resembles that of a light-ray when it approaches, at the critical angle, the boundary of a medium having a lower refractive index. The P-wave will continue its course downwards through the intermediate layer and, subsequently, arrive at the base of that layer, where a fraction of its energy may be transmitted through the interface and thus enter the lower layer.

Turning next to the part of the Sg-wave which is initially propagated downwards, at the interface of the upper and intermediate layers it gives rise to an S^x-wave, the energy of which is partly transmitted in the intermediate layer near the periphery and partly refracted into the lower layer. The refracted part travels in the lower layer near the boundary, as in the case of the P-wave mentioned above.

In this way it is seen that a P-type, for example, may emerge

as a direct wave at a certain distance from its focus, or suffer one or more refractions and thus reappear at increasing angular distances from the epicentre.

An earthquake accordingly involves a number of pulses, of which the Pg and Sg travel near the surface of the earth, the P^x and S^x near the periphery of the intermediate layer, and the P and S near the periphery of the lower layer. With a disturbance at O in Fig. 125, a seismogram taken at B would indicate the arrival of the various 'phases', characterized by sudden or gradual changes in the amplitude, or sudden changes in the period, or both. These changes always occur in regular order, and some are exhibited on all seismograms. The relatively slow rate at which the amplitude diminishes is attributed to the presence of irregular interfaces; and the scattering of the energy due to irregularities in the structure of the layers has been found to account for the dying out of the pulses at about the distance where they actually disappear in the case of 'near' earthquakes.

The core of the earth, which is supposed to extend rather more than half the radius, possesses elastic properties comparable with those of liquid iron containing a small percentage of nickel, and it is distinguished by the fact that distortional waves do not reappear beyond the core. What happens to a wave which penetrates the core is a matter for conjecture, but this is an irrelevant point as regards engineering. It is here sufficient to remark that a large number of waves of different kinds may be generated by a great earthquake, due to internal reflection and refraction.

Some of the components of the motion may be suppressed, as is illustrated by a series of observations undertaken by N. Nasu.[1] Apart from their intrinsic value, the results throw light also on the basis of the assumption made in the example of Art. 72, as to the tendency of certain harmonic components of a disturbance to travel on a given side of an interface. In these experiments two seismographs were used, one being placed inside a tunnel and the other at the surface 525 ft. vertically above the first instrument; and the rock was of the same kind and compactness throughout the region examined. A comparison of more than 100 records showed that the movement at the surface was usually greater than in the tunnel, but the inequality varied with the period of vibration. For example, when the impressed period was less than one second, the amplitude at the surface was invariably more than twice that in the tunnel; it was 4·8 times as much during a disturbance having a period of 0·3 sec. In the case of tremors with very long periods, between 4 sec. and 5 sec., the amplitudes at both stations were nearly equal, and even slightly less above the surface than below

[1] *Bull. Earthq. Inst. of Japan*, vol. 9, page 454 (1931).

during one shock. Moreover, when the impressed period was less than one second, in some cases quick vibrations were superposed on slow ones at the surface, while the quick oscillations were suppressed in the tunnel.

Further research into the general problem of earthquakes will no doubt elucidate many obscure questions, some of which naturally relate to the properties of materials subjected to intense pressures in the interior of the earth. The immediate consequences of a fracture, for instance, are still imperfectly understood, as is to be gathered from a discussion by H. Jeffreys.[1] The accumulation of more data may therefore necessitate modifications in our picture of the sequence of waves in specified circumstances. Nevertheless, there are now good reasons to suppose that the *P*- and *S*-types of waves do not always start simultaneously from the same focus. As a matter of fact, the information derived from great earthquakes supplies, according to present ideas, strong evidence in support of the view that a certain degree of independence exists between the generation of the *P*- and *S*-types, to an extent which is in some way related to the conditions at the focus.

From the engineering point of view, then, we may say that for regions near the earth's surface the *P*-wave may travel with a speed of from 7 km. to 8 km. per sec.; and the *S*-wave, as demonstrated above, travels at approximately 0·57 times that speed. The periods of the corresponding vibrations may, in the case of a near focus, be as short as one second for the *P* and two seconds for the *S* types of disturbance. The Rayleigh and Love waves may be regarded as forming the third train; the greatest amplitude for a given shock is associated with this complex system of oscillations, having a period within the range of 40 sec. and 60 sec., and a velocity of from 3 km. to 4 km. per sec.

The resultant disturbance, which alone is of interest to engineers, has a period of the order of one second, a wave-length that lies between the limits of 5 miles and 40 miles, and a minimum speed of about 2 miles a second. It may help in the way of visualizing this velocity if we compare it with that of an ocean swell, which travels at about 65 ft. per sec.

A building is therefore not in general at rest when the serious part or 'phase' of an earthquake strikes it, for at that instant the structure is already executing small vibrations under the influence of the fast-moving components of the disturbance. Thus the immediate result of a severe shock depends on the relative phases of the oscillation and the impressed force at the instant of impact.

79. Stresses produced by Impacts. In the main this chapter has been restricted to an account of forces which vary

[1] *Proc. R.S. Edin.*, vol. 46, page 158 (1936).

according to a harmonic law, but there are obviously many instances where the disturbing force is suddenly applied to a structure. The general characteristics of the motion which ensues from an impulse are, however, well exhibited in the work of designing buildings to withstand earthquakes. This is so because, in the opinion of the author, an inspection of seismograms for 'near' earthquakes shows that the serious phase of such disturbances can, for many practical purposes, be regarded as an impact. But the assumed impact is by no means instantaneous, as it may last for an interval of time which, though short, is comparable with the natural period of vibration of a principal part of the buildings affected. Under these conditions the consequent motion of a structure is not a free vibration, as is widely supposed, but a forced vibration. It is difficult to avoid the complicated problem thus presented, for a survey of devastated regions does not generally lead to the view that resonance alone was the cause of the damage done to the buildings concerned, since that could occur only if the natural periods of all the structures were approximately equal to the impressed period.

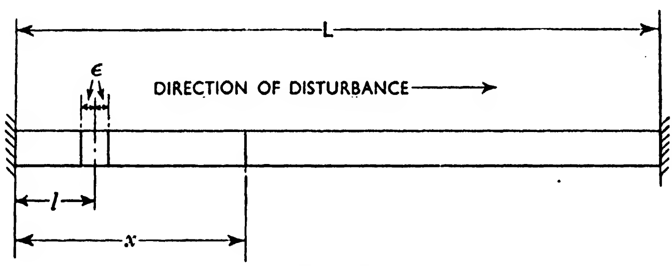


FIG. 126.

To examine the question with reference to an important part of buildings in general, let Fig. 126 represent the dynamical equivalent of a truss of length L, of which the interval $(l - \varepsilon, l + \varepsilon)$ is covered by a perfectly finished joint. The centre line of this joint is therefore distant l from the end shown in the figure. With the structure initially at rest in the equilibrium-position, suppose the joist to receive the impact by way of the intermediate joint, in the direction of its length, which we shall take as coinciding with the x-axis. We may assume, with a perfect joint, that the impulse is evenly distributed over the metal situated within the axial interval $(l - \varepsilon, l + \varepsilon)$. The joist will, in addition, be taken as having a constant line density ρ, and a dimension ε which is small in comparison with L.

On the suppositions inplied in Art. 63 we can now write

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + X \quad . \quad . \quad . \quad . \quad . \quad . \quad (79.1)$$

for the forced vibration, where u denotes the shift of the particles, at any instant t and position x, due to the plane wave thus associated with an extraneous force X. More generally, X is the x-component of the impressed force. Here a is the velocity with which the wave travels through the material of the joist.

In all questions of impact, and the present one in particular, considerable difficulty attends the formulation of the force X. Let us, then, imagine Fig. 127 to be the force-time graph derived

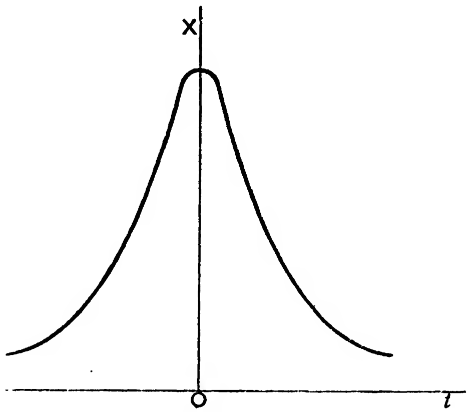


FIG. 127.

when a powerful time-microscope is applied to the 'peak' that matters on the seismogram of a given earthquake, where we neglect what precedes and follows that part of the record. A graph of this type can be expressed by

$$X = \frac{1}{\pi} \frac{\mu\tau}{t^2 + \tau^2}, \qquad (79.2)$$

where μ is the impulse or time integral of the force X from $t = -\infty$ to $t = +\infty$, and τ is a very short interval of time. This equation clearly exhibits the essential characteristic of such disturbances, in so far as a 'sharp' shock may thus be represented by making the interval of time τ sufficiently short. Since we are now interested chiefly in the maximum value of X, an irrelevant point is involved in the fact that equation (79.2) yields no information as to the instants when the impact starts and ceases to operate on the structure. With regard to the magnitude of τ, a study of the seismograms for a given place affords the only means of arriving at a reasonable value, as the personal experience of the author lends strong support to his opinion that τ is in general influenced not only by mechanical factors, but also by geological ones, such as the presence of mountains, and the tendency of the earth at certain places to vibrate like a large flat plate.

On our supposition that the impact is evenly distributed over the metal covered by the joint, and that $\frac{\partial u}{\partial t} = 0$ when $t = 0$ except for points within the interval of length $(l - \varepsilon, l + \varepsilon)$, with the help of equation (68.6) we see that

$$\mu = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi}{L} x, \quad \ldots \ldots \quad (79.3)$$

where, on writing I for the total impulse imparted to the joist,

$$\begin{aligned} F_n &= \frac{2}{L}\int_{l-\epsilon}^{l+\epsilon} \mu \sin \frac{n\pi}{L} x \, dx \\ &= \frac{4\rho\mu\varepsilon}{n\pi\rho\varepsilon} \sin \frac{n\pi l}{L} \sin \frac{n\pi\varepsilon}{L} \\ &= \frac{2I}{n\pi\rho\varepsilon} \sin \frac{n\pi l}{L} \sin \frac{n\pi\varepsilon}{L} \quad . \quad . \quad . \quad . \quad (79.4) \end{aligned}$$

In numerical applications it is well to notice that here ρ refers to the weight per unit *length* of the given member.

When only the first two or three harmonic components are of account, F_n is thus approximately determined by

$$F_n = \frac{2I}{\rho L} \sin \frac{n\pi l}{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (79.5)$$

in that $\sin \frac{n\pi\varepsilon}{L} \rightarrow \frac{n\pi\varepsilon}{L}$ as $\frac{\varepsilon}{L} \rightarrow 0$ for small values of n.

But this simple result does not hold in cases where n tends to infinity. To proceed, we then assume

$$u = \sum_{n=1}^{\infty} \phi_n \sin \frac{n\pi}{L} x, \quad . \quad . \quad . \quad . \quad . \quad . \quad (79.6)$$

with ϕ_n a function of the time alone. If this expression for the shift u be utilized in connection with equations (79.1), (79.2) and (79.3), it will be found that

$$\frac{d^2\phi_n}{dt^2} + \frac{n^2\pi^2a^2}{L^2}\phi_n = \frac{1}{\pi}\frac{F_n\tau}{t^2 + \tau^2}.$$

This, as may be proved by differentiation, has the solution

$$\begin{aligned} \phi_n = \frac{F_nL\tau}{n\pi^2a}\Big(&\sin \frac{n\pi a}{L}t \int^t \frac{1}{t^2 + \tau^2} \cos \frac{n\pi a}{L} t \, dt \\ &- \cos \frac{n\pi a}{L}t \int^t \frac{1}{t^2 + \tau^2} \sin \frac{n\pi a}{L} t \, dt\Big), \quad . \quad . \quad (79.7) \end{aligned}$$

where we omit terms of the type $\left(A \cos \frac{n\pi a}{L}t + B \sin \frac{n\pi a}{L}t\right)$ because the lower limits of the integrals are indeterminate, for the reason already mentioned.

In order to ensure that the joist is initially at rest in the equilibrium-position, it is evident that both ϕ_n and $\dot{\phi}_n$ must vanish as $t \to -\infty$. In these circumstances the expression

$$\phi_n = \frac{F_n L\tau}{n\pi^2 a}\left(\sin\frac{n\pi a}{L}t\int_{-\infty}^{t}\frac{1}{t^2+\tau^2}\cos\frac{n\pi a}{L}t\,dt - \cos\frac{n\pi a}{L}t\int_{-\infty}^{t}\frac{1}{t^2+\tau^2}\sin\frac{n\pi a}{L}t\,dt\right)$$

follows from equation (79.7). When the disturbing force has sensibly ceased to operate, we can by this means write

$$\phi_n = A\cos\frac{n\pi a}{L}t + B\sin\frac{n\pi a}{L}t,$$

where
$$A = -\frac{F_n L\tau}{n\pi^2 a}\int_{-\infty}^{\infty}\frac{1}{t^2+\tau^2}\sin\frac{n\pi a}{L}t\,dt$$
$$= 0,$$

and
$$B = \frac{F_n L\tau}{n\pi^2 a}\int_{-\infty}^{\infty}\frac{1}{t^2+\tau^2}\cos\frac{n\pi a}{L}t\,dt$$
$$= \frac{F_n L}{n\pi a}e^{-\frac{n\pi a\tau}{L}}.$$

For information on the method of evaluating these integrals the reader may be referred to treatises on the calculus,[1] and to tables.[2]

Hence
$$\phi_n = \frac{F_n L}{n\pi a}e^{-\frac{n\pi a\tau}{L}}\sin\frac{n\pi a}{L}t,$$

which enables us to put equation (79.6) in the form

$$u = \frac{F_n L}{\pi a}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{n\pi a\tau}{L}}\sin\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x, \qquad (79.8)$$

and so deduce, on replacing F_n by the expression (79.4),

$$u = \frac{2IL}{\pi^2\rho a\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{n\pi a\tau}{L}}\sin\frac{n\pi l}{L}\sin\frac{n\pi\varepsilon}{L}\sin\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x \qquad (79.9)$$

This is the general relation for the shift of the particles at the position x and time t.

In cases where a sufficiently accurate result is given by restricting the analysis to small orders of the harmonics denoted by n, we may introduce into equation (79.8) the approximate relation (79.5) for F_n. The procedure leads to

$$u = \frac{2I}{\pi\rho a}\sum\frac{1}{n}e^{-\frac{n\pi a\tau}{L}}\sin\frac{n\pi l}{L}\sin\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x, \qquad (79.10)$$

[1] G. A. Gibson, *Advanced Calculus*, page 469.

[2] D. B. de Haan, *Nouvelles Tables d'Intégrales Définies*, table 202, formulae 9 and 11 (Leyden, 1867).

with appropriate limits assigned to the symbol of summation. To this degree of approximation the fundamental mode, for example, is associated with the shift

$$u = \frac{2I}{\pi\rho a} e^{-\frac{\pi a \tau}{L}} \sin\frac{\pi l}{L} \sin\frac{\pi a}{L} t \sin\frac{\pi}{L} x,$$

obtained by making $n = 1$ in equation (79.10).

These results offer a direct means of tracing the shift u and, consequently, the stress thus induced in the specified member, provided always that the elastic limit of the material is not exceeded. Experimental investigations into the matter may disclose a difference between the theoretical and actual values, since we have neglected the effect of the forces of gravity and damping. Under certain conditions, however, these agencies annul each other in the present sense, depending on the magnitude of the shock and the weight of the complete structure of which the member forms a part. Mention should also be made of our assumption that the joint, at $x = l$ in Fig. 126, is perfect in every respect, as it represents a degree of fixity which is rarely met with in regions subject to frequent tremors of the earth. This emphasizes the need for careful design and manufacture of joints and connections in this class of structure.

In the above treatment we have considered a structural member whose mass and coefficient of stiffness at any point on the longitudinal axis are symmetrically disposed about that axis. If either of those quantities be unsymmetrical in this sense, however, it is not difficult to see, on taking account of the impulse and the manner in which waves have been shown to be propagated in actual materials of finite length, that the ends of the member will in general vibrate in the transverse plane of Fig. 126. The magnitude of the movement will depend on the extent to which the form of the member and its method of support differ from the conditions implied in the foregoing problem. This displacement may be neglected in all but special circumstances; an important case is to be found in a gun or a rifle whose muzzle is vibrating due to the effect of the explosion in the breech, together with that of the non-symmetrical disposition of both the mass and the coefficient of stiffness of the barrel about its longitudinal axis. The disturbance, commonly called 'jump' with guns and 'flip' with rifles, usually involves harmonic components of high frequency which arrive at the muzzle before the shot, and a velocity at right-angles to the centre-line of the bore is thus given to the projectile as it emerges from the gun. A practical significance may then be attached to vibrations of relatively small amplitude, since a deflection of only one minute of arc at the muzzle is approximately equivalent to a deviation of 10·5 inches at a range of 1,000 yards.

80. Continuous Wave Motion. It is a simple matter to extend the preceding theory to the case of an indefinitely repeated application of harmonic forces.

The arrangement shown in Fig. 128 offers an instructive example, involving a uniform pipe filled with liquid, through which pressure

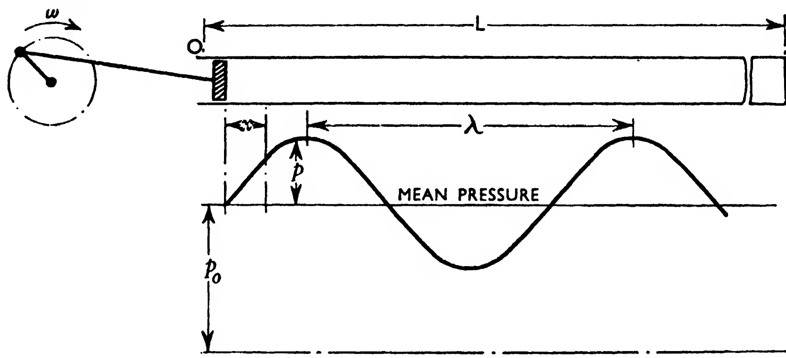


FIG. 128.

waves are propagated by means of a reciprocating plunger or piston having a stroke sufficiently small to secure elastic conditions throughout the cycle. The end B of the pipe is open, and the mechanism at the origin O consists of a crank of throw r, rotating with an angular velocity ω which we shall take to be constant.

If u be the movement of the plunger at any instant t, we see from Chapter I that it can be approximately defined by

$$u = r \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (80\cdot1)$$

Let us, at first, neglect the inherent frictional forces, then, with the liquid specified by its bulk modulus K and density ρ, equation (76.4) gives

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

for the shift u of the fluid-particles at the position x and time t, measured from the origin O in the figure. Here the velocity of propagation $a = \sqrt{\frac{Kg}{\rho}}$. It is easily proved that the solution of this equation may be written in the form

$$u = r \sin\left(\omega t - \frac{\omega}{a}x\right)$$

$$= r \sin\left\{-\frac{\omega}{a}(x - at)\right\}; \quad . \quad . \quad . \quad . \quad . \quad (80.2)$$

thus $u = r \sin \omega t$ at the position $x = 0$, showing that equation (80.1) also is satisfied.

Moreover, according to equation (76.3) the prescribed disturbance in the fluid will induce a pressure

$$p = -K\frac{\partial u}{\partial x}$$

$$= \frac{K\omega r}{a}\cos\left(\omega t - \frac{\omega}{a}x\right),$$

from equation (80.2). If p_0 be the mean or static pressure on the pipe, the total pressure will necessarily amount to $p_0 + p$ at the point x and time t, as indicated by the graph of the resulting pressure wave which is included in Fig. 128.

With a view to calculating the power thus transmitted through the fluid, suppose the cross-sectional area of the pipe to be unity, say 1 sq. ft., then we have the

$$\text{instantaneous value of the power transmitted} = p\frac{\partial u}{\partial t}$$

$$= \frac{K\omega^2 r^2}{a}\cos^2 \omega\left(t - \frac{x}{a}\right).$$

Hence, remembering that the period is $\frac{2\pi}{\omega}$, the

$$\text{mean power transmitted} = \frac{\omega}{2\pi}.\frac{K\omega^2 r^2}{a}\int_0^{\frac{2\pi}{\omega}} \tfrac{1}{2}\left\{1 + \cos 2\omega\left(t - \frac{x}{a}\right)\right\} dt$$

$$= \frac{K\omega^2 r^2}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80.3)$$

In terms of the maximum pressure P and the maximum velocity V, which are given by

$$P = \frac{K\omega r}{a} \text{ and } V = \omega r,$$

$$\text{the mean power transmitted} = \tfrac{1}{2}PV.$$

By way of indicating the magnitude of the wave-length which may be found in practice, we deduce from the general relation

velocity of propagation = wave-length × frequency

the fact that here the

$$\text{wave-length } \lambda = \frac{2\pi a}{\omega}$$

$$= \frac{a}{N}$$

if N is the number of cycles described in unit time. For example, if the number of cycles be 800 a minute, and a = 4,794 ft. per sec., then

$$\lambda = \frac{4{,}794 \times 60}{800}$$

$$= 360 \text{ ft.},$$

approximately. It may be pointed out, by the way, that this figure is about 5·5 per cent. greater than the value which was derived from tests with a particular hydraulic installation, the difference being due partly to the inherent frictional agencies.

These dissipative forces modify the motion in a manner which may now be investigated on the supposition that the energy thus lost to the system is proportional to the velocity $\frac{\partial u}{\partial t}$ of the particles. Writing b for the frictional coefficient, which is positive, we have seen that

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (80.4)$$

is the corresponding equation of motion. This procedure is always practicable in finding a first approximation, even in cases where the friction is not strictly proportional to the velocity of shift, as we can then assign a value to b such as will represent the mean of the actual coefficient within an appropriately small range of values of $\frac{\partial u}{\partial t}$. In instances where Poiseuille's law of fluid friction holds, $b = \frac{32\eta g}{\rho d^2}$ for a pipe of diameter d, containing a fluid having a density ρ, and a coefficient of viscosity η.

With systems in which b is comparatively small, it is not difficult to verify that the solution of equation (80.4) is

$$u = re^{-\frac{1}{2}\frac{b}{a}x}\sin\left(\omega t - \frac{\omega}{a}x\right) \quad . \quad . \quad . \quad . \quad (80.5)$$

Under these conditions the shift u gradually dies away, on account of the exponential term ; this is true also of the related pressure, for

$$p = -K\frac{\partial u}{\partial x}$$

$$= \frac{K\omega r}{a}e^{-\frac{1}{2}\frac{b}{a}x}\cos\left(\omega t - \frac{\omega}{a}x\right) \quad . \quad . \quad . \quad (80.6)$$

if we omit a term in which $\frac{1}{2}\frac{b}{a}$ figures as a factor. Now

$$\text{the instantaneous power transmitted} = p\frac{\partial u}{\partial t}$$

$$= \frac{K\omega^2 r^2}{a}e^{-\frac{b}{a}x}\cos^2\left(\omega t - \frac{\omega}{a}x\right),$$

whence, for the position x, the mean power per stroke of the plunger amounts to

$$\frac{K\omega^2 r^2}{2a}e^{-\frac{b}{a}x}.$$

Since, at x,

the maximum pressure $P = \frac{K\omega r}{a} e^{-\frac{1}{2}\frac{b}{a}x}$

and the maximum shift-velocity $V = \omega r e^{-\frac{1}{2}\frac{b}{a}x}$, it follows that in this notation the maximum power per stroke of the plunger is equal to $\frac{1}{2}PV$.

These results signify, on taking pound-foot-second units, that the horse-power transmitted through a pipe of cross-sectional area A amounts to $\frac{APV}{1{,}100}$; and the horse-power dissipated in friction amounts to $\frac{b}{1{,}100a}APV$, being equal to $-\frac{A}{1{,}100}\frac{\partial}{\partial x}(PV)$.

81. Transmission of Energy by Pressure Waves. It will now be manifest that by fitting a plunger to the open end of the pipe in Fig. 128 we might transmit energy through the fluid, from the left-hand plunger or 'generator' to the right-hand plunger or 'motor' shown in Fig. 129.

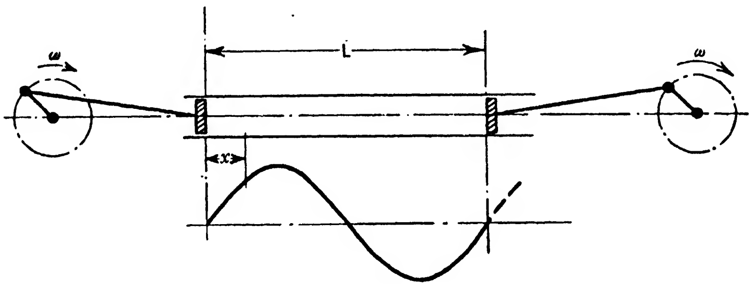


FIG. 129.

Assuming the dissipative forces to be negligibly small, we have, as previously,

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},$$

in the notation of Art. 80. If C, ε refer to arbitrary constants,

$$u = C \sin\left(\frac{\omega}{a}x + \varepsilon\right) \sin \omega t \quad . \quad . \quad . \quad . \quad (81.1)$$

is a solution of the present equation of motion, because it gives

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 C \sin\left(\frac{\omega}{a}x + \varepsilon\right) \sin \omega t,$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\omega^2}{a^2} C \sin\left(\frac{\omega}{a}x + \varepsilon\right) \sin \omega t.$$

To investigate the end conditions, suppose the motor-plunger

to be removed, so that the pipe is open at $x = L$ in Fig. 129. This results in the conditions

$$p = 0 \qquad \text{at } x = L, \; u = r \sin \omega t \text{ at } x = 0$$

for all values of the time t. Since the instant $t = 0$ corresponds to the position $x = 0$, the latter of these conditions requires

$$r = C \sin \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81.2)$$

Further, by virtue of the relation

$$p = -K\frac{\partial u}{\partial x}$$

we obtain, from equation (81.1), the pressure

$$p = -KC\frac{\omega}{a} \cos\left(\frac{\omega}{a}x + \varepsilon\right) \sin \omega t \quad . \quad . \quad . \quad (81.3)$$

Therefore $\qquad 0 = -KC\frac{\omega}{a} \cos\left(\frac{\omega}{a}L + \varepsilon\right) \sin \omega t$

must hold at the point $x = L$. Combining this with equation (81.2) we find

$$0 = -\frac{K\omega r}{a} \frac{\cos\left(\frac{\omega}{a}L + \varepsilon\right)}{\sin \varepsilon} \sin \omega t,$$

i.e. $\qquad 0 = \cos \frac{\omega}{a}L \cot \varepsilon - \sin \frac{\omega}{a}L,$

or $\qquad \cos \frac{\omega}{a}L \cot \varepsilon = \sin \frac{\omega}{a}L,$

or $\qquad \tan \varepsilon = \cot \frac{\omega}{a}L,$

in consequence of which

$$\varepsilon = \frac{\pi}{2} - \frac{\omega}{a}L.$$

Now, from equation (81.3),

$$\begin{aligned} p &= -\frac{K\omega r}{a}\left(\cos \frac{\omega}{a}x \cot \varepsilon - \sin \frac{\omega}{a}x\right) \sin \omega t \\ &= -\frac{K\omega r}{a}\left(\cos \frac{\omega}{a}x \tan \frac{\omega}{a}L - \sin \frac{\omega}{a}x\right) \sin \omega t \quad . \quad (81.4) \end{aligned}$$

determines the pressure at the position x and time t. This pressure is a maximum when $\sin \omega t = 1$ and therefore tends to infinitely large values when

$$\frac{\omega}{a}L = (2n + 1)\frac{\pi}{2},$$

where $n = 0, 1, 2, 3, \ldots$ For this reason we must avoid, so far as possible, having a pipe whose length

$$L = \frac{\pi a}{2\omega}, \frac{3\pi a}{2\omega}, \frac{5\pi a}{2\omega}, \ldots,$$

which correspond to the series of values

$$L = \tfrac{1}{4}\lambda, \tfrac{3}{4}\lambda, \tfrac{5}{4}\lambda, \ldots,$$

since the wave-length $\lambda = \dfrac{2\pi a}{\omega}$.

If the motor-plunger be now replaced, the end conditions become

$$\begin{aligned} u &= 0 \quad\quad\quad \text{at the end } x = \mathrm{L}, \\ u &= r \sin \omega t \text{ at the end } x = 0. \end{aligned}$$

In this system the waves will undergo partial reflection at both ends of the line, so that the effective pressure at any point is obtained by superposing the incident and reflected waves at that point, in accordance with the preceding treatment. On repeating the foregoing operations with the present end-conditions, it will be found that, in the absence of any losses, the wave-pressure on the plunger is always zero in installations where

$$L = \tfrac{1}{4}\lambda, \tfrac{3}{4}\lambda, \tfrac{5}{4}\lambda, \ldots$$

This is so because the original and reflected waves then induce on the plunger the pressures p and $-p$, with the result that the plunger is subjected to the static pressure p_0 alone, as indicated by Fig. 130(*a*). Similarly, in systems where

$$L = \tfrac{1}{2}\lambda, \tfrac{3}{2}\lambda, \tfrac{5}{2}\lambda, \ldots,$$

a reflected wave of intensity p arrives at the plunger just as it completes an in-stroke, so that $2p$ is the wave-pressure on the plunger. This case is represented by Fig. 130(*b*), where the mid-length is consequently a point of zero pressure. It may be demonstrated in a like manner that Fig. 130(*c*) relates to a system in which $L = \lambda$, where the pressure is permanently zero at the points marked N, and the wave-pressure on the plunger amounts to $2p$.

It is clear that a *standing wave* is involved in all these cases, and continuous operation of the generator would lead to a gradual rise of the pressure, at a rate which would depend on the magnitude of the frictional forces. When standing waves are set up in a pipe having a plunger at one end and a reflecting surface at the other, the relative amplitudes of the changes of pressure at nodes and antinodes will, of course, be influenced by the extent to which the energy reaching the surface is thrown back into the pipe.

For systems in which the dissipative forces are negligibly small,

our results together show that very large values of p are liable to occur if the length L is

an *odd* multiple of $\frac{1}{4}\lambda$ with a pipe having an open end,
or an *odd* multiple of $\frac{1}{2}\lambda$ with a pipe having both ends closed.

Such values of L must therefore be avoided in this kind of mechanism.

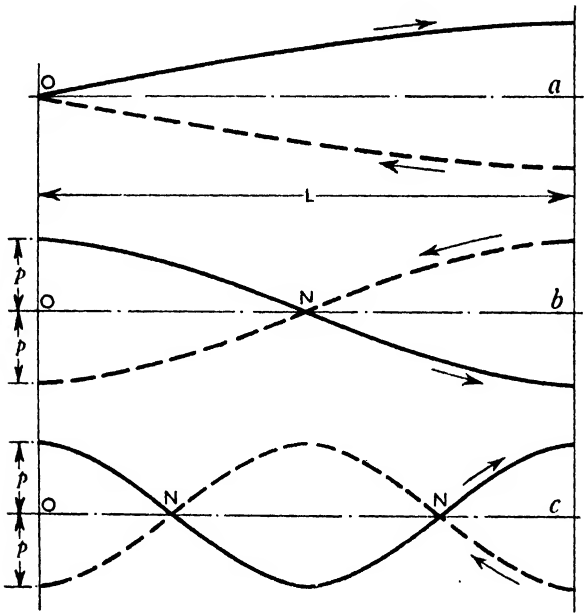


FIG. 130.

These equations have applications in various mechanical devices, and notably in the well-known gear associated with the name of G. Constantinesco. In practice it is usual to maintain the static pressure p_0 by means of a separate pump which is capable of dealing with the leakage losses. It is also advisable to extract the air from the working fluid, for the pressure may fall sufficiently low to release any dissolved air, and thereby destroy the wave motion.

82. Condensers. To prevent the excessively high pressures which we have found to be identified with particular values of the ratio $L:\lambda$ in Art. 81, use is sometimes made of an arrangement which will be referred to as a *condenser*. A simple form of this device consists of a closed vessel having a large volume compared with the displacement of the generator-plunger, and connected to the system as shown in Fig. 131. Here we suppose the right-hand end of the pipe to be closed by a second plunger or a valve.

In order to explain the consequences of this modification, let A be the cross-sectional area of the pipe, and V the volume of the

condenser. As before, the position of the plunger may be specified by

$$u = r \sin \omega t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82.1)$$

where r is the throw of the crank, working with an angular velocity ω which will be taken as constant. With the dimension x measured

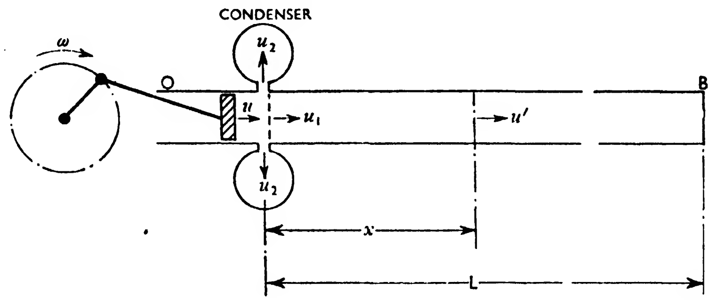


FIG. 131.

from the position shown in Fig. 131, on writing u' for the shift of the fluid-particles at x, the expression

$$u' = C \sin\left(\frac{\omega}{a}x + \varepsilon\right) \sin \omega t \quad . \quad . \quad . \quad . \quad (82.2)$$

follows from equation (81.1), where the arbitrary constants C and ε depend on the circumstances of the motion.

The motion of the fluid-particles will be modified by the action of the condenser, in a manner which may conveniently be considered in three steps.

First, if p' be the pressure associated with the shift u' in the last equation, by the method of Art. 81 we obtain

for the pipe $\quad p' = -K\dfrac{\partial u'}{\partial x}$

$$= -KC\frac{\omega}{a} \cos\left(\frac{\omega}{a}x + \varepsilon\right) \sin \omega t.$$

Therefore, if p_1 denotes the pressure identified with the shift u_1 at the position $x = 0$, when the wave has just traversed the inlet to the condenser its pressure

$$p_1 = -KC\frac{\omega}{a} \cos \varepsilon \sin \omega t. \quad . \quad . \quad . \quad . \quad (82.3)$$

deduced by making $x = 0$ in the expression for p'.

Next, by the aid of equations (82.1) and (82.2), we have

for the condenser $p_1 = K\dfrac{\text{change of volume}}{\text{initial volume}}$ of condenser

$$= \frac{KA(u - u_1)}{V}$$

$$= \frac{KA}{V}(r \sin \omega t - C \sin \varepsilon \sin \omega t),$$

from equations (82.1) and (82.2) when $x = 0$. As this and equation (82.3) relate to the same quantity, the relation

$$-C\frac{\omega}{a}\cos\varepsilon = \frac{A}{V}(r - C\sin\varepsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82.4)$$

must hold.

Finally,

for the pipe $\qquad u' = 0$ at the point $x = L$,

which, on introducing into equation (82.2), leads to

$$0 = C\sin\left(\frac{\omega}{a}L + \varepsilon\right)\sin\omega t.$$

By reason of the fact that the amplitude C cannot be zero for all values of the time t if the installation is working, the last equation enables us to write

$$\sin\left(\frac{\omega}{a}L + \varepsilon\right) = 0,$$

whence $\qquad \frac{\omega}{a}L + \varepsilon = n\pi,$

where $n = 0, 1, 2, 3, \ldots$ Hence

$$\varepsilon = n\pi - \frac{\omega}{a}L \quad . \quad . \quad . \quad . \quad . \quad (82{\cdot}5)$$

defines one of the constants.

With this expression for ε substituted in equation (82.4), we find

$$-C\frac{\omega}{a}\cos\left(n\pi - \frac{\omega}{a}L\right) + \frac{AC}{V}\sin\left(n\pi - \frac{\omega}{a}L\right) = \frac{Ar}{V},$$

hence

$$C = \frac{r\sec\left(n\pi - \frac{\omega}{a}L\right)}{\tan\left(n\pi - \frac{\omega}{a}L\right) - \frac{V\omega}{Aa}} \quad . \quad . \quad . \quad . \quad . \quad (82.6)$$

determines the remaining constant. Thus all the quantities in equation (82.3) are known for a given system, whence the pressure p_1 can be calculated.

In the important case where L is a multiple of the wave-length λ, for example, the expression reduces to

$$C = \frac{Ara}{V\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (82.7)$$

since $\qquad L = m\lambda = \frac{2\pi a}{\omega}m,$

provided $m = 1, 2, 3, \ldots$ If, in these circumstances, P denote

the maximum permissible pressure on the line, $\cos \varepsilon = 1$ in equation (82.3), so that the maximum pressure

$$P = KC\frac{\omega}{a},$$

i.e.

$$P = \frac{KAr}{V}$$

according to equation (82.7). To ensure that the pressure shall not exceed a prescribed value P, we therefore require a condenser with a volume

$$V = \frac{Kv}{2P}, \quad \ldots \ldots \quad (82.8)$$

where v is the volume displaced by the generator-plunger.

The conditions at a node are easily deduced from these results, for, on making $x = 0$ in equation (82.2), we obtain

$$u_1 = C \sin \varepsilon \sin \omega t.$$

Therefore the position of a node is determined by writing $\sin \varepsilon = 0$ in equation (82.5), since this condition secures zero shift for all values of the time t. Consequently, by equation (82.3)

$$p_1 = KC\frac{\omega}{a}\sin \omega t$$

gives the pressure at a node, which is clearly in quadrature with the velocity

$$\frac{\partial u}{\partial t} = r\omega \cos \omega t.$$

At a node, therefore,

$$\text{the mean power transmitted per cycle} = KCA\frac{\omega^2}{2\pi a}r\int_0^{\frac{2\pi}{\omega}} \tfrac{1}{2} \sin 2\omega t dt$$
$$= 0,$$

showing that no power would be transmitted through a branch situated at a node on the line, as is to be expected.

When the motor is using only a fraction of the power generated, our theory shows that, with perfect reflection at the end of the plungers, the travelling wave

$$u = B \sin\left(\frac{\omega}{a}x - \omega t\right) \quad \ldots \ldots \quad (82.9)$$

is superposed on the standing wave, B being a constant for a specified installation. If u'' be the shift of the fluid-particles due to the

combined effect of the travelling and standing waves at the position x and time t, we thus have

$$u'' = B \sin\left(\frac{\omega}{a}x - \omega t\right) + C \sin\left(\frac{\omega}{a}x + n\pi\right) \sin \omega t$$

$$= B \sin \frac{\omega}{a}x \cos \omega t + \left\{C \sin\left(\frac{\omega}{a}x + n\pi\right) - B \cos \frac{\omega}{a}x\right\} \sin \omega t$$

$$= F \cos \omega t + G \sin \omega t,$$

with $F = B \sin \frac{\omega}{a}x$, $G = \left\{C \sin\left(\frac{\omega}{a}x + n\pi\right) - B \cos \frac{\omega}{a}x\right\}$. More concisely,

$$u'' = H \sin(\omega t + \beta), \quad . \quad . \quad . \quad . \quad . \quad (82.10)$$

where $H = \sqrt{F^2 + G^2}$ and $\tan \beta = \frac{F}{G}$. Writing p'' for the related pressure, our procedure now leads to

$$p'' = -K\frac{\partial u''}{\partial x}$$

$$= -K\frac{\omega}{a}\left\{B \cos\left(\frac{\omega}{a}x - \omega t\right) + C \cos\left(\frac{\omega}{a}x + n\pi\right) \sin \omega t\right\} \quad (82.11)$$

It is a simple matter to demonstrate that the pressure p'' and the velocity

$$\frac{\partial u''}{\partial t} = H\omega \cos(\omega t + \beta)$$

are now not in quadrature, whence we can infer from a previous remark that power is being transmitted along the pipe.

With special reference to the fuel system of an oil engine, a practical point is involved in the result that both the pressure p'' and velocity $\frac{\partial u''}{\partial t}$ are functions of x. On this account the motion of the fluid at the right-hand end of our figures can be modified, within certain limits, by changing the position of the condenser in relation to the generator-plunger, which would correspond to the fuel pump, the fuel valve being at $x = L$.

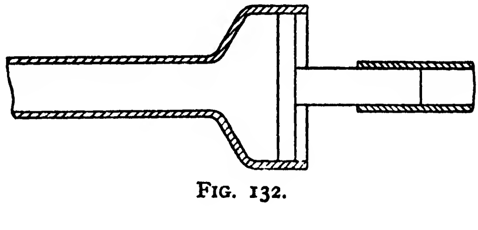


FIG. 132.

It is, of course, possible to devise various forms of condensers, such as the differential piston arrangement shown in Fig. 132, where a predetermined difference of phase may be secured by

utilizing the inertia of the moving parts. The same object may be achieved with the help of a spring-loaded valve, the action of which will be explained in Art. 84.

83. Energy and Momentum of Pressure Waves. It is sometimes most convenient to describe pressure waves in terms of the kinetic and potential energies of the working fluid. The way of approach to this aspect of the problem will be sufficiently elucidated if we consider a system in which the dissipative forces are negligibly small.

Imagine, by way of contrast, the right-hand end of the pipe in Fig. 129 to be fitted with a piston that moves independently of the plunger situated at the other end of the line. Let B in Fig. 133 represent the ' piston ', as distinguished from the ' plunger '

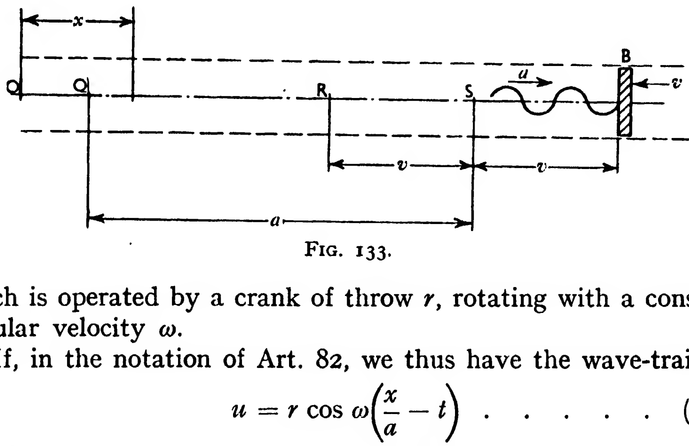


FIG. 133.

which is operated by a crank of throw r, rotating with a constant angular velocity ω.

If, in the notation of Art. 82, we thus have the wave-train

$$u = r \cos \omega\left(\frac{x}{a} - t\right) \quad . \quad . \quad . \quad . \quad . \quad (83.1)$$

travelling through the fluid, then the velocity of shift of the particles at the position x and time t is

$$\frac{\partial u}{\partial t} = -\,\omega r \sin \omega\left(\frac{x}{a} - t\right).$$

Here the velocity of propagation $a = \sqrt{\dfrac{Kg}{\rho}}$ for a fluid having a bulk modulus K and a density ρ. Also, a plane wave is involved, the front of which will be taken as being parallel to the surface of the piston.

Hence, if A be the cross-sectional area of the piston and of the pipe, the weight of fluid in an element of length dx will amount to $A\rho dx$, and the kinetic energy of the element to

$$\frac{A\rho}{2g}\left(\frac{\partial u}{\partial t}\right)^2 dx,$$

i.e.

$$\frac{A\rho\omega^2 r^2}{2g} \sin^2 \omega\left(\frac{x}{a} - t\right) dx.$$

Thus, bearing in mind the relation $\lambda = \frac{a}{\nu}$ between the wave-length λ, velocity of propagation a and frequency ν, we realise that kinetic energy

$$T = \frac{A\rho\omega^2 r^2}{2g}\int_0^\lambda \sin^2 \omega\left(\frac{x}{a} - t\right) dx$$

$$= \frac{A\rho\lambda\omega^2 r^2}{4g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83.2)$$

is associated with a length λ of the fluid-column.

When, as in the case of high-frequency transmission, λ is small in comparison with the other dimensions, the kinetic energy per unit length of the column is accordingly equal to $\frac{A\rho\omega^2 r^2}{4g}$.

Again, since the work done against the force of restitution on the element of length dx amounts to

$$- KA\ dx\int_0^u \frac{\partial^2 u}{\partial x^2}\,du,$$

i.e.

$$\frac{KA\omega^2}{a^2}\,dx\int_0^u u\,du,$$

or

$$\frac{KA\omega^2 u^2}{2a^2}\,dx,$$

it is clear that potential energy

$$V = \frac{KA\omega^2}{2a^2}\int_0^\lambda u^2\,dx$$

$$= \frac{KA\lambda\omega^2 r^2}{4a^2}$$

$$= \frac{A\rho\lambda\omega^2 r^2}{4g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83.3)$$

is associated with a length λ of the fluid-column.

A comparison of equations (83.2) and (83.3) shows that, in the absence of friction, the kinetic and potential energies of this type of wave are, on the average, equal. The total energy of the fluid is of course equal to $T + V$.

To proceed, we shall take the surface of the piston to be a perfect reflector of the wave-energy. Also, let the piston move with a velocity v in a direction *opposite* that of the incident waves, as indicated in the figure, where Q refers to a fixed point which at the time t is distant $a + v$ from the piston.

For definiteness let this configuration correspond to the instant t, and suppose $2n$ waves to pass the point Q during the succeeding 2 sec. At the end of that interval the piston will have advanced to the position R, and the reflected waves to the position Q. If

λ and λ' be the wave-lengths of the incident and reflected waves, respectively, it is obvious that at the instant $(t + 2)$ sec. the distance between the points Q, R will contain $\dfrac{a - v}{\lambda}$ incident waves and $\dfrac{a - v}{\lambda'}$ reflected waves. Stating this in symbols,

$$2n = \frac{2a}{\lambda}$$

$$= \frac{a - v}{\lambda} + \frac{a - v}{\lambda'},$$

in virtue of which the ratio of the wave-lengths

$$\frac{\lambda'}{\lambda} = \frac{a - v}{a + v} \quad \cdots \cdots \quad (83.4)$$

Consider next an instant when the wave-front has just reached the piston B. One second later, the piston will have advanced to the position S, so that the energy in the interval of length $a + v$ will have reached the piston, and the front of the reflected waves will have moved to a position distant $a - v$ to the left of S. In this manner we learn, assuming the amplitude of the waves to be unaffected by the phenomenon of reflection, that in one second the piston will receive an amount

$$\frac{A\rho\omega^2 r^2(a + v)}{2g},$$

or

$$\frac{2\pi^2 A\rho a^2 r^2(a + v)}{g\lambda^2}$$

of incident energy, and will reflect back an amount equal to

$$\frac{2\pi^2 A\rho a^2 r^2(a - v)}{g\lambda'^2}.$$

With continued movement of the piston, the increase per second of the energy of the fluid is, therefore,

$$\frac{2\pi^2 A\rho a^2 r^2}{g}\left(\frac{a - v}{\lambda'^2} - \frac{a + v}{\lambda^2}\right),$$

i.e.

$$\frac{4\pi^2 A\rho v a^2 r^2}{g\lambda^2}\left(\frac{a + v}{a - v}\right),$$

by equation (83.4), or

$$\frac{A\rho v\omega^2 r^2}{g}\left(\frac{a + v}{a - v}\right), \quad \cdots \cdots \quad (83.5)$$

since $\omega = \dfrac{2\pi a}{\lambda}$. But this is equal to the work done on the piston

moving with velocity v against the pressure p, whence we deduce

$$Apv = \frac{A\rho v\omega^2 r^2}{g}\left(\frac{a+v}{a-v}\right).$$

Hence the waves exert on the piston a pressure

$$p = \frac{\rho\omega^2 r^2}{g}\left(\frac{a+v}{a-v}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (83.6)$$

If, in the first place, v is small compared with a, we can consequently write

$$p = \frac{\rho\omega^2 r^2}{g}\left(1 + \frac{v}{a}\right)\left(1 + \frac{v}{a} + \frac{v^2}{a^2} + \ldots\right),$$

and so, to the first order of small quantities, ascertain

$$p = \frac{\rho\omega^2 r^2}{g}\left(1 + 2\frac{v}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (83.7)$$

If, in the second place, $v = 0$, then

$$p = \frac{\rho\omega^2 r^2}{g}$$

determines the pressure on the stationary piston. For unit area of the piston, the 'incident' momentum is accordingly equal to

$$\frac{\rho\omega^2 r^2}{2g},$$

so that the momentum per unit volume of the fluid is

$$\frac{\rho\omega^2 r^2}{2ga},$$

both quantities being reckoned in the direction of the incident waves, that is, from left to right in the figure.

In the general case of a piston moving with the prescribed velocity v, it follows that the incident and reflected waves will together produce the energy

$$\begin{aligned} W &= \frac{2\pi^2\rho a^2 r^2}{g}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) \\ &= \frac{\rho\omega^2 r^2}{2g}\left\{1 + \left(\frac{a+v}{a-v}\right)^2\right\} \quad . \quad . \quad . \quad . \quad (83.8) \end{aligned}$$

in unit volume of the fluid.

If v is small compared with a, according to this expression the energy is approximately

$$\frac{\rho\omega^2 r^2}{g}\left(1 + 2\frac{v}{a}\right),$$

agreeing, as it should, with the value of p in equation (83.7). When

the piston is at rest, $v = 0$, and the energy denoted by W is thus seen to be

$$\frac{\rho\omega^2 r^2}{g}.$$

We have restricted this analysis to the case of undamped plane-waves having fronts parallel to the reflecting surface in question. If the work be extended to cover undamped waves of any form travelling in an enclosure of any shape, we find the pressure $p = \frac{W}{3}$ in equation (83·8).

Moreover, the foregoing results hold even in systems where the surface of the piston is not of the perfect reflecting kind, for the reflecting properties of the plunger, at the opposite end of the pipe, have not entered into our calculations. On this account the equations will not require modification if we then assume perfect reflection at the piston, and ascribe its lack of this ideal property to the inner surface of the plunger.

This analysis relates, with evident restrictions, to the transmission of waves through elastic media in general, and therefore applies to a large number of problems, some of which have already been discussed. It is also worth while to observe that a combination of the methods described here and in Arts. 69, 71, 82 may be employed to examine the relationship between the efficiency of an oil engine and such factors as the length and shape of its exhaust system, one aspect of which has been investigated experimentally by H. O. Farmer.[1] In actual systems, however, the waves are somewhat affected by the inherent dissipative forces, to a degree which may be estimated with the aid of Art. 80, provided an appropriate value can be ascribed to the 'frictional' coefficients.

84. Fuel Systems of Internal Combustion Engines. The treatment of Art. 83 exhibits the essential characteristics of the pressure waves in the fuel system of an internal combustion engine, since the 'plunger' and the 'piston' correspond in order to the fuel-pump and the fuel-valve of the mechanism.

Neglecting the force of gravity and, for a moment, that of viscosity, we have shown that the movement of the plunger will initiate a pressure

$$p = -K\frac{\partial u}{\partial x}$$

$$= -\frac{K}{a}\frac{\partial u}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84.1)$$

As the resulting wave is propagated with the velocity $a = \sqrt{\frac{Kg}{\rho}}$

[1] *Proc. I. Mech. E.*, vol. 138, page 367 (1938).

in an oil having a bulk modulus K and a density ρ, the disturbance will travel from the pump to the nozzle in the interval of time $\frac{L}{a}$ if L be the equivalent length of the pipe. It will be noticed that a comparatively high velocity of propagation is involved, amounting to 4,800 ft. per sec. for oil with a bulk modulus of 267,000 lb. per square inch and a specific density of 0·86.

When the wave-front arrives at the valve, a fraction of its energy is taken up in forcing the oil through the nozzle, the remainder being reflected back towards the pump, on the supposition of perfect reflection. The original and reflected waves will consequently contribute to the total pressure on the pipe.

In order to fix ideas, we shall consider the automatic spring-loaded type of valve, and confine our attention to a single stroke of the pump, but with the engine-crank performing its usual motion under steady conditions of load. For the purpose of reference, let :

$\beta = \dfrac{\text{(constant) cross-sectional area of the pipe}}{\text{cross-sectional area of the nozzle}}$;

v_n = velocity of the oil through the nozzle ;
p = pressure of the original wave ;
p' = pressure of the reflected wave ;
p_c = pressure of the working fluid in the cylinder ;
h = pressure-head on the nozzle.

In terms of pound-foot-second units, when the valve is open

$$v_n = \sqrt{2gh} = \sqrt{\frac{2 \times 144g(p + p' - p_c)}{\rho}},$$

taking the p-terms to be expressed in pounds per square inch.

The velocity of the oil in the pipe then amounts to $\frac{v_n}{\beta}$.

Since the values of p_c and p can be found with the aid of the indicator diagram and a pressure gauge fitted to the pipe, from the relation

$$p' = p - \frac{K}{a}\frac{v_n}{\beta} \quad . \quad . \quad . \quad . \quad . \quad (84.2)$$

it follows that

$$p + p' = 2p - \frac{K}{a}\frac{v_n}{\beta} \quad . \quad . \quad . \quad . \quad . \quad (84.3)$$

determines the total pressure in the pipe when the original wave has, for the first time, suffered perfect reflection at the closed valve. Subsequent reflections would make for corresponding increases in the pressure, and we may thus trace a graph of this quantity.

Suppose the procedure to yield Fig. 134 as illustrating the variation of the pressure on the pipe in relation to the angular

position of the engine-crank, the origin being at O. Here OB is the static pressure on the line, and OA the pressure at which the fuel-valve opens. If the wave starts from the pump at the instant corresponding to B, say $t = 0$, and arrives at the valve at the instant corresponding to B', then the length BB' represents the interval of time L/a. During the initial part of the stroke the pressure will gradually increase due to the successive reflection of the waves at the closed valve on the one hand, and at the pump on the other. The resulting rise of pressure at the plunger, up to the instant when the nozzle opens, may thus be denoted by the curve BA, and that at the valve by $B'A'$. Therefore the full and dotted curves relate to the pressures at the plunger and the valve. The plunger subsequently exerts its maximum pressure, by way of

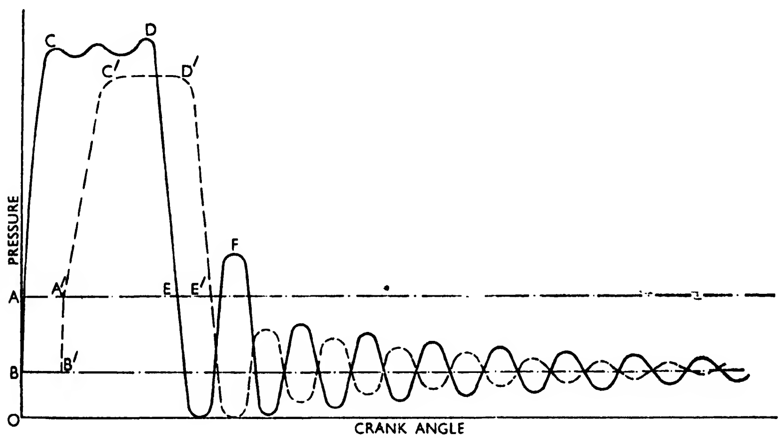


FIG. 134.

the line AC in our figure. As the next part of the stroke of the pump generally involves a number of harmonic components, the pressure on the oil will then vary in a manner which we indicate by the curved line CD. In actual systems the graph for the simultaneous conditions at the valve resembles $A'C'D'$, the difference in height of the ordinates arising from the combined action of viscosity and the partial reflection of the waves which occurs at the open nozzle. The figure also exhibits, in an exaggerated form, the general effect of damping on the wave motion, which is to make the line $C'D'$ smooth compared with CD. After the point D has been passed, the pressures at both the pump and valve fall, along the lines DE and $D'E'$, until the nozzle closes, at the point E which corresponds approximately to the pressure OA. The fuel-pipe being now closed at both ends, in actual systems the phenomenon of reflection takes place under the influence of frictional forces, so

that the pressure wave is of the damped type exhibited by the graphs to the right in the figure.

In order to illustrate a certain point we have here supposed the valve to reopen, at the abnormal crest marked F. This would, of course, adversely affect the thermal efficiency of the engine, but our present interest in the problem is to be found in the implied unstable condition of the valve.

To examine the matter more fully, imagine this irregular motion to occur at the time t, and write:

p = wave-pressure reckoned from the static pressure on the fuel pipe;
y = vertical displacement of the valve in relation to its seat;
A_1 = (constant) cross-sectional area of the valve needle on which p acts in the vertical direction;
c = coefficient of stiffness for the spring, i.e. the force required to compress it unit distance;
M = total weight of the moving parts of the valve;
f = damping coefficient associated with the stuffing box, etc.

In this notation we can at once write down, from Art. 43, the equation

$$\frac{M}{g}\ddot{y} + f\dot{y} + cy = pA_1 \quad . \quad . \quad . \quad . \quad . \quad (84.4)$$

for the valve, assuming the damping force to be proportional to the velocity.

To take account of the elastic properties of the oil, let:

v = volume of oil passing through the nozzle in 1 sec. when y is unity;
δV = decrease in the volume of the valve chamber due to the corresponding motion of the valve.

Thus, on the supposition that the oil flows through the pipe at a uniform rate during the period of injection, we obtain

$$\frac{\partial}{\partial t}(\delta V) = A_1\dot{y} + vy \quad . \quad . \quad . \quad . \quad . \quad . \quad (84.5)$$

Now, as already remarked,

$$p = -\frac{K}{V}\delta V$$

for an oil with a bulk modulus K, whence

$$\begin{aligned}\frac{\partial p}{\partial t} &= -\frac{K}{V}\frac{\partial}{\partial t}(\delta V)\\ &= -\frac{K}{V}(A_1\dot{y} + vy), \quad . \quad . \quad . \quad . \quad (84.6)\end{aligned}$$

by equation (84.5).

Therefore, after differentiating equation (84.4) with respect to the time, and inserting the expression (84.6) for $\frac{\partial p}{\partial t}$, it appears that

$$M\dddot{y} + fg\ddot{y} + g\left(c + \frac{A^2{}_1K}{V}\right)\dot{y} + \frac{A_1Kgv}{V}y = 0 \quad . \quad (84.7)$$

Since this equation is of the same type as was examined in connection with the stability of governors, in Art. 22, a similar method may be used to find a solution of the present question. In short, we assume p and y to vary as $e^{\lambda t}$, or, what amounts to the same thing, write $y = Dpe^{\lambda t}$ in the last equation, with D denoting an arbitrary constant. Making this substitution and cancelling the common factors, we find

$$\lambda^3 + \frac{fg}{M}\lambda^2 + \frac{g}{M}\left(c + \frac{A_1{}^2K}{V}\right)\lambda + \frac{A_1Kgv}{MV} = 0 \quad . \quad (84.8)$$

For stability it is necessary that none of the roots of this cubic in λ has a positive real part, as was pointed out in Art. 22. In the present case, all the coefficients are essentially positive, whence it follows that instability of the valve will be prevented by arranging matters so that

$$\frac{fg^2}{M^2}\left(c + \frac{A^2{}_1K}{V}\right) > \frac{A_1Kgv}{MV},$$

i.e.

$$f\left(c + \frac{A_1{}^2K}{V}\right) > \frac{M}{g}\frac{A_1Kv}{V}. \quad . \quad . \quad . \quad (84.9)$$

Hence, for the specified type of valve, the presence of the abnormal crest F in Fig. 134 signifies that the quantities denoted by f and c are together too small for the purpose of stability. The remedial measures obviously include an increase in either the stiffness (c) of the spring or the frictional coefficient f, or both, to an extent such as will satisfy the inequality (84.9).

If the condenser of Art. 82 be introduced into the system under consideration, the resulting equations would obviously be applicable to installations in which the fuel is supplied to a number of cylinders by the 'common rail' method. In this process of extending the results it is well to bear in mind the previous remark about the way in which the conditions at the valve are influenced by the relative positions of the condenser and valve.

An oscillatory motion of secondary importance takes place on account of the elasticity of the valve-needle, which evidently executes the longitudinal kind of vibration discussed in Art. 63. Moreover, the wall of the fuel-pipe describes vibrations in the manner explained in Art. 76, but this yielding of the material is in general small compared with that of the fluid itself.

The 'critical lengths' specified in Art. 81 should, of course, be avoided so far as may be in installations of this kind. If, in the case of a given pipe, 'time lag' and viscosity are to be examined as distinct variables, the length and the cross-sectional area of the pipe should be considered as independent factors.

For information on the viscous resistance to flow of fuel-oils the reader may be referred to the work of D. H. Alexander,[1] and of L. J. Le Mesurier and R. Stansfield.[2]

[1] *Trans. Inst. Mar. E.*, vol. 39, page 366 (1927).
[2] *Diesel Engine Users' Assoc.*, Paper No. 114 (1933).

CHAPTER V

BEAMS AND PLATES

85. In this and the next chapter we shall take account of the face that in an actual structure the vibratory motion of the constituent members is affected by their flexural stiffnesses, as well as by the degree of fixity associated with the operative constraints. Since a natural period of vibration is, strictly speaking, identified with each member of a complex structure, it is manifest that resonance may occur not only with the system as a whole, but also with any one of the principal parts. If there be n such parts, then $n + 1$ conditions of resonance should be investigated with regard to each of the principal axes. For a particular part of a specified system the work of arranging the axes in an order of relative importance cannot well be effected without recourse to experience. With aircraft, for instance, the 'fore and aft' direction should be carefully examined with reference to the vibration of the airscrews, but that direction may or may not be of any practical account for other parts of the system.

86. Free Vibrations of a Slender Beam having Uniform Cross-section and Density. By way of illustrating the general problem, let Fig. 135 refer to a slender beam fixed in a given manner

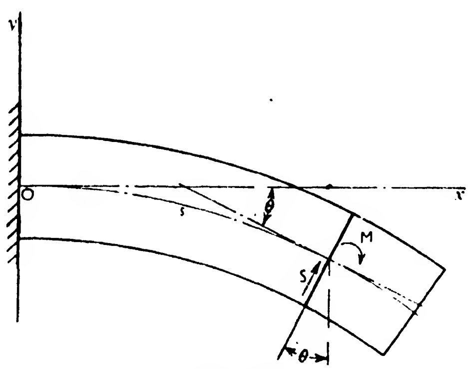


FIG. 135.

at the origin O, and executing small vibrations about the equilibrium position which we take as coinciding with the x-axis. The motion,

greatly exaggerated in the figure, is supposed to take place in the plane of the paper, under the influence of the weight of the beam only, so that this is a case of free oscillations in a natural mode.

To obtain the necessary equations we may consider the forces which then act on an element of unit length, reckoned in the x-direction, whose centre of gravity is distant x from the origin O when the beam is at rest. If, as indicated in the figure, the distance x becomes s when y is the vertical displacement of that centre from its position of rest, on writing θ for the slope at the point in question, we have $\theta = \frac{\partial y}{\partial s}$, and the curvature defined by $\frac{\partial^2 y}{\partial s^2}$. The use of the symbols for partial differentiation is due to the fact that the configuration of the system depends on both the time and the relative position of the element on the x-axis.

If A be the cross-sectional area of the beam, and ρ denotes the density of the material, the weight per unit length of the beam amounts to $A\rho$, so that to the first order of approximation the shear S on the section of the element is given by $\frac{\partial S}{\partial s} = A\rho$, provided θ remains small throughout the motion. Thus it appears that

$$\frac{\partial S}{\partial s} = \frac{A\rho}{g}\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.1)$$

determines the kinetic reaction in the y-direction.

In this notation the longitudinal displacement of the element is $-x\frac{\partial y}{\partial s}$, and if k signifies the radius of gyration of the cross-section about an axis through its centre and at right angles to the plane of the paper, then the corresponding moment of the kinetic reaction is $\frac{A\rho k^2}{g}\frac{\partial^3 y}{\partial s \partial t^2}$. Hence, with the customary symbol M written for the flexural couple or bending moment on the element, we obtain

$$\frac{\partial M}{\partial s} = \frac{A\rho k^2}{g}\frac{\partial^3 y}{\partial s \partial t^2} - S \quad . \quad . \quad . \quad . \quad . \quad (86.2)$$

from consideration of the disturbing agencies.

Writing E for the direct modulus of elasticity of the material, we have, in addition,

$$M = AEk^2\frac{\partial^2 y}{\partial s^2},$$

from the theory of structures, and therefore

$$\frac{\partial M}{\partial s} = AEk^2\frac{\partial^3 y}{\partial s^3},$$

by virtue of which equation (86·2) can be expressed in the more convenient form

$$AEk^2\frac{\partial^3 y}{\partial s^3} = \frac{A\rho k^2}{g}\cdot\frac{\partial^3 y}{\partial s\partial t^2} - S \quad . \quad . \quad . \quad (86.3)$$

This leads, after differentiating with respect to s, to

$$AEk^2\frac{\partial^4 y}{\partial s^4} = \frac{A\rho k^2}{g}\,\frac{\partial^4 y}{\partial s^2\partial t^2} - \frac{\partial S}{\partial s},$$

whence, combining with equation (86·1), we gather that the motion is approximately determined by the equation

$$\frac{\rho}{g}\left(\frac{\partial^2 y}{\partial t^2} - k^2\frac{\partial^4 y}{\partial s^2\partial t^2}\right) = -Ek^2\frac{\partial^4 y}{\partial s^4} \quad . \quad . \quad . \quad (86.4)$$

The second member within the brackets, representing the *rotatory inertia* of the element, is readily shown to be of the second order of small quantities in all but extreme cases.[1] Neglecting this term, it appears that the relations

$$\frac{\partial^4 y}{\partial s^4} = -\frac{\rho}{Egk^2}\,\frac{\partial^2 y}{\partial t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.5)$$

$$S = -AEk^2\frac{\partial^3 y}{\partial s^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.6)$$

then hold good.

Now s is sensibly equal to x so long as θ remains small. Consequently, since the moment of inertia of the section $I = Ak^2$, our equations show that for a beam of constant weight m per unit length

$$\frac{\partial^4 y}{\partial x^4} = -\frac{m}{EgI}\,\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.7)$$

$$S = -EI\frac{\partial^3 y}{\partial x^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.8)$$

In order to simplify the question as far as is possible without sacrificing any of its essential features, we now take advantage of the circumstance that the combined effect of the rotatory inertia and the shear is small in many cases, by neglecting the total displacement associated with these quantities. It may also be pointed out that in the extreme case where the wave-length is ten times the depth of a beam the error thus introduced amounts to only about 2 per cent., hence equation (86·8) commonly relates to an insignificant quantity in the present connection.

To proceed, we assume a solution of the type

$$y = Fe^{nx+qt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.9)$$

[1] S. Timoshenko, *Phil. Mag.*, vol. 41, page 744 (1921), and vol. 43. page 25 (1922).

where F, n, q denote constants, then substitution in equation (86.7) of the consequent relations

$$\frac{\partial^4 y}{\partial x^4} = n^4 y, \quad \frac{\partial^2 y}{\partial t^2} = q^2 y,$$

shows at once that the solution is valid provided

$$n^4 = -\frac{m}{EgI}q^2.$$

As the quantities denoted by m, E, g, I are all essentially positive, it is easily proved that q^2 must be negative, which we may signify by writing $q^2 = -p^2$. Stated otherwise, $n = \pm ip$, where $i = \sqrt{-1}$, so that if

$$n^4 = \alpha^4,$$

then

$$n = \pm\alpha, \pm i\alpha,$$

and

$$\alpha^4 = \frac{m}{EgI}p^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.10)$$

From this it is to be inferred that the solution (86·9) will involve terms of the types $e^{\pm\alpha x}$, $e^{\pm i x\alpha}$, and $e^{\pm ipt}$. If these terms are combined, it will be found that they are comprehended in the pair of expressions $A \cos \alpha x + B \sin \alpha x + C \cosh \alpha x + D \sinh \alpha x$, and $\sin (pt + \varepsilon)$, where A, B, C, D, ε represent arbitrary constants.

Hence

$$y = (A \cos \alpha x + B \sin \alpha x + C \cosh \alpha x + D \sinh \alpha x) \sin (pt + \varepsilon) \quad . \quad . \quad . \quad . \quad (86.11)$$

is the most general solution of equation (86.7), and the beam accordingly describes vibrations with a period

$$\frac{2\pi}{p} = \frac{2\pi}{\alpha^2}\sqrt{\frac{m}{EgI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.12)$$

in a natural mode which is determined by the value of α in equation (86.10).

It may, again, be remarked that the displacement or deflection y in any mode of vibration can be written down by the aid of equation (86.11) and the principle of superposition.

The constants A, B, C, D, ε naturally depend on the end conditions and the initial circumstances of the motion. If, with known values attached to these constants, the expression for y be equated to zero, it is not difficult to realize that the roots of the resulting equation in x correspond to the positions of the nodes or stationary points on the beam for the given mode of vibration.

In the following exemplification of these calculations we shall confine ourselves to the case of slender beams of uniform cross-sections and density, executing small oscillations about the position

of rest. The same significance as before will be ascribed to the symbols E, I, m, and we assume consistent units to be used in numerical applications.

Ex. 1. Derive expressions for the amplitude and the period in a normal mode of vibration for a beam simply supported at the ends shown in Fig. 136.

If L denote the free length of the beam, and the point O be taken as the origin, with the equilibrium position defined by the

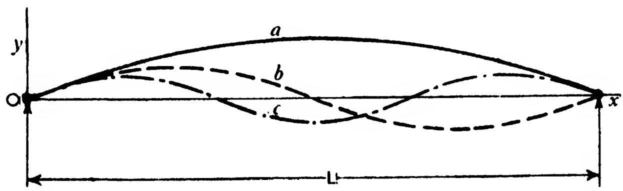


FIG. 136.

x-axis, the boundary conditions to be satisfied by equation (86.11) for all values of t are:

(i) the displacement $y = 0$ at the ends $x = 0$ and $x = L$;

(ii) $\dfrac{\partial^2 y}{\partial x^2} = 0$ at the ends $x = 0$ and $x = L$, because the bending moments there are zero.

To secure the values $y = 0$, $\dfrac{\partial^2 y}{\partial x^2} = 0$ at the point $x = 0$, it is readily deduced from equation (86.11) that we require

$$A + C = 0 \text{ and } C - A = 0,$$

i.e.
$$A = C = 0.$$

The same equation shows that the conditions $y = 0$, $\dfrac{\partial^2 y}{\partial x^2} = 0$ will always be fulfilled at the point $x = L$ provided

$$D \sinh \alpha L + B \sin \alpha L = 0, \text{ and } D \sinh \alpha L - B \sin \alpha L = 0$$

i.e.
$$B \sin \alpha L = 0, \text{ and } D \sinh \alpha L = 0,$$

being the difference and the sum of these expressions. Of the alternative inferences to be drawn from the result $B \sin \alpha L = 0$, we must choose $\sin \alpha L = 0$, since the beam would not execute the implied vibrations if $B = 0$. For the same reason it must be concluded from the result $D \sinh \alpha L = 0$ that $D = 0$.

Therefore the displacement

$$y = B \sin \alpha x \sin (pt + \varepsilon),$$

where the constants B, ε depend on the initial circumstance of the motion, and $p = \alpha^2 \sqrt{\dfrac{EgI}{m}}$, by equation (86.12). The several

modes are accordingly defined by the series of values

$$\alpha = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots,$$

corresponding to the roots of the equation $\sin \alpha L = 0$.

In the fundamental mode, for instance, in which $\alpha = \frac{\pi}{L}$, it follows that at any instant t and point x on the beam the displacement

$$y = B \sin \frac{\pi}{L} x \sin \left\{ \frac{\pi^2}{L^2} \left(\frac{EgI}{m} \right)^{\frac{1}{2}} t + \varepsilon \right\}$$

This equation may be used for the purpose of tracing the configuration of the beam, as illustrated by the curve (a) in Fig. 136, when vibrating with a period $\frac{2L^2}{\pi} \sqrt{\frac{m}{EgI}}$.

It can likewise be demonstrated that

$$y = B \sin \frac{2\pi}{L} x \sin \left\{ \frac{4\pi^2}{L^2} \left(\frac{EgI}{m} \right)^{\frac{1}{2}} t + \varepsilon \right\},$$

$$y = B \sin \frac{3\pi}{L} x \sin \left\{ \frac{9\pi^2}{L^2} \left(\frac{EgI}{m} \right)^{\frac{1}{2}} t + \varepsilon \right\}$$

relate in succession to the second and the third normal modes of vibrations, which are evidently executed with periods $\frac{L^2}{2\pi} \sqrt{\frac{m}{EgI}}$, $\frac{2L^2}{9\pi} \sqrt{\frac{m}{EgI}}$, respectively. The curves (b), (c) may be regarded as the configurations of the beam given by these relations, in which the nodes divide the span into sections of equal length.

The motion in any mode will induce in the material a stress which can now be estimated by inserting the proper value of the deflection y in the usual formula of the theory of structures.

Ex. 2. Determine the periods in a normal mode of vibration for a beam rigidly fixed at one end, and free at the other, as shown in Fig. 137.

Let L denote the free length of the beam, and take the origin O to be at the fixed end, with the x-axis as the position about which the small oscillations are described. For all values of the time the arbitrary constants in equation (86.11) must here agree with the conditions:

(i) $y = 0$, $\frac{\partial y}{\partial x} = 0$ at the fixed end $x = 0$;

(ii) $\frac{\partial^2 y}{\partial x^2} = 0$, $\frac{\partial^3 y}{\partial x^3} = 0$ at the free end $x = L$, where the bending moment and the shearing force are separately zero.

If the conditions (i) be utilized in connection with equation (86.11) it will be found that

$$A + C = 0 \text{ and } B + D = 0,$$

whence
$$A = -C \text{ and } B = -D.$$

A repetition of the foregoing procedure with the conditions (ii) leads next to the simultaneous equations

$$C\,(\cosh \alpha L + \cos \alpha L) + D\,(\sinh \alpha L + \sin \alpha L) = 0,$$
$$C\,(\sinh \alpha L - \sin \alpha L) + D\,(\cosh \alpha L + \cos \alpha L) = 0.$$

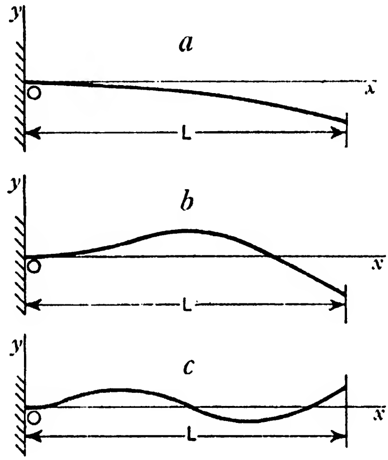


FIG. 137.

A combination of these indicates that in order to fulfil the present conditions we must have either

$$C = 0 \text{ and } D = 0,$$

or the determinantal equation

$$\begin{vmatrix} \cosh \alpha L + \cos \alpha L, & \sinh \alpha L + \sin \alpha L \\ \sinh \alpha L - \sin \alpha L, & \cosh \alpha L + \cos \alpha L \end{vmatrix} = 0,$$

that is, on expanding,

$$\cosh^2 \alpha L + \cos^2 \alpha L + 2 \cosh \alpha L \cos \alpha L - \sinh^2 \alpha L + \sin^2 \alpha L = 0,$$

which reduces to

$$2 + 2 \cosh \alpha L \cos \alpha L = 0,$$

or
$$\cos \alpha L = -\operatorname{sech} \alpha L \quad . \quad . \quad . \quad . \quad . \quad (86.13)$$

It is manifest that the latter of these alternative results is the one to be chosen, as the beam would be at rest if $C = 0$, $D = 0$.

The next step is therefore that of finding the roots of equation (86.13). This may be effected by plotting graphs of the functions $\cos \alpha L$ and $-\operatorname{sech} \alpha L$ in a manner which will be understood from

Fig. 138, or by the usual analytical method, when it will be found that the first six roots are approximately given by

$$\alpha L = 1\cdot8751,\ 4\cdot6941,\ 7\cdot8548,\ 10\cdot996,\ 14\cdot137,\ 17\cdot279.$$

With the foregoing information as to the value of the constants we can now write down the expression for the displacement y in a given mode of 'normal' vibration, by inserting in equation (86.11) the appropriate value of αL. The resulting expressions enable us

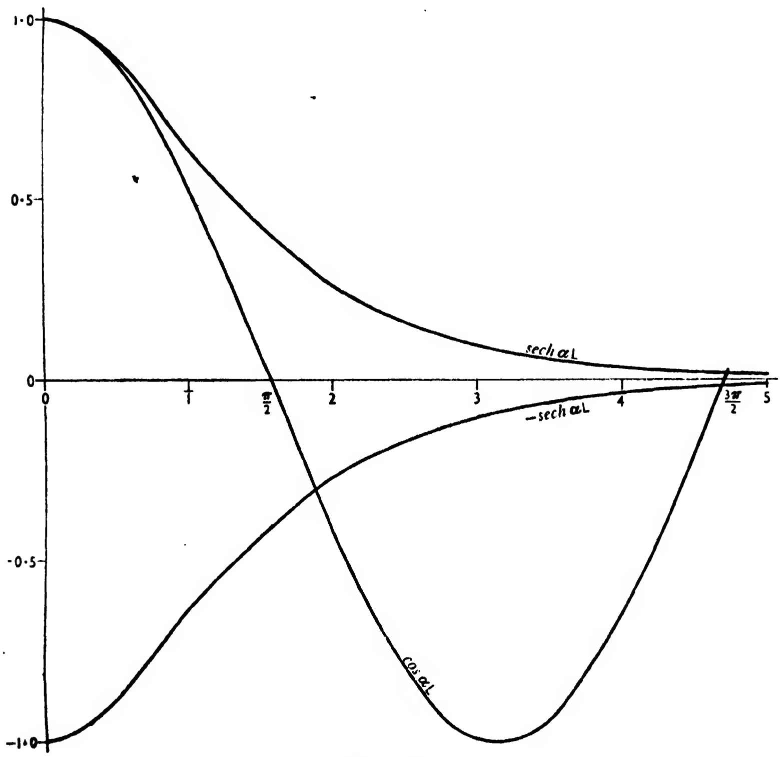


FIG. 138.

to trace the shape of the disturbed beam, and so obtain curves resembling (a), (b), (c) in Fig. 137 for the first three normal modes of oscillation. These, according to equation (86.12), are executed with periods

$$\frac{2\pi L^2}{(1\cdot8751)^2}\sqrt{\frac{m}{EgI}},\quad \frac{2\pi L^2}{(4\cdot6941)^2}\sqrt{\frac{m}{EgI}},\quad \frac{2\pi L^2}{(7\cdot8548)^2}\sqrt{\frac{m}{EgI}}$$

respectively.

If the positions of the nodes be determined by the method already described, it will be seen that they are situated, in relation to the free end of the beam, as follows: $0\cdot2165L$ in Fig. 137 (b);

$0{\cdot}1321L$, $0{\cdot}4999L$ in Fig. 137 (*c*); and $0{\cdot}0944L$, $0{\cdot}3558L$, $0{\cdot}6439L$ in the mode with three nodes.

Ex. 3. Evaluate the periods in a normal mode of vibration for a beam supported at both ends by long bearings that permit the ends to move freely in the horizontal direction of Fig. 139.

We take, as before, the x-axis as the position of rest, with the origin at the left-hand support, and let L be the free length of the

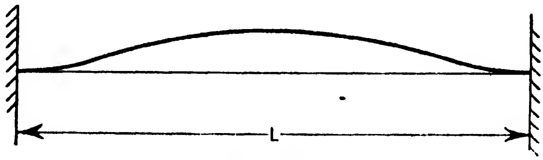


FIG. 139.

beam. From the fact that both the deflection and the slope of the beam are zero at the ends, for all values of the time, it follows that the constants A, B, C, D in equation (86.11) must here comply with the conditions

$$y = 0, \ \frac{\partial y}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L.$$

Our method of analysis shows that this agreement will be fulfilled if

$$1 - \cosh \alpha L \cos \alpha L = 0,$$

that is if
$$\cos \alpha L = \operatorname{sech} \alpha L \quad . \quad . \quad . \quad . \quad . \quad (86.14)$$

The roots of this equation as given by the graphical construction of Fig. 137 are sensibly represented by the series of values

$$\alpha L = 0, \ \frac{3\pi}{2}, \ \frac{5\pi}{2}, \ \frac{7\pi}{2}, \ \frac{9\pi}{2}, \ \ldots$$

A more accurate calculation yields

$$\alpha L = 0, \ 4{\cdot}7300, \ 7{\cdot}8532, \ 10{\cdot}996, \ 14{\cdot}137, \ \ldots$$

In this case the zero root is irrelevant, as it obviously refers to motion in which the beam would vibrate vertically as a rigid body.

Therefore the normal modes, involving the displacement y given by substituting in equation (86.11) the proper value of αL, are executed with periods

$$\frac{2\pi L^2}{(4{\cdot}7300)^2}\sqrt{\frac{m}{EgI}}, \ \frac{2\pi L^2}{(7{\cdot}8532)^2}\sqrt{\frac{m}{EgI}}, \ \frac{2\pi L^2}{(10{\cdot}996)^2}\sqrt{\frac{m}{EgI}}, \ \ldots$$

It is plain that in the second mode the node is identified with the mid-point of the span; and it is not a difficult matter to prove that in the third mode the nodes are situated at points distant $0{\cdot}3593L$ from each end of the beam.

Ex. 4. Obtain expressions for the natural periods of vibration for a beam free at both ends, and attached at its mid-length to a slender wire.

It is of practical interest to remark that a solution of this problem constitutes a first approximation to the vertical vibration of the hull of a ship, in so far as the specified beam represents such a structure.

If the horizontal or x-axis in Fig. 140 be the equilibrium-position, with the origin at the left-hand end of the beam, the statement that the bending moments and the shearing forces are

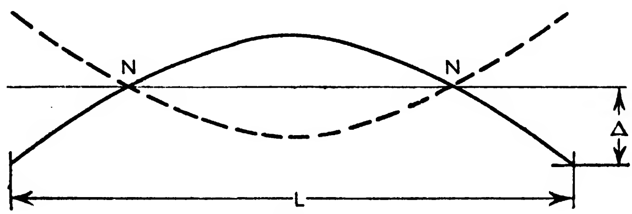


FIG. 140.

separately zero at the end-points $x = 0$, $x = L$ may be expressed by writing

$$\frac{\partial^2 y}{\partial x^2} = 0, \quad \frac{\partial^3 y}{\partial x^3} = 0 \text{ at } x = 0 \text{ and } x = L$$

for all values of the time.

Taken in succession, these conditions imply, through equation (86·11),

$$C - A = 0, \quad D - B = 0,$$

$$-A \cos \alpha L - B \sin \alpha L + C \cosh \alpha L + D \sinh \alpha L = 0,$$

$$A \sin \alpha L - B \cos \alpha L + C \sinh \alpha L + D \cosh \alpha L = 0.$$

Thus, after eliminating the constants between these four equations, we obtain

$$\cos \alpha L \cosh \alpha L = 1,$$

i.e.

$$\cos \alpha L = \operatorname{sech} \alpha L.$$

Although the roots of equation (86·14) apply in the present case, due attention must here be given to the root $\alpha L = 0$, for the beam can move as a rigid body in the vertical direction. In the case of a ship, for instance, the zero root would refer to slight 'heaving' of the vessel in still water, as might be caused by contact with a submerged object.

Proceeding along the same line of reasoning as before, we realize that the mode of vibration exhibited in Fig. 140 is described with a period

$$\frac{2\pi L^2}{(4\cdot7300)^2}\sqrt{\frac{m}{EgI}},$$

and that the nodes N, N occur at points $0{\cdot}2242L$ measured from each end of the beam.

Similarly, in the mode defined by the root $\alpha L = 7{\cdot}8532$ there are three nodes, at points $0{\cdot}1321L$, $0{\cdot}5000L$, $0{\cdot}8679L$ reckoned from one end of the beam. Again, there are four nodes in the next mode, corresponding to the root $\alpha L = 10{\cdot}996$, their positions being $0{\cdot}0944L$, $0{\cdot}3558L$, $0{\cdot}6442L$, $0{\cdot}9056L$ with respect to one end of the beam.

Ex. 5. Apply the preceding theory to a beam rigidly fixed at one end, and simply supported at the other.

Suppose, as previously, the position of rest to coincide with the x-axis, and let L be the free length of the beam. If the fixed end is at $x = 0$, then

$$y = 0, \ \frac{\partial y}{\partial x} = 0 \text{ at } x = 0 \text{ for all values of the time.}$$

These data, when combined with equation (86·11), enable us to write

$$C = -A, \ D = -B.$$

The conditions at the other end, namely

$$y = 0 \text{ and } \frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = L \text{ for all values of the time,}$$

lead, by the same procedure, to

$$A(\cosh \alpha L - \cos \alpha L) + B(\sinh \alpha L - \sin \alpha L) = 0,$$
$$A(\cosh \alpha L + \cos \alpha L) + B(\sinh \alpha L + \sin \alpha L) = 0.$$

Successive addition and subtraction of these expressions gives

$$A \cosh \alpha L + B \sinh \alpha L = 0,$$
$$A \cos \alpha L + B \sin \alpha L = 0\,;$$

and to satisfy these we require either

$$A = B = 0,$$

or
$$\cosh \alpha L \sin \alpha L - \sinh \alpha L \cos \alpha L = 0,$$

i.e.
$$\tan \alpha L = \tanh \alpha L \quad . \quad . \quad . \quad . \quad . \quad (86{\cdot}15)$$

The latter of these alternatives must be chosen because $A = B = 0$ could hold only if the beam were at rest.

The graphical construction shown in Fig. 141 may be used to verify that the roots of equation (86·15) approximately correspond to

$$\alpha L = \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \ldots,$$

where we omit a zero root as of no account in the present problem. It can be shown that the more exact values are

$$\alpha L = 3{\cdot}9270, \ 7{\cdot}0690, \ 10{\cdot}210, \ 13{\cdot}352, \ 16{\cdot}493, \ldots$$

The periods in the corresponding modes are determined by substi-

tuting in equation (86·12) the appropriate value of α. From this series we can also deduce the displacement y and the positions of the nodes for a prescribed mode, in precisely the same way as before.

87. It will be understood from previous remarks that the approximate method of Art. 55 affords a ready means of finding the fundamental period of vibration for the type of system just considered. Since the procedure consists in assuming an expression

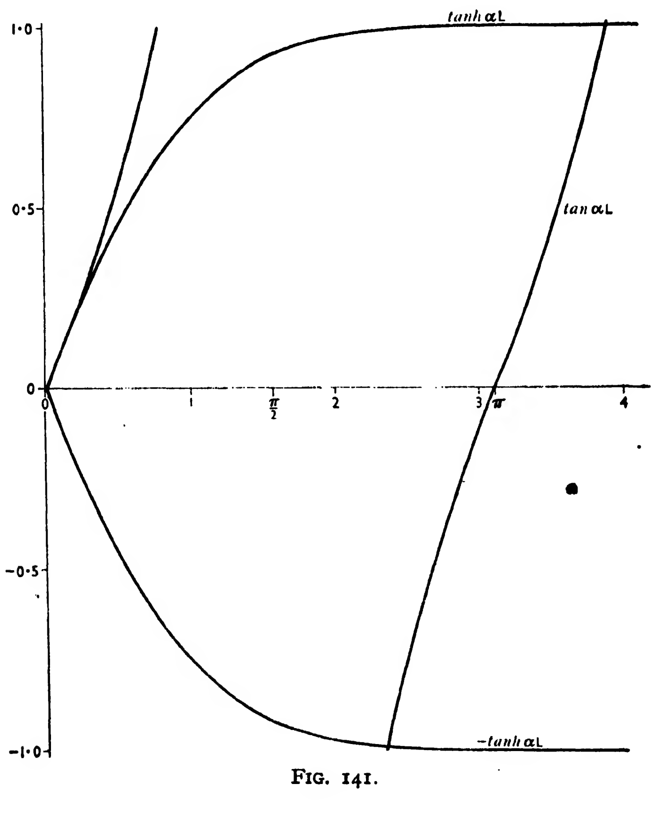


FIG. 141.

for the displacement y which shall agree, at least, with the known end-conditions for a specified beam, the measure of success achieved depends on our knowledge of the actual shape of the disturbed beam. This granted, the next step is that of equating the maximum values of the kinetic and potential energies obtained in this manner, when, as will appear in the following discussion, we arrive at an expression for the fundamental period of oscillation.

General expressions for the potential and kinetic energies may easily be found by considering the element of length dx which

formed the basis of the analysis in Art. 86. If, in Fig. 135, the ends of the element turn through a small angle $d\theta$ due to the action of a bending moment M, the work thus done on the element amounts to $\frac{1}{2}Md\theta$, and this represents the potential energy dV in the case of a perfectly elastic material. Remembering, from Art. 86, the relation

$$d\theta = \frac{d^2y}{dx^2}dx,$$

we next equate the work done to the strain or potential energy of the element, and so ascertain

$$dV = \tfrac{1}{2}M\frac{d^2y}{dx^2}dx.$$

Taking account of the well-known equation $M = EI\frac{d^2y}{dx^2}$ in the theory of structures, we can now state the potential energy V in the form

$$2V = EI\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx \quad . \quad . \quad . \quad . \quad . \quad (87.1)$$

for a beam of length L, made of material specified by its direct modulus of elasticity E, and having a sectional moment of inertia I with respect to bending in the plane of the paper.

Also, if m denote the (constant) weight of the beam per unit length, the kinetic energy T is given by

$$2T = \frac{m}{g}\int_0^L \dot{y}^2 dx, \quad . \quad . \quad . \quad . \quad . \quad (87.2)$$

$\dot{y}$ being the vertical velocity of the element at time t.

A particular problem is completed by equating the maximum values of the expressions (87·1) and (87·2), in a manner which may best be explained by means of a few examples relating, of course, to normal modes of oscillation.

Ex. 1. Find the approximate value of the fundamental period of vibration for the beam of Ex. 2 in Art. 86, fixed at one end and free at the other.

With the origin at the fixed end, and L representing the free length of the beam, it is plain that the conditions

$$y = 0, \quad \frac{\partial y}{\partial x} = 0 \text{ at the point } x = 0,$$

$$\frac{\partial^2 y}{\partial x^2} = 0 \text{ at the point } x = L$$

apply throughout the motion. If we assume the shape of the disturbed beam to be a quarter-cosine curve, simple differentiations suffice to verify that

$$y = F\left(1 - \cos\frac{\pi}{2L}x\right)\cos pt \quad . \quad . \quad . \quad . \quad (87.3)$$

secures the known end-conditions, where F is an arbitrary constant. Here $\frac{2\pi}{p}$ is the required period of oscillation.

Making this substitution in equation (87.1), we see that the potential energy

$$V = \frac{\pi^4 F^2 EI}{32L^4}\int_0^L \cos^2 \frac{\pi}{2L}x\, dx \cos^2 pt,$$

whence $$V_{\text{max.}} = \frac{\pi^4 F^2 EI}{64L^3}.$$

As to the related expression for the kinetic energy, according to equation (87.3)

$$\dot{y} = -Fp\left(1 - \cos\frac{\pi}{2L}x\right)\sin pt,$$

so that $$\dot{y}^2 = F^2p^2\left(1 - \cos\frac{\pi}{2L}x\right)^2 \sin^2 pt,$$

hence $$\begin{aligned}\dot{y}^2_{\text{max.}} &= F^2p^2\\ &= p^2y^2_{\text{max.}}\end{aligned}$$

follows on taking the maximum values of both x and t. Therefore, by equation (87·2),

$$\begin{aligned}T_{\text{max.}} &= \tfrac{1}{2}\frac{m}{g}p^2\int_0^L y^2_{\text{max.}}dx\\ &= \tfrac{1}{2}\frac{m}{g}F^2p^2\int_0^L\left(1 - \cos\frac{\pi}{2L}x\right)^2_{\text{max.}}dx\\ &= \left(\frac{3}{4} - \frac{2}{\pi}\right)\frac{m}{g}F^2Lp^2.\end{aligned}$$

To the degree of approximation implied in the supposition that $V_{\text{max.}} = T_{\text{max.}}$ we accordingly have the fundamental period $\frac{2\pi}{p}$ determined by

$$p = \frac{3{\cdot}66}{L^2}\sqrt{\frac{EgI}{m}},$$

nearly. This value is only about 4 per cent. less than the corresponding period obtained by the more elaborate method of Ex. 2 in Art. 86. The error would, as may be proved without much trouble, have been reduced to 0·5 per cent. of the true value had we taken

$$y = \frac{m}{2EI}\left(\frac{L^2}{2}x^2 - \frac{L}{3}x^3 + \frac{1}{12}x^4\right)$$

as the assumed shape of the disturbed beam. This will be recognized as the equation for a similar cantilever carrying a static load m per unit length in the x-direction.

Ex. 2. Apply the approximate method to the system examined

in Ex. 3 of Art. 86, consisting of a beam of length L, supported by long bearings at both ends.

Since the conditions at the ends $x = 0$ and $x = L$ are

$$y = 0, \frac{\partial y}{\partial x} = 0,$$

we may assume the shape of the beam to be

$$y = F\left(1 - \cos\frac{2\pi}{L}x\right)\cos pt, \quad . \quad . \quad . \quad (87.4)$$

where F is an arbitrary constant.

On repeating the argument used in the previous example, it will be found that

$$V_{\text{max.}} = \frac{4\pi^4 F^2 EI}{L^3}, \; T_{\text{max.}} = \frac{3m}{4g}F^2 Lp^2$$

represent the maximum values of the potential and the kinetic energies, m being the weight per unit length of the beam.

If, as is implied here, $V_{\text{max.}} = T_{\text{max.}}$, it thus appears that the fundamental period of vibration

$$\frac{2\pi}{p} = \frac{2\pi L^2}{22{\cdot}8}\sqrt{\frac{m}{EgI}},$$

approximately. This result is about 1 per cent. greater than that given by the more exact analysis of Ex. 3 in Art. 86, but the error might be diminished by taking, instead of equation (87·4), the expression for a similar beam loaded with a weight m per unit length.

An alternative way of approach to a solution in this case is by way of a sine function in equation (87·4), but we must then take the mid-point of the span as the origin.

Ex. 3. Evaluate, on the same assumptions, the period of vibration for the beam specified in Ex. 4 of Art. 86, in the mode indicated by Fig. 140.

The end-conditions being

$$\frac{\partial^2 y}{\partial x^2} = 0, \; \frac{\partial^3 y}{\partial x^3} = 0 \text{ at the ends } x = 0 \text{ and } x = L,$$

it is feasible to suppose that the beam deflects according to the law

$$y = \left(F\sin\frac{\pi}{L}x - \Delta\right)\cos pt, \quad . \quad . \quad . \quad (87.5)$$

where F is an arbitrary constant, and Δ denotes the deflection of the ends shown in Fig. 140.

The positions of the two nodes involved in the motion are obviously determined by the value of Δ. To express Δ in terms of F and the length of the beam, we note, in the first place, that the

total momentum of the system is zero, because external forces are absent. In the second place, the fact that the beam is of uniform cross-section and density connotes that the momentum of an element of the beam is proportional to py. Moreover, on the present suppositions, py represents both the upward velocity of the ends and the downward velocity of the mid-point of the beam when it passes through the equilibrium-position defined by the x-axis. The momentum of the system is, therefore, zero when the configuration is such that equal lengths of the beam lie above and below the position of rest, that is when

$$\int_0^L y\,dx = 0.$$

If this be combined with equation (87·5), and the value unity be assigned to $\cos pt$, in accordance with the method, we see that

$$L\left(\frac{2}{\pi}F - \Delta\right) = 0$$

must hold, by reason of which

$$\Delta = \frac{2}{\pi}F$$

in the assumed curve, hence

$$y = F\left(\sin\frac{\pi}{L}x - \frac{2}{\pi}\right)\cos pt \quad . \quad . \quad . \quad . \quad (87.6)$$

When this expression is utilized in connection with equations (87·1) and (87·2), and maximum values are ascribed to the variables, we obtain

$$V_{\text{max.}} = \frac{\pi^4F^2EI}{4L^3}, \quad T_{\text{max.}} = \tfrac{1}{4}\left(1 - \frac{8}{\pi^2}\right)\frac{m}{g}F^2Lp^2.$$

Thus

$$p = \frac{22{\cdot}6}{L^2}\sqrt{\frac{EgI}{m}}$$

follows on writing $V_{\text{max.}} = T_{\text{max.}}$. Consequently

$$\frac{2\pi L^2}{22{\cdot}6}\sqrt{\frac{m}{EgI}}$$

is the approximate value of the period in question; and this differs by less than 1 per cent. from the corresponding period in Ex. 4 of Art. 86.

The foregoing solutions show that the fundamental period of vibration for simple systems can readily be determined with a fair degree of accuracy by the method of Art. 55, and this is a matter of some importance because the solution of many practical questions involves only the fundamental period of oscillation.

88. Propagation of the Waves. We may profitably digress, for a moment, and examine the conditions under which waves are transmitted through a stiff bar or beam executing transverse vibrations. A comparison of the corresponding equations of free motion for the wire of Art. 67 and the beam of Art. 86, namely

$$\frac{dy^2}{dt^2} = a^2\frac{dy^2}{dx^2},$$

$$\frac{dy^2}{dt^2} = \frac{EgI}{m}\frac{dy^4}{dx^4},$$

will throw sufficient light on the point. It has been demonstrated that a simple wave travelling with the constant velocity (a) is involved in the former of these equations, but the right-hand member of the latter equation cannot similarly be expressed as a velocity, as is evidenced by the 'dimensions' of the quantities concerned.

A noteworthy consequence of this fact is, as remarked by Lord Rayleigh,[1] that when a train of waves is transmitted through a stiff bar the velocity with which the transverse vibrations are propagated must vary with the wave-length. It actually varies inversely as the wave-length. On this account the harmonic components of a complex wave will travel with different velocities, and the wave will thereby be resolved into its original components. Such a wave, whether incident or reflected, will accordingly be modified by the phenomena of dispersion and distortion which were discussed in Arts. 71–73. The component waves will, therefore, get out of phase with one another during the process of transmission through a dispersive medium, in which the phase-velocities differ slightly for the several components of a group. Hence the resultant amplitude will be comparatively large at a position where for a certain instant all the members of the group agree in phase, as illustrated by Fig. 111.

We can now understand why the group-velocity is in general of much greater significance than the wave- or phase-velocity, for the energy in wave motion is propagated at a rate corresponding to the group-velocity. It may, in one sense, be said that the phase-velocity deserves notice only because it coincides with the group-velocity in the special case where the phase-velocity is independent of the frequency, a notable example of which is offered by electromagnetic waves *in vacuo*.

Systems constructed of composite materials, such as, for instance, reinforced concrete, might well afford the most interesting cases for investigation, by reason of the different ways in which the dispersion of energy may affect the elastic and the plastic properties of the constituent materials.

[1] *Theory of Sound*, vol. I, page 301.

In order to show more explicitly the nature of the modification thus introduced into the wave motion, we may proceed in the following manner. Insert the variables

$$\eta = x - at,\ \tau = x + at \quad . \quad . \quad . \quad . \quad (88\cdot1)$$

of Art. 63 in equation (71·3), namely

$$\frac{\partial^2 U}{\partial t^2} = a^2\frac{\partial^2 U}{\partial x^2} + b^2 U,$$

and so transform it to

$$\frac{\partial^2 U}{\partial\eta\partial\tau} + p^2 U = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (88\cdot2)$$

where $p = \dfrac{b}{\sqrt{2a}}$.

To fix ideas, bearing in mind that u is now a function of x, let the initial conditions be

$$u = \theta(x) \text{ and } \frac{\partial u}{\partial t} = \phi(x) \text{ when } t = 0,$$

which imply that $U, \dfrac{\partial U}{\partial\eta}, \dfrac{\partial U}{\partial\tau}$ are known for all points on the straight line $\eta = \tau$. If, for the present, we suppose that a particular solution of (88·2) is known, and denote it by W, then

$$\frac{\partial^2 W}{\partial\eta\partial\tau} + p^2 W = 0,$$

and
$$W\frac{\partial^2 U}{\partial\eta\partial\tau} - U\frac{\partial^2 W}{\partial\eta\partial\tau} = 0$$

follows from equation (88·2).

An application of Stokes' theorem now discloses the fact that the line integral

$$\int\left(U\frac{\partial W}{\partial\eta}d\eta + W\frac{\partial U}{\partial\tau}d\tau\right)$$

is zero when taken round any closed curve in the plane (η, τ) inside which the functions U, W, and their derivatives are regular. Thus if the boundary of the closed curve be formed by the straight lines $\eta = \eta_1$, $\tau = \tau_1$, drawn through the fixed point P in Fig. 142 and a continuous curve intersecting these lines at Q, R, we learn that

$$\int_Q^R\left(U\frac{\partial W}{\partial\eta}d\eta + W\frac{\partial U}{\partial\tau}d\tau\right) + \int_R^P U\frac{\partial W}{\partial\eta}d\eta + \int_P^Q W\frac{\partial U}{\partial\tau}d\tau = 0 \quad . \quad (88\cdot3)$$

If W can be determined so as to have the value unity over the lines $\eta = \eta_1$, $\tau = \tau_1$, then, by equation (88·3),

$$\int_Q^R\left(U\frac{\partial W}{\partial\eta}d\eta + W\frac{\partial U}{\partial\tau}d\tau\right) + U_Q - U_P = 0 \quad . \quad . \quad (88\cdot4)$$

Furthermore, since equation (88·3) holds good for any two solutions

U and W of equation (88·2), it is legitimate to interchange U and W in equation (88·3), and so deduce

$$\int_Q^R\left(W\frac{\partial U}{\partial\eta}d\eta + U\frac{\partial W}{\partial\tau}d\tau\right) + U_P - U_R = 0.$$

Subtracting this from equation (88·4), we find

$$2U_P = U_Q + U_R + \int_Q^R\left(U\frac{\partial W}{\partial\eta} - W\frac{\partial U}{\partial\eta}\right)\partial\eta$$
$$+ \int_Q^R\left(W\frac{\partial U}{\partial\tau} - U\frac{\partial W}{\partial\tau}\right)d\tau \quad . \quad . \quad (88.5)$$

as the expression for U, at any point P, in terms of its values at Q, R and the values of its partial derivatives referred to the curve over which the integral extends.

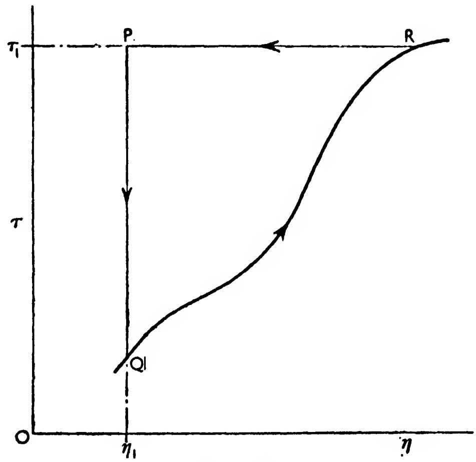


FIG. 142.

To discover a solution W of equation (88·2) which has the value unity along the straight lines $\eta = \eta_1$ and $\tau = \tau_1$, put

$$\xi = (\eta - \eta_1)(\tau - \tau_1).$$

Since the function ξ vanishes along each of the specified straight lines, it is evident that we must find a series in ξ, with a constant term equal to unity, which shall satisfy equation (88·2). On the reasonable supposition that

$$W = 1 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \ldots, \quad . \quad . \quad (88·6)$$

we obtain

$$\frac{\partial W}{\partial\eta} = (c_1 + 2c_2\xi + 3c_3\xi^2 + \ldots)(\tau - \tau_1),$$

$$\frac{\partial^2 W}{\partial\eta\partial\tau} = c_1 + 2c_2\xi + 3c_3\xi^2 + \ldots + (2c_2 + 3.2c_3\xi + \ldots)(\eta - \eta_1)(\tau - \tau_1)$$
$$= c_1 + 2c_2\xi + 3c_3\xi^2 + \ldots + (2c_2 + 3.2c_3\xi + \ldots)\xi \quad . \quad (88·7)$$

It is plain that equation (88·6) will agree with (88·2) for all values of ξ only if the coefficients of ξ, ξ^2, ξ^3, . . . are all zero for the expression given by substituting equations (88·6) and (88·7) in (88·2). Thus it appears that

$$c_1 + p^2 = 0,\ 4c_2 + c_1p^2 = 0,\ 3^2c_3 + c_2p^2 = 0,\ 4^2c_4 + c_3p^2 = 0, \ldots,$$

or $$c_1 = -p^2,\ c_2 = \frac{p^4}{2^2},\ c_3 = -\frac{p^6}{2^2.3^2},\ c_4 = \frac{p^8}{2^2.3^2.4^2}, \ldots,$$

in virtue of which we can put the required function

$$W = 1 - p^2\xi + \frac{p^4\xi^2}{2^2} - \frac{p^6\xi^3}{2^2.3^2} + \frac{p^8\xi^4}{2^2.3^2.4^2} - \ldots$$
$$= J_0\left(\frac{p\xi^{\frac{1}{2}}}{2}\right), \quad \ldots \ldots \ldots \quad (88.8)$$

where J_0 refers to the *Bessel function of zero order.*

To simplify the analysis as much as may be without affecting the general character of the result, we shall now suppose the material through which the waves are transmitted to be infinitely long in the direction of propagation, and take the initial conditions to be

$$u = f(x) \text{ and } \frac{\partial u}{\partial t} = F(x) \text{ when } t = 0.$$

On the assumption implied in equation (71·3), i.e. $u = Ue^{bt}$, we now have

$$U = f(x) \text{ and } \frac{\partial U}{\partial t} = F(x) - bf(x) \text{ when } t = 0,$$

which signify that the relations

$$U = f(\eta),\ \frac{\partial U}{\partial \tau} = \tfrac{1}{2}\left[f'(\eta) + \frac{1}{a}\{F(\eta) - bf(\eta)\}\right]$$

hold along the straight line $\eta = \tau$. In this case the curve is the line $\eta = \tau$, and the η-co-ordinates of the points Q, R are η_1, τ_1, respectively. In virtue of the obvious relation $d\eta = d\tau$ along the straight line of integration defined by equation (88·5), we can write

$$2U(\eta_1, \tau_1) = U(\eta_1) + U(\tau_1) + \int_{\eta_1}^{\tau_1}\left\{U\left(\frac{\partial W}{\partial \eta} - \frac{\partial W}{\partial \tau}\right) - W\left(\frac{\partial U}{\partial \eta} - \frac{\partial U}{\partial \tau}\right)\right\}d\eta$$
$$= f(\eta_1) + f(\tau_1) + \frac{1}{a}\int_{\eta_1}^{\tau_1}\left(W\frac{\partial U}{\partial t} - U\frac{\partial W}{\partial t}\right)d\eta,$$

which means, in terms of the original variables (x, t), that

$$2U(x, t) = f(x - at) + f(x + at) + \frac{1}{a}\int_{x-at}^{x+at} W\{F(\eta) - bf(\eta)\}d\eta$$
$$- \frac{1}{a}\int_{x-at}^{x+at}\left\{f(\eta)\frac{\partial W}{\partial t}\right\}d\eta$$

In these integrals, however, W is stated as a function of ξ which has the value $(\eta - \eta_1)(\eta - \tau_1)$ along the line $\eta = \tau$, hence

$$\begin{aligned}\frac{\partial W}{\partial t} &= a\left(\frac{\partial W}{\partial \tau} - \frac{\partial W}{\partial \eta}\right)\\ &= a(\eta - \eta_1 - \tau + \tau_1)\frac{dW}{d\xi}\\ &= a(\tau_1 - \eta_1)\frac{dW}{d\xi}\end{aligned}$$

along the straight line in question. The final form of the solution is, therefore,

$$2U(x, t) = f(x - at) + f(x + at) + \frac{1}{a}\int_{x-at}^{x+at} W(\xi)\{F(\eta) - bf(\eta)\}d\eta - 2at\int_{x-at}^{x+at} f(\eta)\frac{dW}{d\xi}d\eta, \quad . \quad . \quad . \quad . \quad (88.9)$$

where $\xi = (\eta - \eta_1)(\eta - \tau_1)$, and W is defined by equation (88·8).

This result may be interpreted into more familiar terms by supposing the initial disturbance to be limited to a length $\pm l$ of the structure concerned, reckoned in the x-direction. The last equation clearly indicates that, at any given instant t_1, U will be zero for all values of x such that either $x - at_1 > l$ or $x + at_1 < l$. Both ends of the wave consequently travel with constant velocity a in opposite directions, as in the simple wave of Art. 63. But our expression shows that undamped and damped disturbances differ in one important particular for points situated between the ends of a wave. It has been demonstrated that in the undamped type of wave a particle at x immediately returns to rest after both waves have passed over it, but this is not true of the damped type of wave under examination. The reason is easily found by considering an instant $t_1 > \frac{l}{a}$ and a point x_1 situated between $l - at_1$ and $-l + at_1$, so that $x_1 + at_1 > l$, and $x_1 - at_1 < -l$. If, to simplify matters, we take the initial velocity to be zero, equation (88·9) gives

$$2U = -\frac{b}{a}\int_{x-at}^{x+at} W(\xi)f(\eta)d\eta - 2at\int_{x-at}^{x+at} f(\eta)\frac{dW}{d\xi}d\eta,$$

showing that the displacement does not become zero after the waves have passed over any particular point. Another view of this result is that neither the 'forward' wave nor the 'backward' wave has a definite end, though each has a definite origin.

We conclude, then, that the effect of variation in the velocity of propagation with frequency (Art. 71) is to cause the waves to overlap. A consequence of this phenomenon is that the large damping which is sometimes experienced in practice may exert a considerable influence on the period or wave-length, and thereby

cause vibrations of large amplitude to have periods which are longer than those of vibrations with a small amplitude in a specified material. This imposes a restriction on the conclusion arrived at in Art. 59, where it was established that only a very slight lengthening of the period of free vibration is produced by a small damping force of the prescribed type. The modification brought about by large damping is a matter of practical importance in certain problems, one of which relates to the effect of vibration of the barrel of a gun on the accuracy of aim, for the path of a projectile is naturally affected by the transverse motion of the muzzle at the instant when the shot is ejected.

The special advantage of the foregoing method lies in the fact that we are not restricted to initial conditions that involve the unknown function and its time rate of variation at the instant $t = 0$. It is only necessary to have, referring to Fig. 142, the values of the unknown function and one of its partial derivatives along any curve that intersects two lines (drawn parallel to the axes $O\eta$, $O\tau$) passing through the point (η_1, τ_1) at which the value of the unknown function u is required.

If the frictional term b be neglected, the function W has the constant value unity, and equation (88·9) reduces to

$$2U(x, t) = f(x - at) + f(x + at) + \frac{1}{a}\int_{x-at}^{x+at} F(\eta)d\eta \quad . \quad (88.10)$$

If, in addition, the disturbance starts from rest, $F(\eta)$ is identically zero, and the last expression agrees, as it should, with equation (63·4).

89. Struts and Ties. If either a thrust or a pull is applied, in the longitudinal direction, to any one of the beams examined so far in this chapter, the resulting system will form a strut in the first place and a tie in the second.

The constraint thereby introduced will affect the frequency of vibration in accordance with the general conclusions of Art. 55. To estimate the magnitude and sense of the modification, let us consider the vibrations in a normal mode of a slender beam subjected to the constant thrust P indicated in Fig. 143, where the

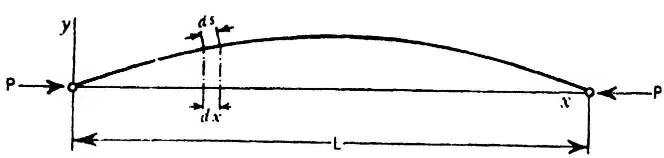


FIG. 143.

ends, distant L apart, are pin-jointed. We shall suppose that the strut remains stable throughout the motion and, for simplicity, that the dissipative agencies are negligibly small.

When the disturbed motion is restricted to slight displacements about the position of rest we may, on the basis of the beam-theory, assume that originally plane sections of the strut remain plane after bending.

Thus, taking the x-axis as the equilibrium-position with the origin at the left-hand joint, and m representing the weight per unit length of the strut, by equation (87·2) we have

$$2T = \frac{m}{g}\int_0^L \dot{y}^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (89{\cdot}1)$$

for the kinetic energy T, the rotatory inertia of the material being regarded as of the second order of small quantities.

Moreover, if V_1 be the potential energy when the thrust is absent, it follows from equation (87·1) that

$$V_1 = \tfrac{1}{2}EI\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx,$$

where I refers to the *least* moment of inertia of the section. Now the total strain energy is the sum of V_1 and the work done by the load P in changing the length of an element from dx to ds. With the positive direction reckoned from left to right in the figure, it is readily seen that in the limit the element will be shortened by an amount $\frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$, hence it appears that the strain energy due to the thrust only may be expressed by

$$V_2 = -\tfrac{1}{2}P\int_0^L \left(\frac{dy}{dx}\right)^2 dx.$$

These relations show that the total potential energy

$$V = \tfrac{1}{2}\int_0^L \left\{EI\left(\frac{d^2y}{dx^2}\right)^2 - P\left(\frac{dy}{dx}\right)^2\right\}dx, \quad . \quad . \quad . \quad (89.2)$$

being the sum of V_1 and V_2.

If we further assume the configuration of the strut to be similar to that found for the beam in Ex. 1 of Art. 86, then at a point x the deflection

$$y = A \sin \frac{r\pi}{L}x \sin pt \quad . \quad . \quad . \quad . \quad . \quad (89.3)$$

in the rth normal mode, where A is an arbitrary constant.

On this supposition the velocity

$$\dot{y} = Ap \sin \frac{r\pi}{L}x \cos pt,$$

so that in equation (89·1)

$$\dot{y}^2 = A^2p^2 \sin^2 \frac{r\pi}{L}x \cos^2 pt.$$

whence
$$2T = \frac{m}{g}.A^2p^2\cos^2 pt\int_0^L \sin^2\frac{r\pi}{L}xdx$$
$$= \frac{mL}{2g}A^2p^2\cos^2 pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89.4)$$

From the same source we also derive the relations

$$\frac{dy}{dx} = \frac{Ar\pi}{L}\cos\frac{r\pi}{L}x\sin pt,$$
$$\frac{d^2y}{dx^2} = -\frac{Ar^2\pi^2}{L^2}\sin\frac{r\pi}{L}x\sin pt,$$

the squares of which give

$$\left(\frac{dy}{dx}\right)^2 = \frac{A^2r^2\pi^2}{L^2}\cos^2\frac{r\pi}{L}x\sin^2 pt,$$
$$\left(\frac{d^2y}{dx^2}\right)^2 = \frac{A^2r^4\pi^4}{L^4}\sin^2\frac{r\pi}{L}x\sin^2 pt,$$

for the expressions required in equation (89·2). Therefore

$$2V = \frac{A^2r^2\pi^2}{L^2}\sin^2 pt\int_0^L\left(\frac{r^2\pi^2EI}{L^2}\sin^2\frac{r\pi}{L}x - P\cos^2\frac{r\pi}{L}x\right)dx$$
$$= \frac{A^2\pi^4EI}{2L^3}\left(r^4 - \frac{r^2L^2P}{\pi^2EI}\right)\sin^2 pt,$$

in the case where EI is uniform along the strut.

Since the absence of dissipative forces implies the condition $T_{\max.} = V_{\max.}$, our results lead to

$$\frac{mL}{g}p^2 = \frac{\pi^4EI}{L^3}\left(r^4 - \frac{r^2L^2P}{\pi^2EI}\right).$$

Thus, on writing a for $\frac{mL}{g}$, and b_r for $\frac{\pi^4EI}{L^3}\left(r^4 - \frac{r^2L^2P}{\pi^2EI}\right)$, we have, from equation (89·3), the deflection in the rth normal mode determined by

$$y = A\sin\frac{r\pi}{L}x\sin\left(\sqrt{\frac{b_r}{a}}t + \varepsilon\right), \quad . \quad . \quad . \quad . \quad (89·5)$$

where ε is an appropriate phase-term. As the frequency

$$\frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{b_r}{a}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (89.6)$$

we learn that the effect of the thrust P is to lower the frequency by an amount equal to the second member in the relation for b_r.

To the present degree of approximation the most general motion, obtained by superposition of the curves (89·5), may be represented by

$$y = \sum_{r=1}^{n} A_r\sin\frac{r\pi}{L}x\sin\left(\sqrt{\frac{b_r}{a}}t + \varepsilon_r\right) \quad . \quad . \quad (89·7)$$

when n 'normal' curves are involved.

It is not difficult to realize that, on the contrary, the effect of a *pull* P is to raise the frequency of vibration, as the corresponding equations for the tie thus formed are given by substituting a positive sign for the negative in the relation for b_r. Moreover, the expression for y will be sensibly the same as that found in the case of a slender wire (Art. 67) if P is very great.

Equation (89.5) indicates that in the fundamental mode, specified by $r = 1$, the deflection y tends to infinitely large values as $\dfrac{L^2P}{\pi^2EI}$ approaches unity, as is to be expected, because this condition corresponds to Euler's crippling load P for the given strut.

An application of Rayleigh's theorem, Art. 55, will serve to illustrate its value in instances where only the fundamental period is required for a system of this type.

Ex. Find the gravest frequency in transverse vibration for the slender strut shown in Fig. 144, rigidly fixed at one end and free

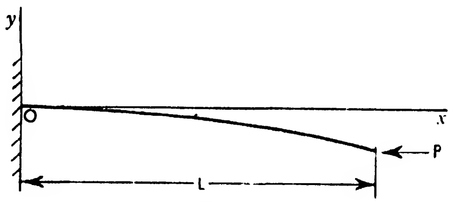


FIG. 144.

at the other. The free length of the strut is L, and the magnitude of the thrust is P.

Taking the origin at O and the x-axis as the position of rest, we shall, from Ex. 1 of Art. 87, assume

$$y = F\left(1 - \cos \frac{\pi}{2L}x\right)$$

for the shape of the disturbed strut at a given instant, F being an arbitrary constant.

By this expression and equation (89.2) the maximum strain energy produced by the thrust P is, with the positive direction as before,

$$\begin{aligned} -\tfrac{1}{2}P\int_0^L \left(\frac{dy}{dx}\right)^2 dx &= -\frac{\pi^2F^2P}{8L^2}\int_0^L \sin^2\frac{\pi}{2L}x\,dx \\ &= -\frac{\pi^2F^2P}{8L^2}\left[\tfrac{1}{2}x - \frac{L}{2\pi}\sin\frac{\pi}{L}x\right]_0^L \\ &= -\frac{\pi^2F^2P}{16L}. \end{aligned}$$

Further, from the previous example, $\frac{\pi^4 F^2 EI}{64L^3}$ is the maximum strain energy when the thrust $P = 0$, hence

$$V_{\text{max.}} = \frac{\pi^4 F^2 EI}{64L^3} - \frac{\pi^2 F^2 P}{16L}$$

relates to the maximum potential energy of the strut.

It is also to be inferred, from the same place, that the maximum kinetic energy

$$T_{\text{max.}} = \left(\frac{3}{4} - \frac{2}{\pi}\right)\frac{m}{g}F^2 L p^2,$$

where m is the weight per unit length of the strut, and $\frac{p}{2\pi}$ is the frequency in question.

On the implied supposition that $T_{\text{max.}} = V_{\text{max.}}$, we obtain

$$\left(\frac{3}{4} - \frac{2}{\pi}\right)\frac{m}{g}F^2 L p^2 = \frac{\pi^4 F^2 EI}{64L^3} - \frac{\pi^2 F^2 P}{16L},$$

and so conclude that

$$p = \frac{3{\cdot}664}{L^2}\left(\frac{EgI}{m} - \frac{4PgL^2}{\pi^2 m}\right)^{\frac{1}{2}}$$

approximately defines the gravest frequency of vibration in the present case.

A slightly more accurate result would have been obtained had we assumed the shape of the disturbed strut to conform to the 'static' curve for a cantilever of weight m per unit length, mentioned in Ex. 1 of Art. 87.

90. Forced Vibration of Beams. The preceding treatment suffices to show that

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{m}{g}\frac{\partial^2 y}{\partial t^2} = Y \quad . \quad . \quad . \quad . \quad . \quad (90.1)$$

will represent the motion which would ensue from the application of a vertical force Y to the beam implied in equation (86.7), the beam being specified by its weight m per unit length, moment of inertia I of the cross-section, and direct modulus of elasticity E of the material.

Taking, as a general case, the force to be periodic and of the type

$$Y = Y_0 \sin\frac{r\pi}{L}x \sin \omega t, \quad . \quad . \quad . \quad . \quad (90.2)$$

where r is an integer and $\frac{\omega}{2\pi}$ is the frequency with which the load

acts on a beam of length L, we have the equation of motion

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{m}{g}\frac{\partial^2 y}{\partial t^2}=Y_0 \sin\frac{r\pi}{L}x \sin \omega t \quad . \quad . \quad . \quad (90.3)$$

If, for the sake of brevity, we let

$$\frac{m}{EgI}=\beta^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (90.4)$$

$$\frac{\pi^2}{L^2}\sqrt{\frac{EgI}{m}}=p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90.5)$$

then the motion is expressed by

$$\frac{\partial^4 y}{\partial x^4}+\beta^4\frac{\partial^2 y}{\partial t^2}=\frac{Y_0}{EI}\sin\frac{r\pi}{L}x\sin\omega t, \quad . \quad . \quad . \quad (90.6)$$

the frequency of the beam being $\frac{p}{2\pi}$.

The forced vibration, corresponding to the 'particular integral' of this equation, is accordingly associated with a displacement

$$y_1=\frac{1}{\frac{\partial^4}{\partial x^4}+\beta^4\frac{\partial^2}{\partial t^2}}\frac{Y_0}{EI}\sin\frac{r\pi}{L}x\sin\omega t$$

$$=\frac{1}{\frac{r^4\pi^4}{L^4}-\beta^4\omega^2}\frac{Y_0}{EI}\sin\frac{r\pi}{L}x\sin\omega t,$$

or, in the original notation,

$$y_1=\frac{L^4}{\pi^4\left(r^4-\frac{\omega^2}{p^2}\right)}\frac{Y_0}{EI}\sin\frac{r\pi}{L}x\sin\omega t$$

$$=\frac{Y_0 g}{m(r^4p^2-\omega^2)}\sin\frac{r\pi}{L}x\sin\omega t \quad . \quad . \quad . \quad (90.7)$$

It is further seen that the free vibrations, corresponding to the 'complementary function' of equation (90.6), will contribute a displacement

$$y_2=\sin\frac{\pi}{L}x(A_1\cos pt+B_1\sin pt)+\sin\frac{2\pi}{L}x(A_2\cos 4pt+B_2\sin 4pt)$$

$$+\ldots+\sin\frac{r\pi}{L}x(A_r\cos r^2pt+B_r\sin r^2pt),$$

where the constants $A_1, A_2, \ldots, A_r, B_1, B_2, \ldots, B_r$ depend on the initial circumstances of the motion.

If y signifies the displacement in general motion, then

$$y=y_1+y_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90.8)$$

To explain the method by which the constants are evaluated in a given case, suppose the beam to be simply supported at both

ends, and initially at rest in the equilibrium-position defined by the x-axis in Fig. 136.

It is a simple matter to prove that the implied condition $y = 0$ when $t = 0$ for all values of x is fulfilled only if $A_1 = A_2 = \ldots = A_r = 0$ in equation (90.8). Hence

$$y = \frac{Y_0 g}{m(r^4 p^2 - \omega^2)} \sin \frac{r\pi}{L}x \sin \omega t + B_1 \sin \frac{\pi}{L}x \sin pt$$

$$+ B_2 \sin \frac{2\pi}{L}x \sin 4pt + \ldots + B_r \sin \frac{r\pi}{L}x \sin r^2 pt.$$

Imposing the remaining condition, $\frac{\partial y}{\partial t} = 0$ when $t = 0$ for all values of x, on the last equation, we find $B_1 = B_2 = \ldots = B_{r-1} = 0$. The limitation to this series of zero values arises from the fact that B_r does not necessarily vanish, since at time $t = 0$ the coefficient of $\sin \frac{r\pi}{L}x$ is

$$\frac{Y_0 g}{m(r^4 p^2 - \omega^2)}\omega + r^2 p B_r.$$

As this quantity must be equal to zero, it follows from the resulting relation

$$B_r = -\frac{Y_0 g}{m(r^4 p^2 - \omega^2)} \frac{\omega}{r^2 p}$$

that the deflection

$$y = \frac{Y_0 g}{m(r^4 p^2 - \omega^2)} \sin \frac{r\pi}{L}x \left(\sin \omega t - \frac{\omega}{r^2 p} \sin r^2 pt \right) \quad . \quad (90.9)$$

This equation for the general motion can be treated in the manner already described, by assigning the proper values to r in the work of summing a given number of harmonic terms.

For example, in the fundamental mode, corresponding to $r = 1$, our equation yields

$$y = \frac{Y_0 g}{m(p^2 - \omega^2)} \sin \frac{\pi}{L}x \left(\sin \omega t - \frac{\omega}{p} \sin pt \right)$$

for the deflection at any point x and time t with the prescribed load. But it is not to be supposed that this relation holds good in all conditions, for if p be very nearly equal to ω, so that $p^2 - \omega^2$ is small, the amplitude of the forced oscillation will be very large. The displacement will, in the absence of friction, increase indefinitely with the time if $p = \omega$, as is indicated by the solution corresponding to the forced oscillation, namely

$$y = Ft \sin \frac{\pi}{L}x \cos \omega t,$$

where F is an arbitrary constant.

It is not difficult to express F in terms of known quantities, since we now have

$$\frac{\partial y}{\partial t} = F \sin \frac{\pi}{L}x(\cos \omega t - \omega t \sin \omega t),$$

$$\frac{\partial^2 y}{\partial t^2} = - F \sin\frac{\pi}{L}x(2\omega \sin \omega t + \omega^2 t \cos \omega t),$$

$$\frac{\partial^4 y}{\partial x^4} = \frac{\pi^4 F}{L^4}t \sin \frac{\pi}{L}x \cos \omega t,$$

whence it is readily inferred that

$$2F\beta^4\omega = - \frac{Y_0}{EI}$$

i.e.

$$F = - \frac{Y_0 g}{2m\omega},$$

remembering the relations $\beta^4\omega^2 = \frac{\pi^4}{L^2}$, $\beta^4 = \frac{m}{EgI}$.

Thus it appears that

$$y = - \frac{Y_0 g}{2m\omega}t \sin \frac{\pi}{L}x \cos \omega t$$

is a particular solution, and

$$y = G \sin\frac{\pi}{L}x \sin \omega t - \frac{Y_0 g}{2m\omega}t \sin\frac{\pi}{L}x \cos \omega t \quad . \quad . \quad (90.10)$$

is, in consequence, the complete solution, where the constant G depends on the initial conditions of motion.

As the beam is initially at rest, the implied condition $\frac{\partial y}{\partial t} = 0$ when $t = 0$ for all values of x is, as can easily be verified, fulfilled if

$$G\omega - \frac{Y_0 g}{2m\omega} = 0$$

i.e. if

$$G = \frac{Y_0 g}{2m\omega^2}.$$

Inserting this value for G in equation (90.10), we find that in a state of resonance

$$y = \frac{Y_0 g}{2m\omega^2} \sin\frac{\pi}{L}x(\sin \omega t - \omega t \cos \omega t),$$

showing, again, that the amplitude will increase indefinitely with the time, and thereby violate our supposition as to small vibrations unless a sufficiently great frictional force be introduced in the system. This fact must be borne in mind when using equation (90.9) for actual systems in which the damping factor is relatively small.

We can proceed in the same way in instances where the load Y

moves across a beam with a specified velocity and varies according to a known law, so that Y is a function of x, by expressing Y in the form

$$B_1 \sin\frac{\pi}{L}x + B_2 \sin\frac{2\pi}{L}x + \ldots + B_r \sin\frac{r\pi}{L}x$$

over the range $0<x<L$ in cases where the load extends over only a fraction of the span L. The method will be understood from the following example.

Ex. Investigate, for the beam above specified, the vibratory motion which will be produced by a periodic load uniformly distributed over the span L, and defined by

$$Y = Y_0 \sin \omega t.$$

To state Y_0 in a convenient form, we recall (Art. 24) that the relation

$$\phi = \frac{4c}{\pi}\left(\sin x + \tfrac{1}{3}\sin 3x + \ldots + \frac{1}{r}\sin rx\right)$$

stands for a function which has a constant value c over the range $0 < x < \pi$, whence it is legitimate to infer that in the present problem

$$Y_0 = \frac{4Y_0}{\pi}\left(\sin\frac{\pi}{L}x + \tfrac{1}{3}\sin\frac{3\pi}{L}x + \ldots + \frac{1}{r}\sin\frac{r\pi}{L}x\right)$$

between the limits $x = 0$ and $x = L$.

Thus, with the origin at the left-hand support, we obtain

$$Y = \frac{4Y_0}{\pi}\left(\sin\frac{\pi}{L}x + \tfrac{1}{3}\sin\frac{3\pi}{L}x + \ldots + \frac{1}{r}\sin\frac{r\pi}{L}x\right)\sin \omega t,$$

so that the general equation (90.1) becomes

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{m}{g}\frac{\partial^2 y}{\partial t^2} = \frac{4Y_0}{\pi}\left(\sin\frac{\pi}{L}x + \tfrac{1}{3}\sin\frac{3\pi}{L}x + \ldots + \frac{1}{r}\sin\frac{r\pi}{L}x\right)\sin \omega t,$$

where m represents the weight per unit length of the beam.

Each term of this series may be treated separately, the 'particular integral' being given by equation (90.7), provided Y_0 is replaced by $\frac{4Y_0}{r\pi}$ and r assumes in turn the values 1, 3, 5, . . . The complete solution may likewise be written down with the aid of equation (90.9).

After effecting these operations, it will be found that the displacements for the forced vibrations in the first, third, and fifth harmonics are respectively

$$\frac{4Y_0 g}{\pi m(p^2 - \omega^2)},\ \frac{4Y_0 g}{3\pi m(p^2 - \omega^2)},\ \frac{4Y_0 g}{5\pi m(p^2 - \omega^2)},$$

where $p = \frac{\pi^2}{L^2}\sqrt{\frac{EgI}{m}}$, so long as a state of resonance does not exist.

It is to be remarked that the series formed by these harmonic terms converges rapidly, owing to the presence of r in the denominators. On this account a sufficiently accurate value of y is obtained, at least for many practical purposes, by neglecting all but the first harmonic. To this order of approximation

$$y = \frac{4Y_0 g}{\pi m(p^2 - \omega^2)} \sin \frac{\pi}{L}x\left(\sin \omega t - \frac{\omega}{p} \sin pt\right),$$

if the dissipative forces are negligibly small.

91. Continuous Beams and Structures. In practice we have frequently to investigate the small vibrations of a system composed of several beams connected together by joints having different degrees of stiffness. The foregoing results show that any number of such beams may be treated as a continuous structural member if the proper values can be assigned to the coefficients of stiffness of the joints concerned. A graphical method usually affords the best means of solving the problem, which is essentially that of finding the effect of an intermediate support having a prescribed degree of rigidity.

(*a*) *Inelastic Supports.* To fix ideas let us examine the beam indicated in Fig. 145 (*a*), simply supported on perfectly rigid abutments at the points marked 0, 1, 2, and carrying (static) concentrated loads P_1, P_2 at distances x_1, x_2 reckoned from the left-hand support.

Let the points x_1, x_2 deflect y_1, y_2, respectively, when the beam is executing small vibrations about the equilibrium-position, which we shall take as corresponding to the horizontal or x-axis. It will be assumed in what follows that the system describes undamped motion with the supports always acting as nodes, and that the weight of the beam is negligibly small in comparison with the loads.

If Q_1, Q_2 denote the inertia forces produced by the loads P_1, P_2 when moving downwards, then

$$Q_1 = -\frac{P_1}{g}\ddot{y}_1,\ Q_2 = -\frac{P_2}{g}\ddot{y}_2, \quad . \quad . \quad . \quad (91.1)$$

with the positive direction measured upwards.

We have also, by equation (51.2), the relations

$$y_1 = \alpha_{11}Q_1 + \alpha_{12}Q_2,\ y_2 = \alpha_{21}Q_1 + \alpha_{22}Q_2, \quad . \quad . \quad (91.2)$$

where the coefficients of stiffness represented by the α-terms remain sensibly constant throughout the prescribed motion.

The signs to be ascribed to the inertia forces will necessarily depend on the mode of vibration under examination. If, as we shall suppose, the system is vibrating in the mode illustrated by Fig. 145 (*b*), the forces will act in opposite senses.

It is not difficult to realize that the work will be greatly simplified

if we imagine the beam as cut at the intermediate support, the operation being indicated in the deflection diagram 145 (*c*). Now the system consists of two freely supported beams, and it would be a simple matter to measure from this diagram the deflections δ_1, δ_2 at the points x_1, x_2, as well as the slopes θ_1, θ_2 at the intermediate

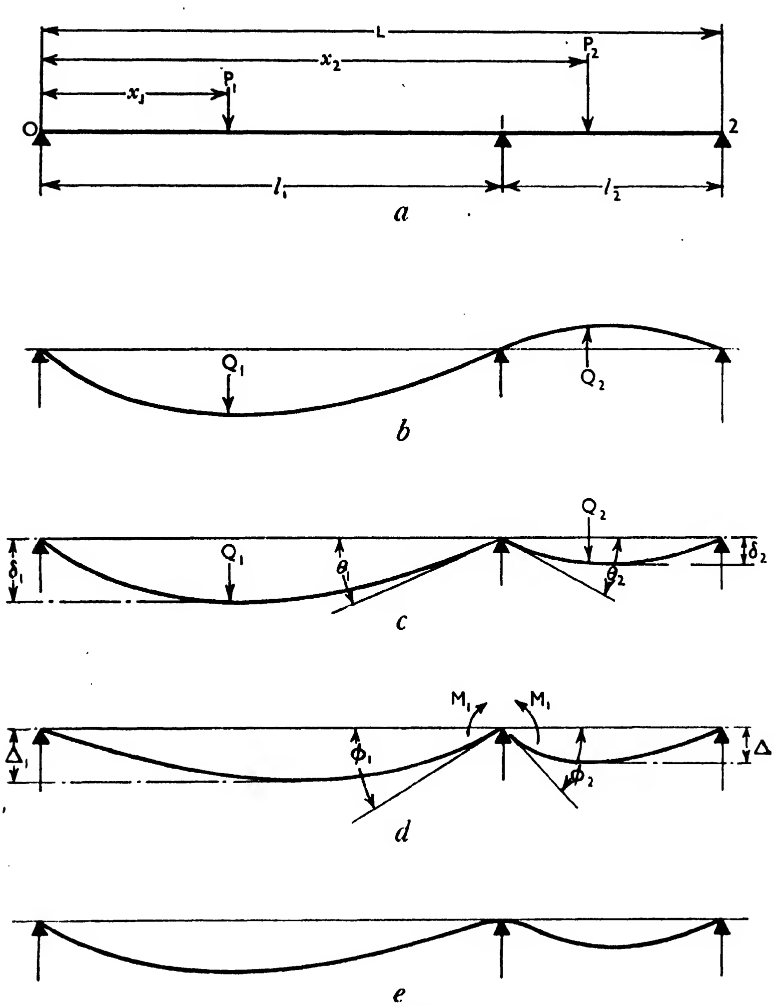


FIG. 145.

supports, which are produced by the loads Q_1, Q_2. We shall therefore take these deflections and slopes to be known quantities.

Let a bending moment M_1 be required at the severed ends to restore the beam to its original form with the loads in position. When the bending moment M_1 acts on the 'cut' beam in the

unloaded condition, let Δ_1, Δ_2 be the deflections at the points x_1, x_2, and ϕ_1, ϕ_2 be the slopes at the intermediate support, as shown in Fig. 145 (*d*). These deflections and slopes are readily determined by analytical or graphical means.

By way of expressing the deflections in terms of *unit* forces and couples, write

$$\alpha_1 = \frac{\delta_1}{Q_1},\ \alpha_2 = \frac{\delta_2}{Q_2},\ \beta_1 = \frac{\Delta_1}{M_1},\ \beta_2 = \frac{\Delta_2}{M_1}, \quad . \quad . \quad (91.3)$$

and so reduce equations (91.2) to the form

$$y_1 = \alpha_1 Q_1 + \beta_1 M_1,\ y_2 = \alpha_2 Q_2 - \beta_2 M_1, \quad . \quad . \quad (91.4)$$

where the negative sign indicates that here the forces act in opposite senses.

The fact that the slopes are proportional to the forces and couples may be signified by the relations

$$\theta_1 = c_1 Q_1,\ \theta_2 = c_2 Q_2,\ \phi_1 = k_1 M_1,\ \phi_2 = k_2 M_1, \quad . \quad (91.5)$$

the constants c_1, c_2, k_1, k_2 being known from the deflection diagrams 144 (*c*) and 144 (*d*).

It follows from considerations of continuity at the intermediate support of the actual beam that the condition

$$\theta_1 + \phi_1 = \theta_2 - \phi_2 \quad . \quad . \quad . \quad . \quad . \quad (91.6)$$

must be fulfilled, which means that

$$c_1 Q_1 + k_1 M_1 = c_2 Q_2 - k_2 M_1,$$

by reason of the relations (91.5). In this notation, therefore, the bending moment

$$M_1 = \frac{c_2 Q_2 - c_1 Q_1}{k_1 + k_2} \quad . \quad . \quad . \quad . \quad . \quad (91.7)$$

Thus, after substituting this expression for M_1 in equations (91.4), we can express the deflections in the form

$$y_1 = \alpha_1 Q_1 + \beta_1\left(\frac{c_2 Q_2 - c_1 Q_1}{k_1 + k_2}\right),\ y_2 = \alpha_2 Q_2 - \beta_2\left(\frac{c_2 Q_2 - c_1 Q_1}{k_1 + k_2}\right),$$

i.e.

$$\left.\begin{aligned} y_1 &= \left(\alpha_1 - \frac{\beta_1 c_1}{k_1 + k_2}\right)Q_1 + \frac{\beta_1 c_2}{k_1 + k_2}Q_2, \\ y_2 &= \frac{\beta_2 c_1}{k_1 + k_2}Q_1 + \left(\alpha_2 - \frac{\beta_2 c_2}{k_1 + k_2}\right)Q_2. \end{aligned}\right\} \quad . \quad . \quad (91.8)$$

A comparison of corresponding coefficients in equations (91.2) and (91.8) at once discloses the fact that

$$\left.\begin{aligned} \alpha_{11} &= \alpha_1 - \frac{\beta_1 c_1}{k_1 + k_2},\ \alpha_{12} = \frac{\beta_1 c_2}{k_1 + k_2} = \frac{\beta_2 c_1}{k_1 + k_2} = \alpha_{21}, \\ \alpha_{22} &= \alpha_2 - \frac{\beta_2 c_2}{k_1 + k_2}, \end{aligned}\right\} \quad . \quad . \quad (91.9)$$

since $\beta_1 = c_1$, $\beta_2 = c_2$, by the reciprocal relation of Art. 49.

Having thus evaluated all the coefficients, we can combine the expressions (91.1), (91.2) and so arrive at

$$y_1 = -\alpha_{11}\frac{P_1}{g}\ddot{y}_1 - \alpha_{12}\frac{P_2}{g}\ddot{y}_2,\ y_2 = -\alpha_{21}\frac{P_1}{g}\ddot{y}_1 - \alpha_{22}\frac{P_2}{g}\ddot{y}_2 \quad . \ (91.10)$$

for the given system.

A solution of these equations may be obtained on the usual supposition that y_1, y_2 vary as e^{ipt}, whence

$$\left.\begin{aligned}\left(\alpha_{11}\frac{P_1}{g}p^2 - 1\right)y_1 + \alpha_{12}\frac{P_2}{g}p^2 y_2 = 0,\\ \alpha_{21}\frac{P_1}{g}p^2 y_1 + \left(\alpha_{22}\frac{P_2}{g}p^2 - 1\right)y_2 = 0\end{aligned}\right\} \quad . \quad . \quad . \ (91.11)$$

Eliminating the ratio $y_1 : y_2$, we ultimately find

$$\left(\alpha_{11}\frac{P_1}{g}p^2 - 1\right)\left(\alpha_{22}\frac{P_2}{g}p^2 - 1\right) - \frac{\alpha_{12}\alpha_{21}P_1P_2}{g^2}p^2 = 0, \quad . \quad . \ (91.12)$$

where p is the only unknown quantity.

This determines the motion, for if p_1, p_2 denote the roots of this quadratic in p^2, arranged in increasing order of magnitude, it is clear that $\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$ represent the free periods of vibrations in the successive modes identified with (*b*) and (*e*) of Fig. 145. To obtain the corresponding deflections signified by y_1, y_2, we insert the appropriate value of p^2 in equations (91.11). It has been explained

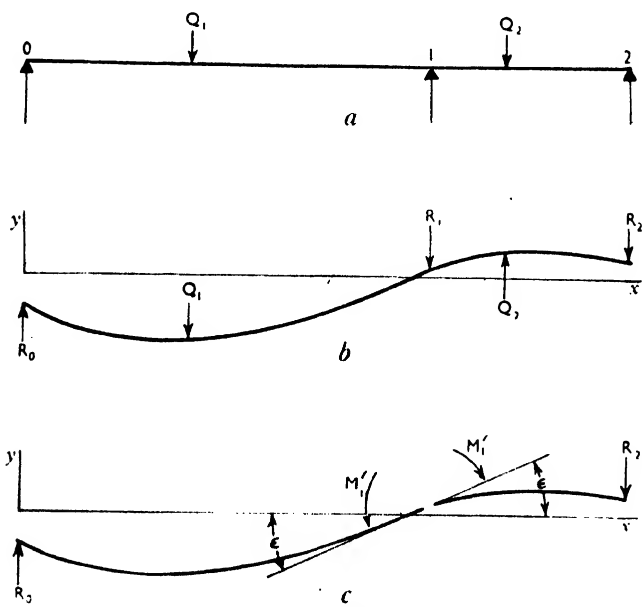


FIG. 146.

that the effect of slight friction is to diminish the deflections derived on the present suppositions, without sensibly affecting the period of vibration.

(*b*) *Elastic Supports.* The introduction of suitable symbols into the preceding analysis enables us to pass to the consideration of the elastic type of support which is associated with actual structures. With a view to exhibiting the similarity of treatment for both cases, let Fig. 146 (*a*) represent the beam and system of loads shown in Fig. 145 when non-rigid supports are involved.

To specify the present coefficients of stiffness, suppose that the successive application of a *unit* load to the supports marked 0, 1, 2 causes them to undergo independent deflections f_0, f_1, f_2 in the vertical direction. We may regard the deflections denoted by these symbols as known quantities, since they can be evaluated experimentally with a given system.

If we signify the bending moment on the beam at the intermediate support by M'_1, and the reactions at the points 0, 1, 2 by R_0, R_1, R_2 taken in turn when the beam is vibrating in the mode of Fig. 146 (*b*), it is easily inferred from equations (91.4) that

$$\left.\begin{aligned} y_1 &= \alpha_1 Q_1 + \beta_1 M'_1 + \frac{f_0 R_0 - f_1 R_1}{2}, \\ y_2 &= \alpha_2 Q_2 - \beta_2 M'_1 + \frac{f_1 R_1 + f_2 R_2}{2}, \end{aligned}\right\} \quad . \quad . \quad . \quad (91.12)$$

define the displacements y_1, y_2 at the points x_1, x_2, respectively. Here the same meaning as before is attached to α_1, α_2, β_1, β_2.

It is also clear that the configuration of the beam will be unchanged if we cut it at the intermediate support and simultaneously apply a bending moment M'_1 at the ends thereby introduced, as indicated in Fig. 146 (*c*).

In this hypothetical system we have

$$\begin{aligned} M'_1 &= l_1 R_0 - \frac{l_1}{x_1} Q_1 \\ &= l_2 R_2 - \frac{l_2}{L - x_2} Q_2, \end{aligned} \quad . \quad . \quad . \quad (91.13)$$

on taking moments of the forces about the intermediate support, and

$$R_0 + Q_2 - R_1 - R_2 - Q_1 = 0, \quad . \quad . \quad . \quad (91.14)$$

on equating to zero the vertical forces, the several dimensions being specified in Fig. 145 (*a*).

These results supply three of the four equations relating to the unknown quantities M'_1, R_0, R_1, R_2. To deduce the remaining expression, let the effect of the elastic supports be such as to change the slopes ϕ_1, ϕ_2 in equations (91.5) by the amounts ϕ'_1,

ϕ'_2, respectively. Thus, after equating the two relations for the slope ε in Fig. 146 (c) which are given by inserting the modified slopes ϕ'_1, ϕ'_2 in equation (91.6), it will be found that

$$\theta_1 + \phi'_1 + \frac{f_0R_0 + f_1R_1}{l_1} = \theta_2 - \phi'_2 + \frac{f_1R_1 - f_2R_2}{l_2} \quad . \quad (91.15)$$

With the help of the deflection diagram 146 (c) it is now practicable to derive a set of relations corresponding to (91.5), and, after introducing the data so obtained, to write equation (91.15) in the more convenient form

$$c_1Q_1 + k_1M'_1 + \frac{f_0R_0 + f_1R_1}{l_1} = c_2Q_2 - k_2M'_1 + \frac{f_1R_1 - f_2R_2}{l_2}, \quad (91.16)$$

the c- and k-coefficients having the same significance as before.

A straightforward process of elimination between equations (91.13), (91.14), (91.16) will lead to expressions corresponding to (91.8), and the latter may be solved in the same way as were equations (91.10).

This being so, the fact that the present method is applicable to a system of the prescribed type with any number of supports, either rigid or elastic, is seen from an inspection of equations (91.8), since they conform to the set of equations

$$\begin{aligned}
y_1 &= \alpha_{11}Q_1 + \alpha_{12}Q_2 + \ldots + \alpha_{1n}Q_n, \\
y_2 &= \alpha_{21}Q_1 + \alpha_{22}Q_2 + \ldots + \alpha_{2n}Q_n, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_n &= \alpha_{n1}Q_1 + \alpha_{n2}Q_2 + \ldots + \alpha_{nn}Q_n,
\end{aligned}$$

which was discussed in the general theory of Art. 52. Moreover, we can by the same method investigate the natural vibrations of a system involving any number of supports and any number of loads, provided the suffix n is equal to the sum of the number of supports and loads. The explanation is that in the general case all the intermediate supports are thus replaced by equivalent loads, and the beam is considered as if it were supported at the extreme ends alone.

In the foregoing analysis we have, for definiteness, taken the beam as being freely supported, but it is evident that the treatment holds equally well in the case of 'built-in' ends.

92. Framed structures with rigid joints offer a noteworthy example of 'continuous' systems, inasmuch as engineers frequently aim at this type of joint in general practice. It should, however, be observed here that the most direct way of approach to a solution depends on the nature of the problem under consideration, owing to the number of variables which may be present in the analysis of actual structures.

We shall formulate a general kind of problem by reference to buildings subject to disturbances in the earth, for then the cover-plates and foundation-joints should be designed so as to secure reasonably rigid connections. In the case of a proposed structure, it is also necessary to avoid having the free period of vibration approximately equal to the predominant period of an earthquake, which for certain regions lies within the range of 1·0 sec. and 1·5 sec. It is not difficult to understand the important bearing which this aspect of the matter has on the repair of a damaged building, since the treatment of Art. 55 shows that the fractured material will usually cause the natural period of the building to approach that associated with earthquakes, regard being had to the free

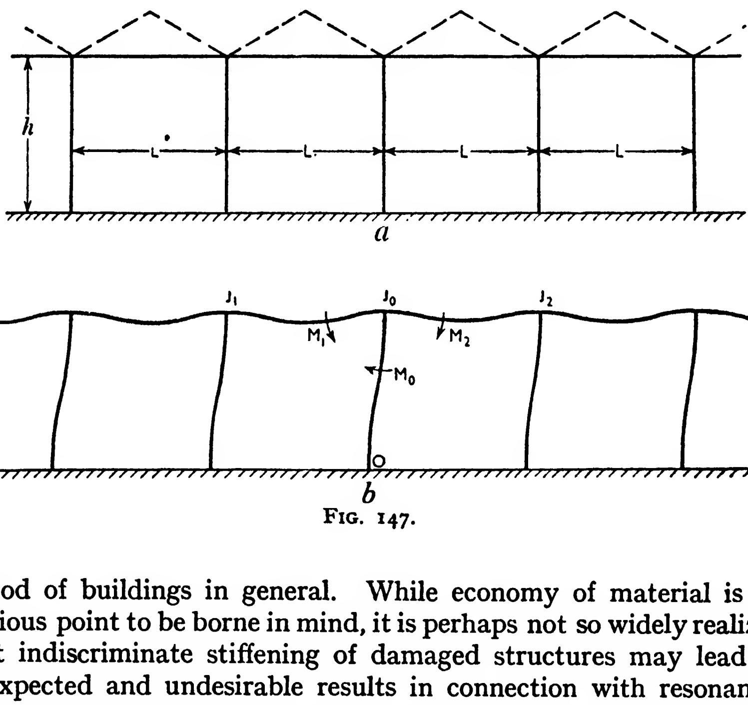


FIG. 147.

period of buildings in general. While economy of material is an obvious point to be borne in mind, it is perhaps not so widely realized that indiscriminate stiffening of damaged structures may lead to unexpected and undesirable results in connection with resonance.

Sufficient light for our present purpose may be thrown on the several questions just raised by finding the relative stiffnesses of the vertical and horizontal members which will give the minimum period in a normal mode of vibration of the structure illustrated by Fig. 147 (*a*).

The roof members may conveniently be combined with the horizontal members, as in this class of building the roof should always be of the lightest possible construction. To retain the symmetrical properties of the structural system, however, it will be assumed that the stiffening effect of the roof is negligibly small.

We may then regard Fig. 147 (*b*) as representing the configuration of the system when executing small vibrations in a normal mode about the equilibrium-position indicated by Fig. 147 (*a*), with each member rigidly fixed at both ends.

To distinguish the two classes of members, let m_1, E_1, I_1, h denote in turn the weight per unit length, the direct modulus of elasticity of the material, the proper moment of inertia of cross-section, and the length of any one of the *columns*; and let m_2, E_2, I_2 L denote the corresponding quantities for any one of the *horizontal beams*.

The effect of friction on the motion of such structures during the serious phase of a shock is, in the opinion of the author, a matter for experimental investigation at the present time. We shall, nevertheless, neglect any dissipative forces which may operate, and suppose the structure to execute transverse vibrations under the influence of gravity alone.

If the system consists of an indefinite number of columns and beams arranged in, and at right angles to, the plane of the paper, in conformity with the figure, consideration of any one of the columns and its beams will yield the necessary and sufficient equations for most practical purposes. This is so because the lack of continuity at the outermost columns will not then greatly affect the vibratory motion of an intermediate column.

To proceed on these suppositions, let us confine our attention to the column OJ_0 together with the rigidly connected beams $J_1J_0J_2$, and refer the motion to the origin O and the usual axes Ox, Oy.

From equation (86.7) we have

$$\frac{\partial^4 x}{\partial y^4} = -\frac{m_1}{E_1 g I_1}\frac{\partial^2 x}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad (92.1)$$

for the column, and

$$\frac{\partial^4 y}{\partial x^4} = -\frac{m_2}{E_2 g I_2}\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad (92.2)$$

for the beam J_0J_1 or J_0J_2.

Now, on writing

$$\left.\begin{aligned} u &= A_1 \cos \alpha_1 y + B_1 \sin \alpha_1 y + C_1 \cosh \alpha_1 y + D_1 \sinh \alpha_1 y, \\ v &= A_2 \cos \alpha_2 x + B_2 \sin \alpha_2 x + C_2 \cosh \alpha_2 x + D_2 \sinh \alpha_2 x, \end{aligned}\right\} \quad . \quad (92.3)$$

it follows from equation (86.11) that in a normal mode

$$x = u \cos pt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92.4)$$

$$y = v \cos pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92.5)$$

are the solutions of equations (92.1) and (92.2), where the arbitrary constants A_1, A_2, . . ., D_2 depend on the end-conditions.

By way of evaluating these constants, consider first the column OJ_0. In virtue of the conditions at the origin O, namely

$$u = 0, \ \frac{du}{dy} = 0 \text{ at the position } y = 0,$$

we deduce from the first of equations (92.3) the relations $C_1 = -A_1$ and $D_1 = -B_1$, so that

$$u = A_1(\cos\alpha_1 y - \cosh\alpha_1 y) + B_1(\sin\alpha_1 y - \sinh\alpha_1 y) \quad . \quad . \quad (92.6)$$

Consideration of the shear indicates that this expression must satisfy the condition

$$\frac{d^3u}{dy^3} = -\frac{m_2Lp^2}{E_1gI_1}u \text{ at the position } y = h \quad . \quad . \quad (92.7)$$

Since at that place

$$u = A_1(\cos\alpha_1 h - \cosh\alpha_1 h) + B_1(\sin\alpha_1 h - \sinh\alpha_1 h),$$

$$\frac{d^3u}{dy^3} = \alpha_1^3\{A_1(\sin\alpha_1 h - \sinh\alpha_1 h) - B_1(\cos\alpha_1 h + \cosh\alpha_1 h)\},$$

equation (92.7) means that

$$A_1(\sin f_1 - \sinh f_1) - B_1(\cos f_1 + \cosh f_1)$$
$$= -f_2\{A_1(\cos f_1 - \cosh f_1) + B_1(\sin f_1 - \sinh f_1)\},$$

i.e. $A_1\{(\sin f_1 - \sinh f_1) + f_2(\cos f_1 - \cosh f_1)\}$

$$= B_1\{(\cos f_1 + \cosh f_1) - f_2(\sin f_1 - \sinh f_1)\}, \quad . \quad (92.8)$$

where $f_1 = \alpha_1 h$, $f_2 = \dfrac{m_2Lp^2}{\alpha_1^3E_1gI_1}$.

We also notice that the bending moment M_0, at the joint J_0, is defined by

$$M_0 = -E_1gI_1\frac{d^2u}{dy^2} \text{ at the position } y = h \quad . \quad (92.9)$$

Turning next to the horizontal members in question, and remembering that the column and beams remain mutually at right angles throughout the motion, we have, to the first approximation,

$$v = 0, \ \frac{dv}{dx} = \frac{du}{dy} \text{ at the position } x = 0, \ y = h.$$

According to equations (92.3), the first of these conditions implies that $C_2 = -A_2$, hence

$$v = A_2(\cos\alpha_2 x - \cosh\alpha_2 x) + B_2\sin\alpha_2 x + D_2\sinh\alpha_2 x;$$

and it is easily proved that this expression leads to

$$\alpha_2(B_2 + D_2) = \alpha_1(-A_1\sin f_1 + B_1\cos f_1 + C_1\sinh f_1 + D_1\cosh f_1)$$

when the second of the present conditions is imposed on the motion.

Combining this result with equation (92.6), we gather that

$$f_3(B_2 + D_2) = -A_1(\sin f_1 + \sinh f_1) + B_1(\cos f_1 - \cosh f_1), \quad (92.10)$$

where $f_3 = \dfrac{\alpha_2}{\alpha_1}$.

The horizontal beams are subject also to the conditions

$$v = 0, \ \frac{dv}{dx} = \frac{du}{dy} \text{ at the positions } x = L, \ y = h.$$

If the first of these relations be inserted in the second of equations (92.3), and account taken of the result $C_2 = -A_2$, the operations lead to

$$A_2 (\cos \alpha_2 L - \cosh \alpha_2 L) + B_2 \sin \alpha_2 L + D_2 \sinh \alpha_2 L = 0.$$

By means of the second of the present conditions and equations (92.3), we likewise obtain the related expression

$$\begin{aligned} &\alpha_2 \{ -A_2 (\sin \alpha_2 L + \sinh \alpha_2 L) + B_2 \cos \alpha_2 L + D_2 \cosh \alpha_2 L \} \\ &= \alpha_1 \{ -A_1 (\sin f_1 + \sinh f_1) + B_1 (\cos f_1 - \cosh f_1) \}. \end{aligned}$$

It will make for brevity in working if we now put $\alpha_2 L = f_4$, then the last pair of equations become

$$A_2 (\cos f_4 - \cosh f_4) + B_2 \sin f_4 + D_2 \sinh f_4 = 0, \quad . \quad . \quad (92.11)$$

$$\begin{aligned} &f_3 \{ -A_2 (\sin f_4 + \sinh f_4) + B_2 \cos f_4 + D_2 \cosh f_4 \} \\ &= -A_1 (\sin f_1 + \sinh f_1) + B_1 (\cos f_1 - \cosh f_1) \\ &= f_3 (B_2 + D_2), \end{aligned}$$

by equation (92.10). The latter relation may be expressed more concisely in the form

$$-A_2(\sin f_4 + \sinh f_4) + B_2(\cos f_4 - 1) + D_2(\cosh f_4 - 1) = 0,$$

or

$$A_2(\sin f_4 + \sinh f_4) + B_2(1 - \cos f_4) + D_2(1 - \cosh f_4) = 0 \, . \, (92.12)$$

It is manifest that the bending moments are further specified by

$$M_1 = -E_2 g I_2 \frac{d^2 v}{dx^2} \text{ at the position } x = L, \ y = h, \quad . \quad . \quad (92.13)$$

$$M_2 = -E_2 g I_2 \frac{d^2 v}{dx^2} \text{ at the position } x = 0, \ y = h, \quad . \quad . \quad (92.14)$$

and, from considerations of continuity,

$$M_2 + M_0 - M_1 = 0 \, . \quad . \quad . \quad . \quad . \quad (92.15)$$

What remains of the problem will consist of elimination, since we have exhausted the relations between the variables.

Starting the process by combining equations (92.9) and (92.6), we gather that

$$\begin{aligned} M_0 &= -E_1 g I_1 \frac{d^2 u}{dy^2} \\ &= \alpha_1^2 E_1 g I_1 \{ A_1(\cos f_1 + \cosh f_1) + B_1(\sin f_1 + \sinh f_1) \} \end{aligned}$$

at the position $x = 0, \ y = h$. A repetition of the operation with

equations (92.14) and (92.3) shows that, at the position $x = 0$, $y = h$,

$$M_2 = -E_2 g I_2 \frac{d^2 v}{dx^2}$$
$$= \alpha^2_2 E_2 g I_2 (A_2 - C_2)$$
$$= 2\alpha^2_2 E_2 g I_2 A_2,$$

since $C_2 = -A_2$. Equations (92.13) and (92.3) in a like manner give, for the position $x = L$, $y = h$,

$$M_1 = -E_2 g I_2 \frac{d^2 v}{dx^2}$$
$$= \alpha^2_2 E_2 g I_2 (A_2 \cos f_4 + B_2 \sin f_4 - C_2 \cosh f_4 - D_2 \sinh f_4)$$
$$= \alpha^2_2 E_2 g I_2 \{A_2(\cos f_4 + \cosh f_4) + B_2 \sin f_4 - D_2 \sinh f_4\}$$

since $C_2 = -A_2$.

To ensure that the result shall exhibit the moments of inertia I_1, I_2 in the form of a ratio, we put $\frac{\alpha^2_2 E_2 I_2}{\alpha^2_1 E_1 I_1} = f_5$ in the work of substituting these expressions for M_0, M_1, M_2 in equation (92.15), and so obtain

$$A_1(\cos f_1 + \cosh f_1) + B_1(\sin f_1 + \sinh f_1)$$
$$+ f_5\{A_2(\cos f_4 + \cosh f_4 - 2) + B_2 \sin f_4 - D_2 \sinh f_4\} = 0 \quad . (92.16)$$

It is clear that equations (92.8), (92.10), (92.11), (92.12), (92.16) now afford means of eliminating the constants A_1, B_1, A_2, B_2, D_2. On effecting this reduction, it will be found, finally, that

$$2f_5\{(\cos f_1 \sinh f_1 + \sin f_1 \cosh f_1) + f_2(\cos f_1 \cosh f_1 - 1)\}$$
$$\{\sin f_4(\cosh f_4 - 1) + \sinh f_4(1 - \cos f_4)\}$$
$$= f_3\{(\cos f_1 \cosh f_1 + 1) + f_2(\cos f_1 \sinh f_1 - \sin f_1 \cosh f_1)\}$$
$$(\cos f_4 \cosh f_4 - 1), \quad (92.17)$$

where, to repeat,

$$f_1 = \alpha_1 h,\ f_2 = \frac{m_2 L p^2}{\alpha^3_1 E_1 g I_1},\ f_3 = \frac{\alpha_2}{\alpha_1},\ f_4 = \alpha_2 L,\ f_5 = \frac{\alpha^2_2 E_2 I_2}{\alpha^2_1 E_1 I_1}.$$

Here α_1, α_2 represent the only unknown quantities for a specified structure, and this pair of terms fix the mode of vibration, in accordance with equation (86.10). In the fundamental mode, for example, $\alpha_1 = \alpha_2 = 1$.

If the same material is used throughout the system, then $E_1 = E_2$, and m_1, m_2 are proportional to the cross-sectional areas of the respective members.

Remembering that the fundamental mode of vibration of buildings in general is of primary importance in connection with earthquakes, light will be thrown on a practical problem if we consider the

case of $\alpha_1 = \alpha_2 = 1$, $E_1 = E_2 = E$ in equation (92.17). In these circumstances $\frac{2\pi}{p}$ is the fundamental period in a normal mode of vibration for the structure shown in Fig. 147.

Let k_1, k_2 denote the appropriate radii of gyration associated with I_1, I_2, respectively. The following Table shows the magnitudes of $\frac{2\pi}{p}\sqrt{\frac{E_1gI_1}{m_1h^4}}$ which are obtained by assigning the stated values to the quantities $\frac{h}{L}, \frac{I_1k^2_2}{I_2k^2_1}$ in equation (92.17), where for brevity we have written R for $\sqrt{\frac{E_1gI_1}{m_1h^4}}$.

VALUES OF $\frac{2\pi}{p}R$.

$\frac{h}{L}$ \ $\frac{I_1k_2^2}{I_2k_1^2}$	$\frac{1}{2}$	1	2	3	5	7	9	10
$\frac{1}{2}$ . .	12·48	11·02	10·47	11·24	12·90	13·13	13·14	13·15
1 . .	10·79	9·759	9·515	10·40	11·91	12·27	12·34	12·35

This information has been used to plot the graphs shown in Fig. 148, whence we see that the minimum period for a given value

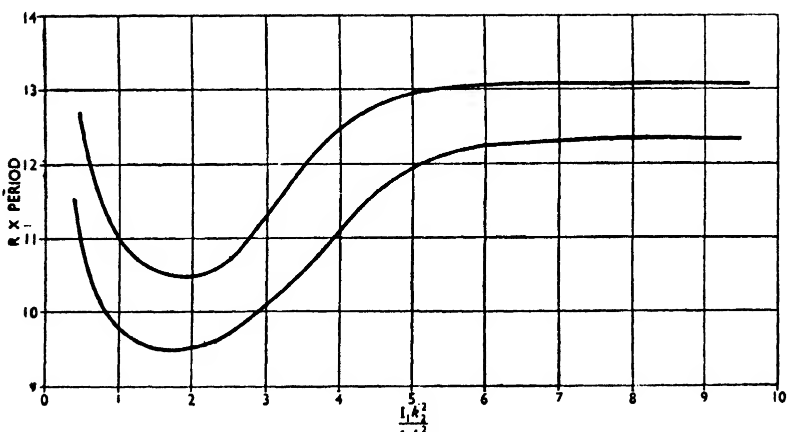


FIG. 148.

of $\frac{h}{L}$ is here influenced solely by the *ratio* of the stiffnesses of the columns and beams. It will be noticed from these graphs that the mere addition of material to a damaged building does not

necessarily increase the degree of safety as regards resonance with a probable earthquake.

Although framed structures with rigid joints are generally characterized by graphs of a similar form, each system must be examined on its merits because of the several variables which are thus seen to affect the relative position of the minimal point on the graph for a given mode of vibration with a building having a given number of storeys. Moreover, the approximate method of Art. 89 suffices to indicate that the effect of the vertical thrust on the columns will be different for different storeys.

Furthermore, there is no reason for the supposition that a normal mode of vibration will invariably result from an earthquake. This type of motion would, strictly speaking, occur only if the disturbance in the earth were propagated in a direction parallel to the plane of the paper in the case of the symmetrical system represented by Fig. 147. With a building having an unsymmetrical plan, involving wings and annexes, the consequences of a serious shock approaching the structure from any direction might well lead to failure of the joints between the main building and annexes, owing to the relative movement of these parts about a vertical axis. hence the shape of a building in plan is also a factor in the general question of design.

It is of practical interest to notice that equation (92.17) has applications in the work of designing *foundations* for tanks and machinery. The supporting structure of an elevated reservoir may correspond to the case where the ratio $E_2I_2{:}E_1I_1$ is very large, for which equation (92.17) gives a period of vibration that is practically the same as was found in Ex. 3 of Art. 86. If, on the contrary, the ratio $E_2I_2{:}E_1I_1$ is very small, the period of vibration of the system shown in Fig. 147 approximates to that found in Ex. 2 of Art. 86. These limiting cases therefore relate to a beam with 'built-in' ends in the first place, and in the second to a beam with one end fixed and the other free.

93. Membranes. We now pass to an examination of the small vibrations of a tightly stretched membrane of elastic material having a uniform surface-density ρ, a good illustration of which is offered by a drumhead. This is an instructive problem of motion which involves two variables.

If a line is drawn in any direction on the surface of the membrane, the mutual forces acting between the two parts thus marked out will be sensibly at right-angles to the line, and the magnitude of the force will be independent of the direction, being proportional to the length of the line. The force per unit length of the line is the *tension*, and it will be denoted by P. We also notice that, for slight displacements about the position of rest, the value of P will remain

practically constant if, as will be supposed, the tension is great. To the same degree of accuracy the squares of the slope at any point on the disturbed surface may be neglected as of the second order of small quantities.

Let Fig. 149 represent a rectangular element of the membrane when at rest in the plane of the paper, the corners A, B, C, D being

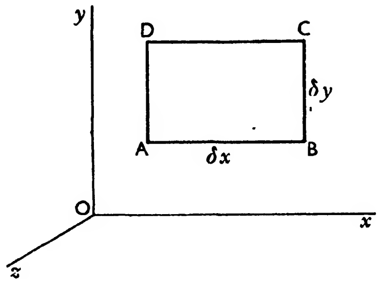


FIG. 149.

situated at the points $(x,\ y)$, $(x + \delta x,\ y)$, $(x + \delta x,\ y + \delta y)$, $(x,\ y + \delta y)$, respectively.

In this notation, with the vibrations taking place in a direction parallel to the z-axis, the force acting across the line AD is equal to $-P\delta y$, and its z-component is therefore $-P\dfrac{\partial z}{\partial x}\delta y$. Similarly, the z-component of the force across the line BC is

$$P\left(\frac{\partial z}{\partial x}\right)_{x+\delta x}\delta y = P\left(\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2}\delta x\right)\delta y.$$

It follows that the sum of these forces on the element is equivalent to

$$P\frac{\partial^2 z}{\partial x^2}\delta x\delta y.$$

By a similar argument we obtain

$$P\frac{\partial^2 z}{\partial y^2}\delta x\delta y$$

for the sum of the z-components of the forces on AD and BC.

Hence, since the product of mass and acceleration is equal to the total force,

$$\frac{\rho}{g}\frac{\partial^2 z}{\partial t^2}\delta x\delta y = P\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\delta x\delta y,$$

i.e.

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (93.1)$$

where $a = \sqrt{\dfrac{Pg}{\rho}}$.

A comparison of this result with equation (66.1) at once shows that a membrane is a two-dimensional generalization of a slender wire. It is consequently legitimate to infer that an infinite number of natural modes will be associated with a membrane, and that

$$2T = \frac{\rho}{g}\iint \dot{z}^2 dx\, dy, \quad \ldots \ldots \ldots \quad (93.2)$$

$$2V = P\iint\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}dx\, dy \quad \ldots \quad (93.3)$$

will give the kinetic and potential energies of the membrane, provided the limits of integration cover the complete surface.

The application of these results may best be described with reference to rectangular and circular membranes.

(*a*) *Rectangular Membrane.* Let Fig. 150 illustrate a rectangular membrane with a fixed boundary along the lines $x = 0$, $x = b$,

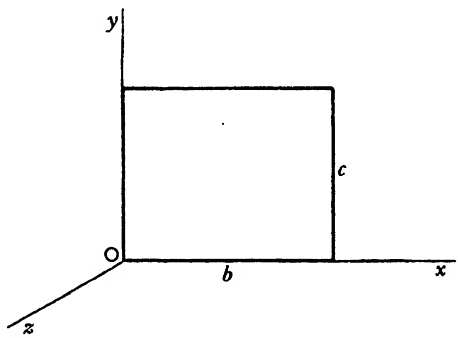


FIG. 150.

$y = 0$, $y = c$, and a plane of equilibrium coinciding with that of the paper.

Since equation (93.1) means that the displacement z at any point (x, y) on the membrane is a function of x, y, t, with x', y', t' representing quantities which are separately functions of x, y, t, so that

$$z = x'y't',$$

we have, after differentiating,

$$\frac{\partial^2 z}{\partial t^2} = x'y'\frac{d^2t'}{dt^2}, \quad \frac{\partial^2 z}{\partial x^2} = y't'\frac{d^2x'}{dx^2}, \quad \frac{\partial^2 z}{\partial y^2} = x't'\frac{d^2y'}{dy^2}.$$

On substituting these expressions in (93.1) and dividing throughout by $x'y't'$, it will be found without difficulty that

$$\frac{\dfrac{d^2t'}{dt^2}}{t'} = a^2\left(\frac{\dfrac{d^2x'}{dx^2}}{x'} + \frac{\dfrac{d^2y'}{dy^2}}{y'}\right) \quad \ldots \ldots \quad (93.4)$$

As the right-hand members are, by definition, independent of t,

it follows that the left-hand member also must be independent of t, therefore it must be constant, and we may put

$$\frac{d^2t'}{dt^2} = -p^2t',$$

provided p^2 is a constant.

A similar line of reasoning enables us to write

$$\frac{d^2x'}{dx^2} = -r^2x', \ \frac{d^2y'}{dy^2} = -s^2y',$$

on the understanding that r^2 and s^2 are constants.

The quantities denoted by p, r, s evidently must satisfy the relation

$$p^2 = a^2(r^2 + s^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (93.5)$$

this being the result of making the substitutions in equation (93.4).

Solving the equation for x', we find

$$x' = A'e^{irx} + B'e^{-irx},$$

where A', B' are constants, and $i = \sqrt{-1}$. Now it is easily proved that the boundary-condition $x' = 0$ when $x = 0$ is fulfilled if $B' = -A'$, hence the expression reduces to

$$x' = A'(e^{irx} - e^{-irx}).$$

This equation must satisfy the boundary-condition $x' = 0$ when $x = b$, and a simple substitution suffices to show that it will do so only if r is of the form $\frac{m\pi}{b}$, where m is an integer. Therefore

$$x' = C \sin \frac{m\pi x}{b},$$

where C is a constant.

The same method of analysis may be used to establish the related expressions

$$y' = D \sin \frac{n\pi y}{c}, \ t' = A'' \cos pt + B'' \sin pt,$$

where D, A'', B'' are constants, and n is an integer.

Inserting these relations in $z = x'y't'$, we conclude that one solution of equation (93.1) has the form

$$z = (A \cos pt + B \sin pt) \sin \frac{m\pi}{b}x \sin \frac{n\pi}{c}y, \quad . \quad . \quad . \quad (93.6)$$

where A, B are arbitrary constants, and

$$p^2 = \pi^2a^2\left(\frac{m^2}{b^2} + \frac{n^2}{c^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (93.7)$$

Our result relates to a normal mode of vibration with period

$$\frac{2\pi}{p} = \frac{2}{a}\left(\frac{m^2}{b^2} + \frac{n^2}{c^2}\right)^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (93.8)$$

the symbol a being specified in equation (93.1).

The most general expression for the displacement z is in this way seen to be

$$z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{m,n}\cos pt + B_{m,n}\sin pt)\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y, \quad . \quad . \quad (93.9)$$

where the constants $A_{m,n}$, $B_{m,n}$ depend on the initial circumstances of the motion.

For example, if the membrane starts from rest, all the B-terms must vanish, because $\dot{z} = 0$ when $t = 0$; and if, concurrently, the initial displacement can be represented by $z = f(x, y)$, then, by equation (93.9),

$$z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{m,n}\cos pt\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y, \quad . \quad . \quad (93.10)$$

with p given by equation (93.8). Now

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{m,n}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y$$

when $t = 0$, so that the constant is determined by

$$A_{m,n} = \frac{4}{bc}\int_0^b\int_0^c f(u, v)\sin\frac{m\pi}{b}u\sin\frac{n\pi}{c}v\,du\,dv,$$

where u, v denote the corresponding displacements of the point (x, y).

Furthermore, on solving the equation given by making $z = 0$ in equation (93.6), it will be found that in a normal mode the stationary points on the membrane lie on the *nodal lines* specified by the pairs of values

$$x = \frac{b}{m}, \frac{2b}{m}, \ldots, \frac{(m-1)b}{m},$$

$$y = \frac{c}{n}, \frac{2c}{n}, \ldots, \frac{(n-1)c}{n}.$$

These nodal lines accordingly divide the membrane into mn equal parts, in each of which the numerical value of z is repeated.

But the nodal lines are not necessarily straight lines for a combination of modes. To establish this point, suppose the membrane to be square, with $c = b$, and that it is vibrating in the combination of modes signified by $m = 1$, $n = 2$. Then, by equation (93.6), the displacement

$$z = \cos(pt + \varepsilon)\left(\sin\frac{2\pi}{b}x\sin\frac{\pi}{b}y + \lambda\sin\frac{\pi}{b}x\sin\frac{2\pi}{b}y\right)$$

$$= 2\cos(pt + \varepsilon)\sin\frac{\pi}{b}x\sin\frac{\pi}{b}y\left(\cos\frac{\pi}{b}x + \lambda\cos\frac{\pi}{b}y\right),$$

ε, λ being arbitrary constants. Thus, putting $z = 0$ for all values

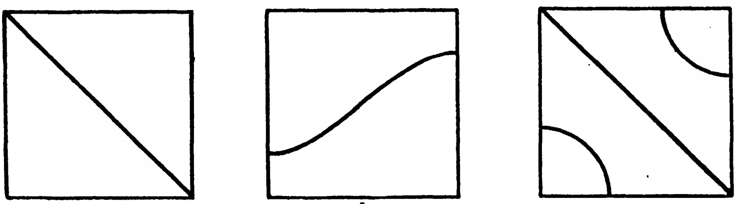
of t, the nodal lines must agree with the equation

$$\sin\frac{\pi}{b}x \sin\frac{\pi}{b}y\left(\cos\frac{\pi}{b}x + \lambda\cos\frac{\pi}{b}y\right) = 0.$$

This condition is fulfilled if either $\sin\frac{\pi}{b}x$ or $\sin\frac{\pi}{b}y$ is zero, and these values obviously refer to the nodal lines at the fixed boundary. The remaining nodal lines must, therefore, be given by

$$\cos\frac{\pi}{b}x + \lambda\cos\frac{\pi}{b}y = 0, \quad . \quad . \quad . \quad . \quad (93.11)$$

which represents a curved line. The nodal patterns corresponding to $\lambda = 1$ and $\lambda = -2$ are exhibited in succession by Figs. 151 (*a*) and Fig. 151 (*b*), showing that the nodal lines then pass through



a *b* *c*

FIG. 151.

the centre of the membrane. But this is not always so, for if $m = 2, n = 3, \lambda = 1$, according to equation (93.11) the nodal pattern is as shown in Fig. 151 (*c*).

(*b*) *Circular Membrane.* Let us next consider the transverse vibration of the circular membrane illustrated by Fig. 152, with a fixed boundary of radius R.

It is naturally most convenient to express the motion in terms of the polar co-ordinates r, θ indicated in the figure, by writing

$$x = r\cos\theta,\ y = r\sin\theta$$

in equation (93.1), whence

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right), \quad . \quad . \quad . \quad (93.12)$$

where, as before, $a = \sqrt{\dfrac{Pg}{\rho}}$.

If, in much the same manner as previously, we put $z = r'\theta't'$, where r', θ', t' are separately functions of r, θ, t only, and divide the resulting expression throughout by $r'\theta't'$, then equation (93.12) becomes

$$\frac{\dfrac{d^2t'}{dt^2}}{t'} = \frac{a^2}{r'}\left(\frac{d^2r'}{dr^2} + \frac{1}{r}\frac{dr'}{dr}\right) + \frac{a^2}{r^2\theta'}\frac{d^2\theta'}{d\theta^2} \quad . \quad . \quad (93.13)$$

Since the right-hand members are independent of t, the left-hand member also must be independent of t, hence the latter is a constant. For a normal mode we may further infer from the analysis of a rectangular membrane that here

$$t' = C \cos pt + D \sin pt,$$

and thus conclude that $-p^2$ is the constant in question. This means that

$$\frac{d^2t'}{dt^2} = -p^2t' \quad . \quad . \quad . \quad . \quad . \quad . \qquad (93.14)$$

Also, the last member of equation (93.13) is independent of θ,

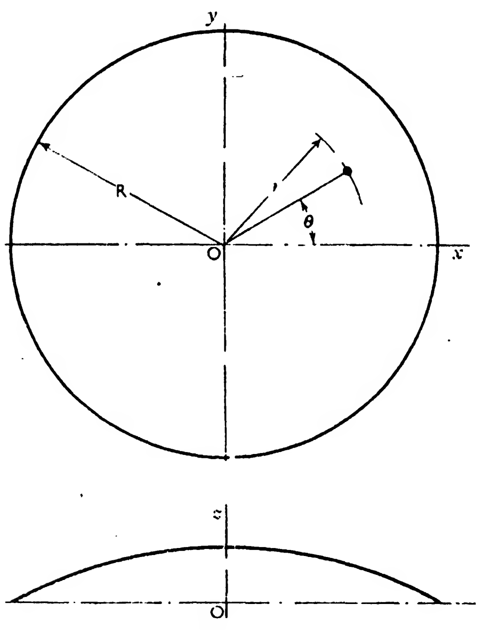


FIG. 152.

with the result that $\frac{1}{\theta'}\frac{d^2\theta'}{d\theta^2}$ must be a constant. But θ' is essentially periodic in θ, of period 2π, in consequence of which we can write

$$\frac{d^2\theta'}{d\theta^2} = -n^2\theta', \quad . \quad . \quad . \quad . \quad . \quad . \qquad (93.15)$$

provided n is an integer and

$$\theta' = A \cos n\theta + B \sin n\theta,$$

where A, B relate to arbitrary constants.

Now, inserting in equation (93.13) the expressions (93.14) and (93.15), we learn that

$$-p^2 = \frac{a^2}{r'}\left(\frac{d^2r'}{dr^2} + \frac{1}{r}\frac{dr'}{dr}\right) - \frac{a^2n^2}{r^2},$$

or

$$r^2\frac{d^2r'}{dr^2} + r\frac{dr'}{dr} + (k^2r^2 - n^2)r' = 0, \quad . \quad . \quad . \quad (93.16)$$

with $k = \frac{p}{a}$.

This is known as *Bessel's equation,* for a full examination of which the reader must refer to treatises on the subject.[1] It is sufficient to notice here that its solution is

$$r' = EJ_n(kr) + FY_n(kr),$$

where E, F represent arbitrary constants, and $J_n(kr)$, $Y_n(kr)$ are called *Bessel's functions* of the first and second kinds of order n, respectively. Hence

$$z = (A\cos n\theta + B\sin n\theta)(C\cos pt + D\sin pt)\{EJ_n(kr) + FY_n(kr)\} \quad . \quad . \quad (93.17)$$

is a solution of equation (93.12), n being an integer, and $p = ka$.

It follows from the theory of these functions that F must be zero for a membrane having a fixed boundary at a circle of radius $r = R$, because $Y_n(kr)$ is infinitely great when $r = 0$; and $J_n(kR)$ must vanish for all values of θ and t, on account of the fixed boundary. The membrane consequently deflects according to the law

$$z = (A\cos n\theta + B\sin n\theta)(C\cos pt + D\sin pt)J_n(kr), \quad . \quad (93.18)$$

where $J_n(kR) = 0$.

In the special case where the vibrations are symmetrical in relation to the centre of the membrane, and therefore independent of θ, n must be zero in the last equation, whence we have

$$z = (C\cos pt + D\sin pt)J_0(kr), \quad . \quad . \quad . \quad (93.19)$$

together with $p = ka$, and $J_0(kR) = 0$.

For readers unfamiliar with Bessel's functions it may be worth while to remark at this stage that, with positive values of the integer n, the general expression for

$$J_n(x) = \frac{(\frac{1}{2}x)^n}{n!}\left\{1 - \frac{(\frac{1}{2}x)^2}{1(n+1)} + \frac{(\frac{1}{2}x)^4}{1.2(n+1)(n+2)} - \ldots\right\},$$

since by the aid of this formula we can at once write down

$$J_0(x) = 1 - (\tfrac{1}{2}x)^2 + \frac{(\frac{1}{2}x)^4}{2^2} - \frac{(\frac{1}{2}x)^6}{2^2.3^2} + \ldots,$$

$$J_1(x) = \tfrac{1}{2}x - \frac{(\frac{1}{2}x)^3}{2} + \frac{(\frac{1}{2}x)^5}{2^2.3} - \ldots,$$

and so on. Such series might be used to evaluate the positive

[1] G. N. Watson, *Treatise on the Theory of Bessel Functions.*

zeros of $J_0(x)$. $J_1(x)$, . . . by plotting graphs of the functions in a manner which will be understood from Fig. 153. The procedure is, however, unnecessary in so far as tabulated values of these zeros are available,[1] and Table 4 contains a sufficient number of them for many practical purposes.

The graphs exhibit in a general way two characteristics which deserve mention in connection with certain problems:

(i) $J_0(x)$ is sensibly equal to $\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)$ for large values of x;

(ii) the positive roots of $J_0(x) = 0$ and of $J_1(x) = 0$ are spaced approximately π units apart.

Returning to the main problem, if k_1, k_2, k_3, . . . represent in

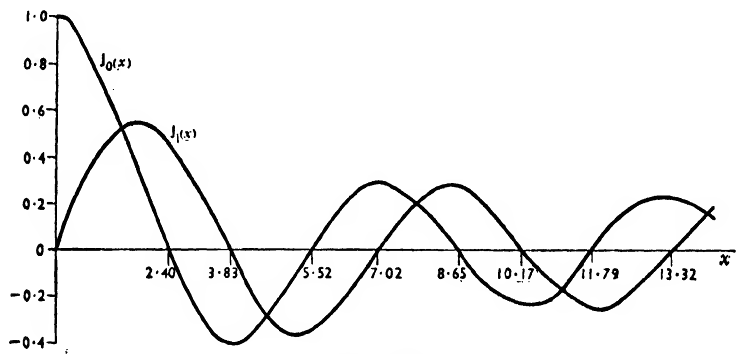


FIG. 153.

succession the positive zeros of $J_0(kR)$, we now see that the normal vibrations of the circular membrane are determined by substituting the proper values for k in equation (93.19). Thus the mth mode will involve, besides the boundary of radius $r = R$, $(m - 1)$ nodal circles, because $J_0(k_m r)$ vanishes at the radii

$$r = \frac{k_1 R}{k_m}, \frac{k_2 R}{k_m}, \frac{k_3 R}{k_m}, \ldots, \frac{k_{m-1} R}{k_m} \quad . \quad . \quad . \quad (93.20)$$

If the membrane starts from rest, then in equation (93.19) $\frac{\partial z}{\partial t} = 0$ initially, hence D must vanish; and if, in addition, the initial displacement is symmetrical about the centre of the surface, so that $z = f(r)$, we can write

$$z = \sum_{m=1}^{\infty} C_m \cos(p_m t) J_0(k_m r), \quad . \quad . \quad . \quad . \quad (93.21)$$

with
$$f(r) = \sum_{m=1}^{\infty} C_m J_0(k_m r) . \quad . \quad . \quad . \quad . \quad (93.22)$$

when $t = 0$, and $p_m = a k_m$.

[1] *Brit. Assoc. Report for 1922*, page 271.

TABLE 4

Positive Zeros of $J_n(x)$.

m \ n	0	1	2	3	4	5	6	7	8	9	10
1	2·405	3·832	5·136	6·380	7·588	8·772	9·936	11·086	12·225	13·354	14·476
2	5·520	7·016	8·417	9·761	11·065	12·339	13·589	14·821	16·038	17·241	18·433
3	8·654	10·173	11·620	13·015	14·372	15·700	17·004	18·288	19·554	20·807	22·047
4	11·792	13·324	14·796	16·223	17·616	18·980	20·321	21·642	22·945	24·234	25·509
5	14·931	16·471	17·960	19·409	20·827	22·218	23·586	24·935	26·267	27·584	28·887
6	18·071	19·616	21·117	22·583	24·019	25·430	26·820	28·191	29·546	30·885	32·212
7	21·212	22·760	24·270	25·748	27·199	28·627	30·034	31·423	32·796	34·154	35·500
8	24·352	25·904	27·420	28·908	30·371	31·812	33·233	34·637	36·026	37·400	38·762
9	27·494	29·047	30·569	32·065	33·537	34·989	36·422	37·839	39·240	40·628	42·004
10	30·635	32·190	33·716	35·219	36·699	38·160	39·603	41·031	42·444	43·844	45·232

The mode of vibration under consideration is accordingly executed with a period $\frac{2\pi}{p_m}$, where

$$p_m = \frac{1}{R}\sqrt{\frac{Pg}{\rho}}\{\text{corresponding zero of } J_0(k_m R)\} \quad . \quad (93.23)$$

For example, in the normal mode specified by $n = 0$, $m = 1$, it is seen from Table 4 that the period $\frac{2\pi}{p}$ is defined by

$$p = \frac{2 \cdot 405}{R}\sqrt{\frac{Pg}{\rho}},$$

and that the fixed boundary constitutes the only nodal circle.

In the next mode, corresponding to the values $n = 0$, $m = 2$, we likewise find

$$p = \frac{5 \cdot 520}{R}\sqrt{\frac{Pg}{\rho}},$$

and ascertain, from (93.20), that there will be an intermediate nodal circle at the radius

$$\frac{k_1}{k_2}R = \frac{2 \cdot 405}{5 \cdot 520}R = 0 \cdot 436R.$$

For the succeeding mode in which $n = 0$, $m = 3$, we also see that

$$p = \frac{8 \cdot 654}{R}\sqrt{\frac{Pg}{\rho}},$$

and that there are nodal circles at the radii

$$\frac{2 \cdot 405}{8 \cdot 654}R \text{ and } \frac{5 \cdot 520}{8 \cdot 654}R, \text{ i.e., } 0 \cdot 278R \text{ and } 0 \cdot 638R,$$

as well as at the fixed boundary.

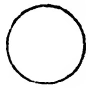
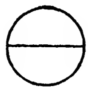
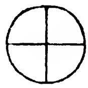
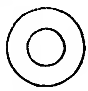
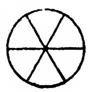
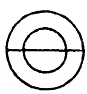
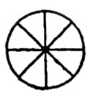
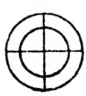
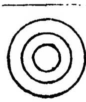
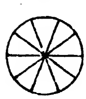
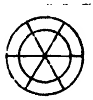
Similarly, for the combination of modes $n = 2$, $m = 1$ Table 4 gives

$$p = \frac{5 \cdot 136}{R}\sqrt{\frac{Pg}{\rho}}.$$

By substituting this value for p_m in equation (93.21) and equating the resulting expression to zero, it will be seen that the nodes consist of two diametral lines at right angles, as well as the fixed boundary.

Proceeding in this manner to the higher modes, we obtain results which are summarized in Table 5, where it will be noticed that the symbols m and n represent in turn the number of nodal *circles* and of nodal *diameters*. This correspondence enables us to ascertain, from a knowledge of m and n only, the nodal pattern in a symmetrical mode of vibration.

TABLE 5

Root of $J_n(x)$.		Nodal Pattern.	
2·405	$n = 0$ $m = 1$		Circle at boundary of radius R.
3·832	$n = 1$ $m = 1$		One diameter, and circle at boundary.
5·136	$n = 2$ $m = 1$		Two diameters, and circle at boundary
5·520	$n = 0$ $m = 2$		Two circles of radii 0·436R, and R.
6·380	$n = 3$ $m = 1$		Three diameters, and circle at boundary.
7·016	$n = 1$ $m = 2$		One diameter, and two circles of radii 0·546R, and R.
7·588	$n = 4$ $m = 1$		Four diameters, and circle at boundary.
8·417	$n = 2$ $m = 2$		Two diameters, and two circles of radii 0·601R, and R.
8·654	$n = 0$ $m = 3$		Three circles of radii 0·278R, 0·638R, and R.
8·772	$n = 5$ $m = 1$		Five diameters, and circle at boundary.
9·761	$n = 3$ $m = 2$		Three diameters, and two circles of radii 0·654R, and R.

The analysis necessarily becomes more complicated when a membrane of circular form describes unsymmetrical vibrations. If we

suppose the membrane to start from rest when $t = 0$, and put $z = f(r, \theta)$ initially, then our line of argument leads to

$$z = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta)\cos(p_m t)J_n(k_m r), \quad (93.24)$$

where the positive zeros of $J_n(kR)$ are denoted by $k_1, k_2, k_3 \ldots$, in succession, and $p_m = ak_m$. Also

$$f(r, \theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta)J_n(k_m r) \quad . \quad (93.25)$$

when $t = 0$, and it can be proved that this implies the relation

$$A_{n,m} = \frac{2\int_0^{2\pi}\int_0^{R} f(r, \theta)J_n(k_m r)rdr\cos n\theta d\theta}{\pi R^2\{J_n'(k_m R)\}^2}, \quad . \quad . \quad (93.26)$$

where $J_n'(k_m r) = \frac{d}{dr}J_n(k_m r)$, and 2π is used instead of π when $n = 0$. The same formula gives the expression for $B_{n,m}$ if $\cos n\theta$ is replaced by $\sin n\theta$.

In the special case of an annular membrane bounded by circles of radii $r = R$ and $r = R_0$, where $R_0 < R$, we find

$$z = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta)\cos(p_m t)$$
$$\times\{J_n(k_m r)G_n(k_m R_0) - G_n(k_m r)J_n(k_m R_0)\}, \quad (93.27)$$

where $p_m = ak_m$, and $k_1, k_2, k_3, \ldots$ represent in turn the positive zeros of

$$J_n(kR)G_n(kR_0) - G_n(kR)J_n(kR_0)$$

when the expression is treated as a function of k. Here the symbol $G_n(x)$ refers, as is customary, to *Bessel's function of the second kind of order n.*

For membranes having boundaries of elliptical, triangular, and other shapes the reader may be referred to the investigations of Lord Rayleigh.[1]

The present problem offers an opportunity of extending the approximate method implied in Art. 55, in a manner which will be understood from the following example.

Ex. Calculate the fundamental period of small vibrations for a circular membrane of surface-density ρ, clamped at a boundary of radius R, and subjected to a uniform tension P.

Taking the symmetrical displacements as being parallel to the z-axis, we have, from equations (93.2) and (93.3),

$$\left.\begin{aligned} 2T &= \frac{2\pi\rho}{g}p^2\int_0^R z^2 rdr, \\ 2V &= 2\pi P\int_0^R\left(\frac{\partial z}{\partial r}\right)^2 rdr, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (93.28)$$

[1] *Theory of Sound,* vol. I, page 343.

for the kinetic and potential energies, with $\frac{2\pi}{p}$ equal to the period of vibration.

Since the measure of success attainable by this method depends on our knowledge of the shape of a disturbed membrane, it is to be noted that the related problem of thin flat plates having circular boundaries has been investigated by S. D. Poisson.[1] His expressions for the 'static' displacements contain the factor $(R^2 - r^2)^2$ when the load is uniformly distributed and the plate is either simply supported or clamped at the edge. Furthermore, the same investigator, as well as B. de Saint-Venant,[2] obtained equations for z which involve the factors $(R^2 - r^2)$ and $r^2 \log \frac{R}{r}$ in cases where the load is concentrated at the centre of a circular surface or is uniformly distributed over an annular surface, with both types of support.

In view of these results it is legitimate to assume

$$z = A\left(1 - \frac{r^2}{R^2}\right)$$

as a first approximation to the shape of the disturbed membrane, where A is an arbitrary constant.

On this supposition we obtain, from equations (93.28),

$$\begin{aligned} 2T &= \frac{2\pi\rho}{g} A^2 p^2 \int_0^R \left(r - 2\frac{r^3}{R^2} + \frac{r^5}{R^4}\right) dr \\ &= \frac{\pi\rho}{3g} A^2 R^2 p^2, \\ 2V &= 8\pi A^2 P \int_0^R \frac{r^3}{R^4} dr \\ &= 2\pi A^2 P. \end{aligned}$$

Since our method involves the assumption $T = V$, it follows that

$$p^2 = \frac{6Pg}{\rho R^2},$$

whence the period

$$\frac{2\pi}{p} = 2{\cdot}5649 R \sqrt{\frac{\rho}{Pg}},$$

to the present degree of approximation.

This result will be improved if we next assume the shape to be

$$z = A\left(1 - \frac{r^2}{R^2}\right)\left(1 + \beta\frac{r^2}{R^2}\right), \quad . \quad . \quad . \quad . \quad (93{\cdot}29)$$

and adjust β so as to make the resulting period a maximum, in

[1] *Mém. de l'Acad. de Paris*, vol. 8, page 357 (1829).
[2] *'Annotated Clebsch'*, note du § 45 (1883).

accordance with the treatment of Art. 55, where it was demonstrated that in a normal mode p is stationary for slight variations in the value of β.

Now

$$z^2r = A^2\left(r - 2\frac{r^3}{R^2} + \frac{r^5}{R^4} + 2\beta\frac{r^3}{R^2} - 4\beta\frac{r^5}{R^4} + 2\beta\frac{r^7}{R^6} + \beta^2\frac{r^5}{R^4} - 2\beta^2\frac{r^7}{R^6} + \beta^2\frac{r^9}{R^8}\right),$$

$$\left(\frac{\partial z}{\partial r}\right)^2 r = \frac{4A^2}{R^4}\left(r^3 - 2\beta r^3 + 4\beta\frac{r^5}{R^2} + \beta^2 r^3 - 4\beta^2\frac{r^5}{R^2} + 4\beta^2\frac{r^7}{R^4}\right),$$

so that, by equations (93·28),

$$2T = \frac{2\pi\rho}{g}p^2\int_0^R z^2r\,dr$$

$$= \frac{2\pi\rho}{g}A^2R^2p^2(\tfrac{1}{6} + \tfrac{1}{12}\beta + \tfrac{1}{60}\beta^2),$$

$$2V = 2\pi P\int_0^R\left(\frac{\partial z}{\partial r}\right)^2 r\,dr$$

$$= 8\pi A^2P(\tfrac{1}{4} + \tfrac{1}{6}\beta + \tfrac{1}{12}\beta^2).$$

Hence we deduce, on the supposition that $T = V$,

$$\frac{\rho}{g}R^2p^2(\tfrac{1}{3} + \tfrac{1}{6}\beta + \tfrac{1}{30}\beta^2) = 4P(\tfrac{1}{2} + \tfrac{1}{3}\beta + \tfrac{1}{6}\beta^2)$$

and so conclude that

$$p^2 = \frac{4Pg}{\rho R^2}\frac{(\tfrac{1}{2} + \tfrac{1}{3}\beta + \tfrac{1}{6}\beta^2)}{(\tfrac{1}{3} + \tfrac{1}{6}\beta + \tfrac{1}{30}\beta^2)} \quad . \quad . \quad . \quad (93{\cdot}30)$$

To secure a minimum value of p we must obviously have

$$\frac{\partial}{\partial\beta}\left(\frac{\tfrac{1}{2} + \tfrac{1}{3}\beta + \tfrac{1}{6}\beta^2}{\tfrac{1}{3} + \tfrac{1}{6}\beta + \tfrac{1}{30}\beta^2}\right) = 0.$$

Thus, after differentiating by parts and confining ourselves to the numerator of the resulting expression, it appears that the required condition will be fulfilled if

$$(\tfrac{1}{3} + \tfrac{1}{6}\beta + \tfrac{1}{30}\beta^2)(\tfrac{1}{3} + \tfrac{1}{3}\beta) - (\tfrac{1}{2} + \tfrac{1}{3}\beta + \tfrac{1}{6}\beta^2)(\tfrac{1}{6} + \tfrac{1}{15}\beta) = 0,$$

i.e. if $\qquad \beta^2 + \tfrac{14}{3}\beta + \tfrac{5}{3} = 0.$

The roots of this quadratic in β are $-0{\cdot}3897$ and $-4{\cdot}2270$, and it is plain that the first of these values is the one to be chosen for our present purpose. Consequently, with $\beta = -0{\cdot}3897$ in equation (93·30),

$$p^2 = 5{\cdot}7800\frac{Pg}{\rho R^2},$$

and therefore the more accurate value of the period

$$\frac{2\pi}{p} = 2{\cdot}6124R\sqrt{\frac{\rho}{Pg}}.$$

It may be noted, by way of indicating the degree of accuracy thus attained, that the exact expression for the period contains the factor 2·61253, showing that our final result is correct to the third decimal place.

94. Thin Flat Plates. This is perhaps the point at which we may profitably introduce the related, but more difficult, problem of a thin flat plate. The analysis will, however, be restricted to an approximate solution, for an exact determination is impracticable in the general case.

Bearing this in mind, let Fig. 154 represent the configuration of such a plate when slightly displaced from its position of equilibrium, which we take as being parallel to the plane xOy. The implication is that the amplitude of vibration is small compared with the thickness of the plate. It will, for simplicity, be supposed that the material is homogeneous, isotropic, and of uniform thickness $2h$.

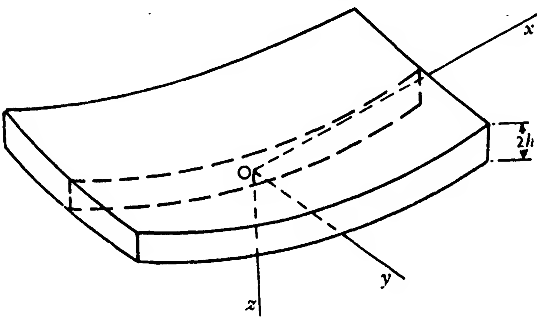


FIG. 154.

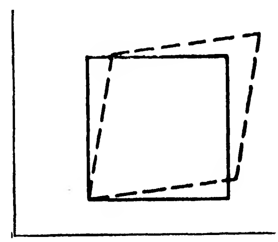


FIG. 155.

We have proved a membrane to be a two-dimensional generalization of a slender wire, and a similar argument may be used to demonstrate that a thin flat plate is a two-dimensional generalization of the beam implied in Art. 86. Let us, then, imagine a *middle plane* to exist in the given plate, comparable with the *neutral axis* of the theory of beams, so that no transverse displacements of points on the middle plane take place throughout the motion in question. We shall take a point on the middle plane as the origin O, and choose the rectangular axes Ox, Oy, Oz so that the principal radii of curvature lie in the planes xOz, yOz of the figure. To this degree of accuracy an initially plane section of the plate will remain plane, and the stress in any layer of the material will be directly proportional to the distance z of that layer from the middle plane.

To investigate the two-dimensional distribution of stress over the plate, we may consider the forces which act on an element distant z from the middle plane, and regard the full and broken lines in Fig. 155 as indicating the undisturbed and disturbed configurations of the element.

With the displacements of the element parallel to the x- and y-axes denoted in succession by u and v, we have, from the theory of structures,

$$\left.\begin{aligned} \text{stretch in the } x\text{-direction } e_{xx} &= \frac{\partial u}{\partial x}, \\ \text{stretch in the } y\text{-direction } e_{yy} &= \frac{\partial v}{\partial y}, \\ \text{slide or shear deformation } e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \end{aligned}\right\} \quad . \quad . \quad (94.1)$$

Furthermore, if the stress-components parallel to the x- and y-axes be X and Y, and the shear be S, it is known from the same source of reference that

$$\left.\begin{aligned} Eze_{xx} &= X - \frac{1}{m}Y, \\ Eze_{yy} &= Y - \frac{1}{m}X, \\ Nze_{xy} &= S, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (94.2)$$

where, as usual, E, $\frac{1}{m}$, N refer in turn to the direct modulus of elasticity, Poisson's ratio, and the shear modulus of the material. These quantities are therefore connected by the relation

$$N = \frac{mE}{2(m + 1)}.$$

A combination of equations (94.1) and (94.2) leads to the formulae

$$\left.\begin{aligned} X &= \frac{m^2}{m^2 - 1}Ez\left(\frac{\partial u}{\partial x} + \frac{1}{m}\frac{\partial v}{\partial y}\right), \\ Y &= \frac{m^2}{m^2 - 1}Ez\left(\frac{\partial v}{\partial y} + \frac{1}{m}\frac{\partial u}{\partial x}\right), \\ S &= \frac{m}{2(m + 1)}Ez\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (94.3)$$

Hence, if w signifies the displacement of the element in the z-direction of Fig. 154, so that

$$u = \frac{\partial w}{\partial x} \text{ and } v = \frac{\partial w}{\partial y}$$

in equations (94.3), then

$$\left.\begin{aligned} X &= \frac{m^2}{m^2 - 1}Ez\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{m}\frac{\partial^2 w}{\partial y^2}\right), \\ Y &= \frac{m^2}{m^2 - 1}Ez\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{m}\frac{\partial^2 w}{\partial x^2}\right), \\ S &= \frac{m}{m + 1}Ez\frac{\partial^2 w}{\partial x \partial y}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (94.4)$$

Confining ourselves to undamped motion, and writing ρ for the density of the material, we can, in view of the preceding analysis, express the kinetic energy, T, of the plate in the form

$$T = \frac{h\rho}{g}\iint\left(\frac{\partial w}{\partial t}\right)^2 dx\,dy, \quad . \quad . \quad . \quad . \quad (94.5)$$

provided the limits of integration are chosen so as to cover the complete surface. It is also clear that the potential energy of the element

$$\begin{aligned} dV &= \tfrac{1}{2}(e_{xx}X + e_{yy}Y + e_{xy}S)dx\,dy\,dz \\ &= \frac{m^2}{2(m^2 - 1)}Ez^2\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{2}{m}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right. \\ &\qquad\qquad \left. + \frac{2(m-1)}{m}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}dx\,dy\,dz, \end{aligned}$$

from equations (94.4), in virtue of which we have the strain energy of the complete plate specified by

$$\begin{aligned} V &= \frac{m^2 E}{m^2 - 1}\iint\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{2}{m}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right. \\ &\qquad\qquad \left. + \frac{2(m-1)}{m}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}dx\,dy\int_{-h}^{h} z^2 dz \\ &= D\iint\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{2}{m}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right. \\ &\qquad\qquad \left. + \frac{2(m-1)}{m}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}dx\,dy, \quad . \quad . \quad . \quad (94.6) \end{aligned}$$

where the *flexural rigidity* $D = \dfrac{2m^2Eh^3}{3(m^2 - 1)}$ and, as before, appropriate limits are to be assigned to the symbols of integration.

The present theory would only agree with fact if the constraints on the actual plate were modified in such a way as to secure the condition that the middle plane is free from transverse stretch. A consequence of this is that the frequency of vibration obtained from the above equations will, in general, be less than the true value for a specified plate, as is readily inferred from Art. 55. Calculations based on the above equations nevertheless give reasonably accurate results when the amplitude of vibration is less than about one-third the thickness of a given plate. The degree of accuracy has been investigated experimentally by J. H. Powell and J. H. T. Roberts,[1] who found, as was to be expected, that both the stretch which actually takes place in the middle plane and the external pressure on a plate contribute to the increase of frequency.

A more complete solution of the problem involves Bessel functions, as will appear in Ex. 2 of Art. 124, though for a full treatment

[1] *Proc. Phys. Soc., London*, vol. 35, page 170 (1923).

of the matter the reader must consult the work of Professor A. E. H. Love.[1]

(*a*) *Rectangular Plate.* To proceed by way of the stationary property of normal modes we shall suppose that the deflection-curve of a thin flat plate *simply supported* at its edge approximates to that of a membrane having a boundary of the same form. On this supposition it follows from equation (93.9) that the relation

$$w = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(A_{rs}\cos pt + B_{rs}\sin pt)\sin\frac{r\pi}{b}x\sin\frac{s\pi}{c}y$$

$$= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}C_{rs}\cos(pt+\varepsilon_r)\sin\frac{r\pi}{b}x\sin\frac{s\pi}{c}y \quad . \quad . \quad . \quad (94.7)$$

holds good at any point (x, y) on a rectangular plate with edges b, c measured along the x- and y-axes, respectively. This expression evidently satisfies the boundary-conditions $w = 0$, $\frac{\partial^2 w}{\partial x^2} = 0$, $\frac{\partial^2 w}{\partial y^2} = 0$.

If, for brevity in working, we put

$$C_{rs}\cos(pt+\varepsilon_r) = \theta_{rs}, \quad . \quad . \quad . \quad . \quad . \quad (94.8)$$

so that θ_{rs} is an explicit function of the time, then

$$w = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\theta_{rs}\sin\frac{r\pi}{b}x\sin\frac{s\pi}{c}y \quad . \quad . \quad . \quad . \quad (94.9)$$

Since in this notation

$$\left(\frac{\partial w}{\partial t}\right)^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\dot{\theta}_{rs}^2\sin^2\frac{r\pi}{b}x\sin^2\frac{s\pi}{c}y,$$

we ascertain from equation (94.5) that the kinetic energy of a plate of thickness $2h$ is

$$T = \frac{h\rho bc}{g}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\dot{\theta}_{rs}^2,$$

remembering that θ_{rs} is a function of the time alone. Similarly, a combination of the relation for θ_{rs} and equation (94.6) leads to

$$V = \pi^4 bcD\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\theta_{rs}^2\left(\frac{r^2}{b^2}+\frac{s^2}{c^2}\right)^2$$

as the formula for the strain energy, where $D = \dfrac{2m^2Eh^3}{3(m^2-1)}$.

On reverting to the original symbolism in which

$$\theta_{rs} = C_{rs}\cos(pt+\varepsilon_r)$$

and, therefore,

$$\theta_{rs}^2 = C_{rs}^2\cos^2(pt+\varepsilon_r),\ \dot{\theta}_{rs}^2 = p^2C_{rs}^2\sin^2(pt+\varepsilon_r),$$

it is seen that

$$T_{\max.} = \frac{h\rho bc}{g}C_{rs}^2p^2,\ V_{\max.} = \pi^4 bcDC_{rs}^2\left(\frac{r^2}{b^2}+\frac{s^2}{c^2}\right)^2.$$

[1] *Mathematical Theory of Elasticity*, page 497, fourth edition.

As these quantities are equal, in the implied absence of friction, we thus obtain

$$p^2 = \frac{\pi^4 Dg}{h\rho}\left(\frac{r^2}{b^2} + \frac{s^2}{c^2}\right)^2,$$

and so learn that the value of p involved in the expression for frequency $\left(\frac{p}{2\pi}\right)$ in a normal mode of vibration is approximately given by

$$p = \pi^2\left(\frac{r^2}{b^2} + \frac{s^2}{c^2}\right)\sqrt{\frac{Dg}{h\rho}}.$$

The corresponding displacement w may be determined in a straightforward manner by inserting this value for p in equation (94.7), the various constants being evaluated with the aid of the conditions at the boundary and at the instant $t = 0$. The nodal pattern is, as already explained with reference to membranes, defined by the roots of the equation which is then given by making $w = 0$.

In the case of a square plate, with $c = b$, it appears that the displacement

$$w = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} C_{rs} \cos(pt + \varepsilon_r) \sin\frac{r\pi}{b}x \sin\frac{s\pi}{b}y,$$

where $$p = \frac{\pi^2}{b^2}(r^2 + s^2)\sqrt{\frac{Dg}{h\rho}}.$$

The fundamental mode, corresponding to the values $r = s = 1$, accordingly involves the motion

$$w = C\cos(pt + \varepsilon)\sin\frac{\pi}{b}x \sin\frac{\pi}{b}y,$$

and a frequency

$$\frac{p}{2\pi} = \frac{\pi}{b^2}\sqrt{\frac{Dg}{h\rho}}.$$

It may be remarked that W. Voigt [1] has considered the case of a rectangular plate with two opposite edges supported, and the question has been discussed also by B. de Saint-Venant.[2] Approximate methods of solution for plates of the same shape, but supported in different ways, have been devised by W. Ritz, and by Lord Rayleigh, in a manner which is described in the references appended to Art. 55. In more recent years the same problem has been examined by C. G. Knott,[3] M. Paschoud,[4] and S. Timoshenko.[5]

[1] *Göttinger Nachrichten a. d. Jahre 1893*, page 225.
[2] '*Annotated Clebsch*', note du § 73 (Paris, 1883).
[3] *Proc. Roy. Soc., Edin.*, vol. 32, page 390 (1912).
[4] Mentioned in *Ann. des Ponts et Chaussées*, vol. 31, page 319 (1916).
[5] *Proc. Lond. Math. Soc.*, vol. 20, page 389 (1922), and *Phil. Mag.*, vol. 47, page 1095 (1924).

An instructive investigation into the effect of viscous resistance on the gravest mode of vibration for a square plate has been undertaken by H. Lamb,[1] who found, with an edge of length b,

$$p = \frac{10{\cdot}21}{b^2(1+\alpha)^{\frac{1}{2}}}\sqrt{\frac{Dg}{h\rho}},$$

where $\alpha = 0{\cdot}6689\frac{b\rho_0}{h\rho}$, and $\frac{\rho_0}{\rho} = \frac{\text{density of the fluid}}{\text{density of the plate}}$. The result disclosed a twofold effect of the fluid; the frequency is lowered due to the increased inertia, and a certain amount of energy is dissipated by the viscous resistance.

(*b*) *Circular Plate.* We may likewise arrive at an approximate solution for a circular plate, though the work is simplified by making use of the polar co-ordinates

$$x = r\cos\theta,\ y = r\sin\theta,$$

so that $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x}$.

Since
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x},$$

we have, by differentiation,

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial^2 r}{\partial x^2}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial^2 \theta}{\partial x^2}\frac{\partial w}{\partial \theta} + 2\frac{\partial^2 w}{\partial r\partial \theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x},$$

where $\frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \cos\theta$, $\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$,

and, in consequence,

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{\sin^2\theta}{r},\quad \frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} = 2\frac{\sin\theta\cos\theta}{r^2}.$$

That is to say,

$$\frac{\partial^2 w}{\partial x^2} = \cos^2\theta\frac{\partial^2 w}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial w}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial w}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 w}{\partial r\partial \theta}.$$

A similar argument leads to

$$\frac{\partial^2 w}{\partial y^2} = \sin^2\theta\frac{\partial^2 w}{\partial r^2} + \frac{\cos^2\theta}{r}\frac{\partial w}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial w}{\partial \theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 w}{\partial r\partial \theta},$$

$$\frac{\partial^2 w}{\partial x\partial y} = \sin\theta\cos\theta\frac{\partial^2 w}{\partial r^2} - \frac{\sin\theta\cos\theta}{r}\frac{\partial w}{\partial r} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{\cos 2\theta}{r^2}\frac{\partial w}{\partial \theta} + \frac{\cos 2\theta}{r}\frac{\partial^2 w}{\partial r\partial \theta}.$$

[1] *Proc. Roy. Soc.*, vol. 98, page 205 (1921).

Thus, on introducing these expressions into equations (94.5) and (94.6), it follows that, for a plate of radius R,

$$T = \frac{h\rho}{g}\int_0^{2\pi}\int_0^R\left(\frac{\partial w}{\partial t}\right)^2 r d\theta dr, \quad . \quad . \quad . \quad . \quad . \quad (94.10)$$

$$V = D\int_0^{2\pi}\int_0^R\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 - 2\frac{m-1}{m}\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right.$$
$$\left. + 2\frac{m-1}{m}\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\}^2\right] r d\theta dr \quad . \quad (94.11)$$

are the formulae for the kinetic and potential energies.

These results must be adjusted so as to agree with the initial and boundary conditions of a given plate. For example, if the plate starts from rest, then $\frac{\partial w}{\partial t} = 0$ initially; and if, in addition, the motion is restricted to symmetrical vibrations, then w is independent of θ and, in consequence, $\frac{\partial w}{\partial \theta}, \frac{\partial^2 w}{\partial \theta^2}$ both vanish.

If, to mention another case, the edge of the plate is rigidly clamped, we require both w and $\frac{\partial w}{\partial r}$ to be zero at the radius $r = R$. Under these conditions

$$T = \frac{h\rho}{g}\int_0^{2\pi}\int_0^R\left(\frac{\partial w}{\partial t}\right)^2 r d\theta dr,$$

$$V = D\int_0^{2\pi}\int_0^R\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 r d\theta dr;$$

and if the motion is restricted to symmetrical vibrations, then $\frac{\partial^2 w}{\partial \theta^2}$ is zero in these formulae, hence they reduce to

$$T = \frac{2\pi h\rho}{g}\int_0^R\left(\frac{dw}{dt}\right)^2 r dr, \quad . \quad . \quad . \quad . \quad . \quad (94.12)$$

$$V = 2\pi D\int_0^R\left(\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)^2 r dr, \quad . \quad . \quad . \quad (94.13)$$

where the symbols for ordinary differentiation imply that w is now a function of r only.

The symmetrical vibration of circular plates has been discussed by S. D. Poisson,[1] and G. Kirchhoff[2] has given a more general theory of the subject. Readers who are interested in the related question of elliptical plates may profitably study the work of E. Mathieu,[3] and of A. Barthélémy.[4]

[1] *Mém. de l'Acad. de Paris,* vol. 8, page 357 (1829).
[2] *Jour. für Math.* (*Crelle*), vol. 40, page 51 (1850).
[3] *Jour. de Math.* (*Liouville*), vol. 14, page 241 (1869).
[4] *Mém. de l'Acad. de Toulouse,* vol. 9, page 175 (1877).

Ex. 1. Find, with the help of equations (94.12) and (94.13), the fundamental period in a symmetrical mode of free vibration for a thin circular plate of uniform thickness $2h$, rigidly clamped at its boundary of radius $r = R$.

If, with the same notation, we assume

$$w = w_0 \cos pt$$

to be the shape of the disturbed plate, w_0 being a function of r alone, the investigations of S. D. Poisson and of B. de Saint-Venant, already referred to in the example of Art. 93, show that we may with sufficient accuracy write

$$w_0 = k_1\left(1 - \frac{r^2}{R^2}\right)^2 + k_2\left(1 - \frac{r^2}{R^2}\right)^3 + k_3\left(1 - \frac{r^2}{R^2}\right)^4 + \ldots, \quad (94.14)$$

where the coefficients k_1, k_2, k_3, . . . must have values such as will make the resulting frequency of vibration a minimum.

On taking maximum values in the process of equating (94.12) and (94.13), it is easily proved that the minimal condition will be fulfilled if

$$\frac{\partial}{\partial k_n}\int_0^R\left\{\left(\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right)^2 - \frac{h\rho p^2}{Dg}w_0^2\right\}rdr = 0 \quad . \quad . \quad (94.15)$$

with regard to the nth coefficient, $\dfrac{p}{2\pi}$ being the frequency in question.

If we proceed to a solution by the aid of the first term in the series (94.14), namely

$$w_0 = k^2{}_1\left(1 - \frac{r^2}{R^2}\right)^2,$$

then

$$w_0{}^2r = k^2{}_1\left(r - 4\frac{r^3}{R^2} + 6\frac{r^5}{R^4} - 4\frac{r^7}{R^6} + \frac{r^9}{R^8}\right),$$

whence we derive

$$\int_0^R w_0{}^2rdr = \tfrac{1}{10}k_1{}^2R^2,$$

for use in equation (94.15). We have also

$$\frac{dw_0}{dr} = 4\frac{k_1}{R^2}\left(-r + \frac{r^3}{R^2}\right)$$

and, consequently,

$$\frac{d^2w_0}{dr^2} = 4\frac{k_1}{R^2}\left(-1 + 3\frac{r^2}{R^2}\right),$$

which lead to the sum

$$\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr} = 8\frac{k_1}{R^2}\left(-1 + 2\frac{r^2}{R^2}\right),$$

and so to the integral

$$\int_0^R\left(\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right)^2 rdr = \frac{32}{3}\frac{k^2{}_1}{R^2}.$$

To this degree of accuracy the formula (94.15) obviously implies that

$$\frac{\partial}{\partial k_1}\left(\frac{32}{3}\frac{k^2{}_1}{R^2} - \frac{1}{10}\frac{h\rho R^2}{Dg}p^2k^2{}_1\right) = 0,$$

which, after differentiating with regard to k_1 and cancelling common factors, gives

$$\frac{32}{3} - \frac{1}{10}\frac{h\rho R^4}{Dg}p^2 = 0,$$

i.e.

$$p^2 = \frac{320}{3}\frac{Dg}{h\rho R^4},$$

showing that, to the first order of approximation, the fundamental period

$$\frac{2\pi}{p} = 0{\cdot}6082R^2\sqrt{\frac{h\rho}{Dg}}.$$

An improved estimate of the period in the fundamental mode of vibration may be obtained in a like manner by taking into consideration the first two terms in the series (94.14). Then

$$w_0 = k_1\left(1 - \frac{r^2}{R^2}\right)^2 + k_2\left(1 - \frac{r^2}{R^2}\right)^3,$$

whence we deduce, for a corresponding interpretation of the condition (94.15),

$$w_0{}^2 = \left(1 - \frac{r^2}{R^2}\right)^2\left\{k^2{}_1 + 2k_1k_2\left(1 - \frac{r^2}{R^2}\right) + k^2{}_2\left(1 - \frac{r^2}{R^2}\right)^2\right\}$$

$$= k^2{}_1\left(1 - 4\frac{r^2}{R^2} + 6\frac{r^4}{R^4} - 4\frac{r^8}{R^8} + \frac{r^{16}}{R^{16}}\right)$$

$$+2k_1k_2\left(1 - 5\frac{r^2}{R^2} + 10\frac{r^4}{R^4} - 10\frac{r^8}{R^8} + 5\frac{r^{16}}{R^{16}} - \frac{r^{32}}{R^{32}}\right)$$

$$+k^2{}_2\left(1 - 6\frac{r^2}{R^2} + 15\frac{r^4}{R^4} - 20\frac{r^8}{R^8} + 15\frac{r^{16}}{R^{16}} - 6\frac{r^{32}}{R^{32}} + \frac{r^{64}}{R^{64}}\right),$$

$$\frac{dw_0}{dr} = 4k_1\left(-\frac{r}{R^2} + \frac{r^3}{R^4}\right) + 6k_2\left(-\frac{r}{R^2} + 2\frac{r^3}{R^4} - \frac{r^5}{R^6}\right),$$

$$\frac{d^2w_0}{dr^2} = 4k_1\left(-\frac{1}{R^2} + 3\frac{r^2}{R^4}\right) + 6k_2\left(-\frac{1}{R^2} + 6\frac{r^2}{R^4} - 5\frac{r^4}{R^6}\right),$$

$$\frac{1}{r}\frac{dw_0}{dr} = 4k_1\left(-\frac{1}{R^2} + \frac{r^2}{R^4}\right) + 6k_2\left(-\frac{1}{R^2} + 2\frac{r^2}{R^4} - \frac{r^4}{R^6}\right).$$

From these equations it follows that

$$\int_0^R w_0{}^2 r\,dr = \tfrac{1}{10}k_1{}^2R^2 + \tfrac{1}{6}k_1k_2R^2 + \tfrac{1}{14}k^2{}_2R^2,$$

$$\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr} = 8\frac{k_1}{R^2}\left(-1 + 2\frac{r^2}{R^2}\right) + 12\frac{k_2}{R^2}\left(-1 + 4\frac{r^2}{R^2} - 3\frac{r^4}{R^4}\right),$$

and therefore

$$\int_0^R\left(\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right)^2 r\,dr = \frac{32}{3R^2}\left(k_1{}^2 + \frac{3}{2}k_1k_2 + \frac{9}{10}k^2{}_2\right).$$

Substitution of these expressions in the formula (94.15) shows that the minimal conditions to be satisfied are

$$\frac{\partial}{\partial k_n}\left\{\frac{32}{3R^2}\left(k^2_1+\frac{3}{2}k_1k_2+\frac{9}{10}k^2_2\right)-\frac{h\rho R^2}{Dg}p^2\left(\frac{1}{10}k^2_1+\frac{1}{6}k_1k_2+\frac{1}{14}k^2_2\right)\right\}=0,$$

where $n = 1, 2$;

$$\text{i.e. } \frac{\partial}{\partial k_n}\left\{\frac{32}{3R^2}\left(k^2_1+\frac{3}{2}k_1k_2+\frac{9}{10}k^2_2\right)-\frac{\beta}{R^2}\left(\frac{1}{10}k^2_1+\frac{1}{6}k_1k_2+\frac{1}{14}k^2_2\right)\right\}=0,$$

with β written for $\dfrac{h\rho R^4}{Dg}p^2$, and $n = 1, 2$.

On effecting in succession the operations of differentiation with respect to k_1 and k_2, it will be found that

$$\frac{32}{3}\left(2k_1+\frac{3}{2}k_2\right)-\beta\left(\frac{1}{5}k_1+\frac{1}{6}k_2\right)=0,$$

$$\frac{32}{3}\left(\frac{3}{2}k_1+\frac{9}{5}k_2\right)-\beta\left(\frac{1}{6}k_1+\frac{1}{7}k_2\right)=0,$$

i.e.

$$k_1\left(\frac{64}{3}-\frac{1}{5}\beta\right)+k_2\left(16-\frac{1}{6}\beta\right)=0,$$

$$k_1\left(16-\frac{1}{6}\beta\right)+k_2\left(\frac{96}{5}-\frac{1}{7}\beta\right)=0,$$

whence, on eliminating the ratio $k_1 : k_2$, we obtain

$$\beta^2-1{,}958\beta+193{,}800=0,$$

in round numbers.

If the roots of this quadratic in β be denoted by β_1 and β_2, it is readily proved that

$$\beta_1 = 104{\cdot}3,\quad \beta_2 = 1{,}854,$$

nearly, which mean that the corresponding values of p are

$$p^2_1=\frac{104{\cdot}3Dg}{h\rho R^4},\quad p^2_2=\frac{1{,}854Dg}{h\rho R^4}.$$

Hence we conclude that the related periods of vibration of the given plate are

$$\frac{2\pi}{p_1}=0{\cdot}6152R^2\sqrt{\frac{h\rho}{Dg}},\quad \frac{2\pi}{p_2}=0{\cdot}1459R^2\sqrt{\frac{h\rho}{Dg}}.$$

The first of these values refers to the fundamental mode, and the second may be regarded as a rough approximation to the period in the next mode. It will be noticed that our improved expression for the gravest mode contains the factor 0·6152, compared with 0·6082 in the previous calculation.

The displacement in the fundamental mode of vibration can now be determined by the corresponding equation

$$w=\left\{k_1\left(1-\frac{r^2}{R^2}\right)^2+k_2\left(1-\frac{r^2}{R^2}\right)^3\right\}\cos p_1t,$$

the coefficients k_1, k_2 being adjusted to satisfy the conditions at the boundary of the plate under consideration.

We might similarly derive a still more accurate result, based on the implied suppositions, by taking account of the first three terms in the series (94.14), but it is clear that such an extension of the method would involve lengthy calculations which could only be justified in special circumstances.

Ex. 2. Apply the same method to the general case of a thin plate, of any shape, executing free vibrations in a normal mode.

If we suppose the displacement to be given with sufficient accuracy by

$$w = w_0 \cos pt,$$

the fact that w_0 must be a function of x and y may be indicated by writing

$$w_0 = f_1(x,y) + f_2(x,y) + f_3(x,y) + \ldots$$

It is obvious that this expression affords a means of satisfying the boundary-conditions of a given plate. A second set of parameters must, however, be introduced for the purpose of making the resulting frequency a minimum, in accordance with the stationary property of normal modes (Art. 55). If $k_1, k_2, k_3, \ldots$ signify the second set of parameters, then

$$w_0 = k_1f_1(x,y) + k_2f_2(x,y) + k_3f_3(x,y) + \ldots \quad . \quad . \quad (94.16)$$

The result of inserting an appropriate number of terms of this series in equations (94.5) and (94.6), with maximum values assigned to the proper variables, may be written in the form

$$T_{\text{max.}} = \frac{h\rho}{g}p^2\iint w_0{}^2dxdy,$$

$$V_{\text{max.}} = D\iint\left\{\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + \frac{2}{m}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + \frac{2(m-1)}{m}\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)^2\right\}dx\,dy,$$

the limits of integration being arranged to cover the complete plate.

We thus have, to the degree of approximation implied in the supposition that $T_{\text{max.}} = V_{\text{max.}}$, the frequency $\frac{p}{2\pi}$ defined by

$$p^2 = \frac{g}{h\rho}\frac{V_{\text{max.}}}{\iint w^2{}_0dxdy}. \quad . \quad . \quad . \quad . \quad (94.17)$$

From the foregoing demonstration that the values of this quan-

tity, with the first n terms of the series (94.16) included in the analysis, will be a minimum if

$$\frac{\partial}{\partial k_n}\left(\frac{V_{\max.}}{\iint w^2_0 dxdy}\right) = 0,$$

we can, after differentiation by parts, express the minimal condition in the symbolic form

$$\iint w^2_0 dxdy \frac{\partial V_{\max.}}{\partial k_n} - V_{\max.}\frac{\partial}{\partial k_n}\iint w^2_0 dxdy = 0, \quad . \quad . \quad (94.18)$$

this being the numerator of the resulting expression. But it has been proved that, on the present assumptions,

$$V_{\max.} = \frac{h\rho}{g}p^2\iint w^2_0 dxdy.$$

Hence the formula (94.18) implies the proviso

$$\frac{\partial}{\partial k_n}\left(V_{\max.} - \frac{h\rho}{g}p^2\iint w^2_0 dxdy\right) = 0,$$

or, when written in full,

$$\frac{\partial}{\partial k_n}\iint\left\{\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + \frac{2}{m}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + \frac{2(m-1)}{m}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 - \frac{h\rho}{Dg}p^2 w^2_0\right\}dxdy = 0. \quad . \quad . \quad . \quad (94.19)$$

The preceding numerical example suffices to explain the way in which formula (94·19) will supply a set of n equations linear in $k_1, k_2, \ldots, k_n$ for a given plate. The solution of these equations likewise gives the values of $p_1, p_2, \ldots, p_n$, which approximately determine the frequencies $\frac{p_1}{2\pi}, \frac{p_2}{2\pi}, \ldots, \frac{p_n}{2\pi}$ in the corresponding modes of vibration.

Attention may, in conclusion, be drawn to a common characteristic of our equations for both rectangular and circular plates of the prescribed type, namely that the expression for p has the form

$$p = \frac{d}{l^2}\sqrt{\frac{Dg}{h\rho}}, \quad . \quad . \quad . \quad . \quad . \quad (94.20)$$

where d refers to a constant, and l to the boundary-dimension of a given plate.

95. Beams of Non-Uniform Cross-Section. We may well generalize the approximate method just exemplified and, by way of contrast, apply it to the case of a beam having a cross-section that changes gradually from point to point in the longitudinal direction. It will be assumed that the usual theory of beams holds good when no sudden changes of cross-section are present and, to

this order of approximation, that the beam behaves as if it consisted of a number of elements of length or thin discs which remain plane throughout small vibration, or, what amounts to the same thing, that the beam is equivalent to a number of rigid discs which move under the influence of the elastic forces of the system.

It must, however, again be admitted that the method to be described involves lengthy computations for all but the fundamental mode of transverse vibrations of beams in general, and therefore an application to higher modes cannot always be justified. There are, nevertheless, instances where a generalization may be utilized in connection with systems such as are exemplified by the blades of an airscrew. Of equal importance is the type of system which conforms in certain particulars to a slender beam, by reason of the fact that the extreme cases include the hull of a ship, and the fuselage of an aeroplane.

Imagine the dynamical equivalent of such a system to be a beam having, at any point x on the longitudinal axis, a cross-section specified by its area A and moment of inertia I with reference to the transverse motion under examination. It is always practicable to express both A and I as functions of x when recourse is had to the drawings of an actual structure.

If ρ be the density of the material, and y the displacement of the beam at x, reference to the theory of structures shows that

$$A\rho = \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)$$

relates to the beam when carrying a static load $A\rho$ per unit length, E being the direct modulus of elasticity of the material. Since this load will produce an inertia force $\frac{A\rho}{g}\frac{\partial^2 y}{\partial t^2}$ in the direction of motion when the system is vibrating slightly about the equilibrium position, we have

$$\frac{A\rho}{g}\frac{\partial^2 y}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) \quad . \quad . \quad . \quad . \quad (9.51)$$

The stiffness denoted by EI must, in the general case, be placed within the brackets because it is a function of x.

On the basis of the preceding treatment the assumed shape of the disturbed beam may be expressed by a relation of the type

$$y = v \cos pt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (95.2)$$

where v is a function of x.

To fix ideas without sensibly affecting the general character of the discussion, let both A and I vary symmetrically in relation to the mid-point of the length L of the beam. If the origin be taken at the mid-length and $L = 2l$, substitution of the formula

(95.2) in equations (87.1) and (87.2) leads, in the notation of equation (95.2), to

$$\left.\begin{aligned} 2T_{\text{max.}} &= \frac{\rho}{g}p^2\int_{-l}^{l} Av^2dx, \\ 2V_{\text{max.}} &= E\int_{-l}^{l} I\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (95.3)$$

as the expressions for the maximum potential and kinetic energies of the beam when ρ, E both refer to constant quantities.

96. To exemplify these formulae, let us investigate the transverse vibrations in a normal mode of the beam shown in Fig. 156 (*a*), having a parabolic profile such that the cross-sectional area A

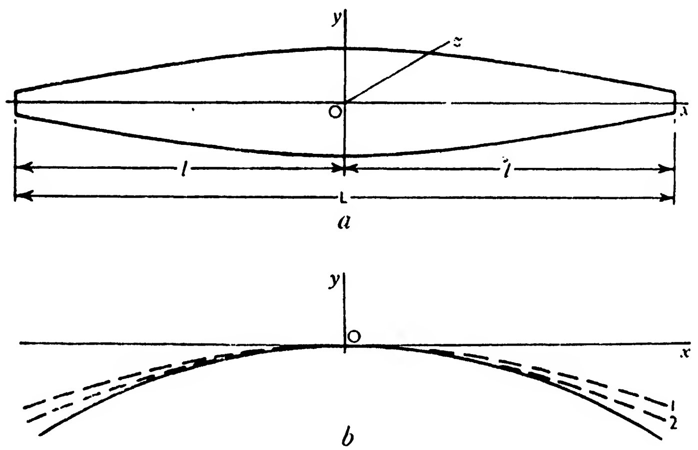


FIG. 156.

and the moment of inertia, I, of the cross-section at any point distant x from the origin O vary according to the laws

$$A = A_0(1 - ax^2), \; I = I_0(1 - bx^2) \quad . \quad . \quad . \quad (96.1)$$

Here the symbols a, b relate to constants, and A_0, I_0 to the values of the corresponding quantities for the mid-section specified by $x = 0$ in the figure. The beam is fixed at the origin O, and both ends are free to execute small vibrations about the position of rest which coincides with the x-axis.

Although the actual shape of the centre-line of a disturbed beam is usually unknown, we may for the purpose of reference here imagine the curve shown full in Fig. 156 (*b*) as indicating the line in question. In the present notation such a curve can be expressed as the sum of a series in x, say

$$v = f_1(x) + f_2(x) + \ldots + f_n(x), \quad . \quad . \quad . \quad (96.2)$$

with $f_1, f_2, \ldots, f_n$ denoting variable parameters in the general case.

We might, to take the next step, insert the formulae (96.1) and (96.2) in equations (95.3), then the resulting expressions would lead, on the implied supposition $T_{\text{max.}} = V_{\text{max.}}$, to the line integral

$$\int_{-l}^{l}\left\{I\left(\frac{d^2v}{\partial x^2}\right)^2 - \frac{\rho p^2}{Eg}Av^2\right\}dx = 0 \quad . \quad . \quad . \quad . \quad (96.3)$$

If, for example, only the first term in the series (96.2) were included in the calculations, then the integral would relate to the energy associated with the first approximation to the deflection-curve of the beam, for the given end-conditions could be fulfilled in the usual manner with the help of the f_1-variable.

But the symbols at our disposal do not, so far, offer an analytical means of securing a minimum value for p thus deduced from the integral (96.3), as is required by the method of Art. 55. To attain this end it is essential to introduce a second set of variable parameters into the series (96.2), and if these be signified by $k_1, k_2, \ldots, k_n$, then

$$v = k_1f_1(x) + k_2f_2(x) + \ldots + k_n f_n(x) \quad . \quad . \quad (96.4)$$

In the general case we can, with an assumed deflection-curve having its ends fixed by the functions $f_1(x), f_2(x), \ldots, f_n(x)$, now adjust the curve to comply with the stationary property of normal modes. This is so because the k-variables enable us to pass from one curve to another having the same end-points, with the object of ensuring a minimum expenditure of energy in a given mode of vibration and, consequently, a minimum value of p.

The ends of the beam being free, it follows from Ex. 4 of Art. 86 that in the present circumstances

$$v = C_1(\cos \alpha x + \cosh \alpha x) + C_2(\cos \alpha x - \cosh \alpha x), \quad (96.5)$$

where C_1, C_2 are arbitrary constants. Furthermore, the fact that the bending moments and the shearing forces are both zero at the ends of the beam implies the conditions

$$\frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^3 v}{\partial x^3} = 0 \text{ at the points } x = \pm l.$$

Imposing these conditions on the appropriate differential forms of equation (96.5), we find

$$\left.\begin{aligned} C_1(-\cos \alpha l + \cosh \alpha l) - C_2(\cos \alpha l + \cosh \alpha l) &= 0,\\ C_1(\sin \alpha l + \sinh \alpha l) + C_2(\sin \alpha l - \sinh \alpha l) &= 0.\end{aligned}\right\} \quad (96.6)$$

Elimination of the ratio $C_1 : C_2$ between these equations thus leads to

$$\tan \alpha l + \tanh \alpha l = 0 \quad . \quad . \quad . \quad . \quad . \quad (96.7)$$

Again, an expression for the general coefficient C_r may be

gathered from a combination of the first of equations (96.6) and (96.5), whence

$$v_r = C_r(\cos \alpha_r x \cos \alpha_r l + \cosh \alpha_r x \cos \alpha_r l),$$

where $r = 1, 2, \ldots, n$; hence

$$C_r = \frac{v_r}{\cos \alpha_r x \cosh \alpha_r l + \cosh \alpha_r x \cos \alpha_r l} \quad . \quad . \quad . \quad (96.8)$$

From this equation we may, remembering the restrictions placed on the motion under examination, infer that the more convenient form

$$C_r = \frac{1}{(\cos^2 \alpha_r l + \cosh^2 \alpha_r l)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (96.9)$$

applies in a first approximation. Since, as already noticed, zero is a root of equation (96.7), it follows that a solution to the present degree of accuracy is given by writing $\alpha_r l = 0$, with $r = 1$, in equation (96.9), whence $C_1 = \frac{1}{\sqrt{2}}$ and, therefore,

$$v = \frac{k_1}{\sqrt{2}}$$

in the corresponding mode of vibration.

The accuracy of a result based on the same assumptions will evidently be improved if we introduce the quantity given by putting $r = 2$ in the related equations (96.8) and (96.9). Thus

$$v = \frac{k_1}{\sqrt{2}} + k_2 \frac{\cos \alpha_2 x \cosh \alpha_2 l + \cosh \alpha_2 x \cos \alpha_2 l}{(\cos^2 \alpha_2 l + \cosh^2 \alpha_2 l)^{\frac{1}{2}}} \quad . \quad (96.10)$$

refers to the second approximation, and the curve marked 1 in Fig. 156 (*b*) may be regarded as the corresponding displacement-curve.

It is manifest that the expression which results from inserting equation (96.10) in the integral (96.3) will be a minimum, when the integral is varied by a change of the parameters k, if

$$\frac{\partial}{\partial k_r}\int_{-l}^{l}\left\{I\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - \frac{\rho p^2}{Eg}Av^2\right\}dx = 0, \quad . \quad . \quad . \quad (96.11)$$

where $r = 1, 2$. This means, when account is taken of the relations (96.1), and ρ, E are both treated as constant quantities, that

$$\frac{\partial}{\partial k_m}\left\{I_0\int_{-l}^{l}(1 - bx^2)\sum_{r=1}^{2}\sum_{s=1}^{2}k_r k_s f_r'' f_s'' dx \right.$$
$$\left. - \frac{\rho p^2 A_0}{Eg}\int_{-l}^{l}(1 - ax^2)\sum_{r=1}^{2}\sum_{s=1}^{2}k_r k_s f_r f_s dx\right\} = 0, \quad . \quad . \quad . \quad (96.12)$$

where $m = r$, $s = 1, 2$, and the accents signify differentiation with respect to x.

If, for brevity in working, we write

$$\left.\begin{aligned}\int_{-l}^{l}(1-bx^2)f_r''f_s''dx &= D_{rs},\\ \int_{-l}^{l}(1-ax^2)f_rf_sdx &= F_{rs},\\ \frac{\rho p^2 A_0}{EgI_0} &= \beta,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (96.13)$$

then

$$\left.\begin{aligned}(D_{11}-\beta F_{11})k_1 + (D_{21}-\beta F_{21})k_2 &= 0,\\ (D_{12}-\beta F_{12})k_1 + (D_{22}-\beta F_{22})k_2 &= 0\end{aligned}\right\} \quad . \quad (96.14)$$

follow from the integral (96.12) when the values 1, 2 are in turn ascribed to the suffixes r, s.

The arithmetical work of evaluating the D- and F-coefficients in equations (96.14) has, for convenience, been transferred to an appendix at the end of the volume. It is there demonstrated that, in approximate numbers,

$$\left.\begin{aligned}&D_{11}=0,\ D_{12}=0,\ D_{21}=0,\ D_{22}=\frac{31\cdot28}{l^3}(1-0\cdot0875bl^2),\\ &F_{11}=l(1-0\cdot3333al^2),\ F_{12}=0\cdot2970al^3=F_{21},\\ &F_{22}=l(1-0\cdot4590al^2).\end{aligned}\right\} \quad . \quad (96.15)$$

Substitution of the values $D_{11}=D_{12}=D_{21}=0$, and elimination of the ratio $k_1 : k_2$ between the equations (96.14), at once disclose the relation

$$(F^2_{12}-F_{11}F_{22})\beta^2 + F_{11}D_{22}\beta = 0,$$

since $F_{21}=F_{12}$. Since one of the roots of this quadratic in β is obviously zero, the remaining root must be

$$\beta = \frac{F_{11}D_{22}}{F_{11}F_{22}-F^2_{12}}.$$

Thus, introducing the numerical values (96.15), we have

$$\begin{aligned}\beta &= \frac{31\cdot28\{1-(0\cdot0875b+0\cdot3333a)l^2+0\cdot0292abl^4\}}{(1-0\cdot7823al^2+0\cdot0648a^2l^4)l^4}\\ &= \frac{\rho A_0}{EgI_0}p^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96.16)\end{aligned}$$

by equations (96.13).

Therefore a beam of the specified form will execute transverse vibrations in a normal mode with a fundamental frequency $\frac{p}{2\pi}$ defined, to the second order of approximation, by

$$p^2 = \frac{31\cdot28EgI_0\{1-(0\cdot0875b+0\cdot3333a)l^2+0\cdot0292abl^4\}}{\rho A_0(1-0\cdot7823al^2+0\cdot0648a^2l^4)l^4}, \quad . \quad (96.17)$$

the proper value of the moment of inertia of the section being assigned to I_0.

A still more accurate value of the fundamental frequency may, on the same suppositions, be arrived at by repeating the analysis with the first three terms of the series (96.4). In this way we should obtain an improved approximation to the curve shown full in Fig. 156 (*b*), which may be illustrated by the curve marked 2 in that figure, but laborious calculations would be incurred by such an extension of the method. This procedure is, however, unnecessary, to the extent that a solution involving only the first two terms of the series stated above will usually suffice for most practical purposes, provided an appropriate expression for v can be found. The measure of success achieved in this part of the work depends chiefly on a careful examination of the drawings of the structure in question, combined with a knowledge of the related theory. In the general case, however, the most troublesome points to settle are associated with the constraints at the ends and the conditions at points where the cross-section changes suddenly.

The hull of a ship offers a noteworthy application of the foregoing analysis, but then we must consider separately the vibrations parallel to the axes Oy and Oz in Fig. 156, for different values are generally identified with the corresponding moments of inertia of the section about those axes of reference.

Mention should also be made of the equally important example offered in this connection by the blade of an airscrew. If, with the system at rest, the section of the blade be specified by its flexural rigidity EI_x in relation to bending parallel to the chord, flexural rigidity EI_y in relation to bending normal to the chord, and cross-sectional area A_z at a point distant z from the axis of rotation, an inspection of actual airscrews shows that as a first approximation we may assume

$$A_z = az, \quad I_x = bzA_z, \quad I_y = czA_z, \quad . \quad . \quad (96.18)$$

where a, b, c denote constants for a given blade. In this hypothetical blade A_z varies as z, and I_x, I_y both vary as z^2. An estimate of the fundamental frequency in a normal mode of vibration for a blade conforming with this specification may be found by the foregoing method, the x- and y-axes being examined separately. The value thus deduced will, however, be affected by rotation under service conditions, in the manner explained in Art. 57. It is to be pointed out that considerable difficulty attends an experimental determination of the constraint or degree of fixity at the root of actual blades, for this variable greatly affects the frequency of vibration.

97. Combined Transverse and Torsional Vibrations. Some consideration should next be given to the type of 'built-up' structure which can be treated as being practically equivalent to a slender beam.

Let Fig. 157 (*a*) represent such a system, consisting of a hollow structure of length L, and of circular form in cross-section with a diameter that changes gradually from point to point along the x-axis. The system is built-up by a number of bulkheads or 'formers' connected together by a number of longitudinal members or 'stringers', and is supported at both ends. It will also be supposed that the system simultaneously performs transverse and torsional oscillations about the position of rest which we take as coinciding with the x-axis, the origin being at O.

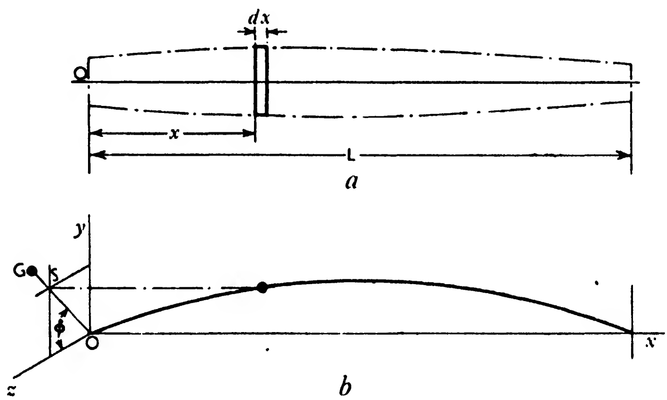


FIG. 157.

On the usual supposition that originally plane sections of the beam remain plane throughout the small vibrations in question, we may divide the structure into a finite number of sectional elements or discs, and so investigate the problem by Lagrange's method. The centre-lines of the bulkheads may conveniently be identified with the mid-lengths of the hypothetical discs into which the structure is thus divided in the x-direction, on the understanding that the centre of gravity of a disc may or may not coincide with the geometrical centre of the section. The implication is that the component elements of length are separate masses which can vibrate as such under the influence of the elastic forces associated with the material.

To include the several variables mentioned, let dx in the figure relate to one of the component discs, having its centre of gravity at G and its geometrical centre at S in the plane formed by the fixed axes Oy, Oz in Fig. 157 (*b*). The curve in the latter figure may be regarded as the centre-line of the disturbed beam. It will, for simplicity of treatment, be assumed that the points O, S, G always lie on a straight line, and that the end-points of the beam remain stationary throughout the motion.

The procedure consists in finding the equations of motion for a

disc specified by its centres S, G, and considerations of continuity then enable us to study the motion of the complete system with reference to the fixed plane yOz.

If at time t the co-ordinates of S, G be (y, z), (y_g, z_g), respectively, and the relative position of the line OSG be defined by the angle ϕ in the figure, it is clear that

$$y_g = y + e\sin\phi,\ z_g = z + e\cos\phi, \quad . \quad . \quad . \quad (97.1)$$

where e signifies the distance between the centres S and G on the line OSG. If any variations in the eccentricity e be regarded as of the second order of small quantities, differentiation with respect to the time gives the velocities

$$\dot{y}_g = \dot{y} + e\dot{\phi}\cos\phi,\ \dot{z}_g = \dot{z} - e\dot{\phi}\sin\phi \quad . \quad . \quad (97.2)$$

To obtain a suitable expression for the kinetic energy of the representative disc at x, write J for its polar moment of inertia about a line through S, and k for its radius of gyration about a parallel line through G. Then, with the weight of the element denoted by M,

$$J = \frac{M}{g}(k^2 + e^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (97.3)$$

The kinetic energy, T, of the element is given by

$$2T = \frac{M}{g}(k^2\dot{\phi}^2 + \dot{y}_g{}^2 + \dot{z}_g{}^2)$$

when referred to the centre G, and by

$$2T = \frac{M}{g}\{(k^2 + e^2)\dot{\phi}^2 + \dot{y}^2 + \dot{z}^2 + 2e\dot{\phi}(\dot{y}\cos\phi - \dot{z}\sin\phi)\} \quad . \quad (97.4)$$

when referred to the centre S.

On writing Q_y, Q_z, Q_ϕ for the generalized components of force associated in turn with the co-ordinates y, z, ϕ, the corresponding equations of motion follow at once from Lagrange's formula (19.8), in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = Q_y,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = Q_z,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = Q_\phi.$$

Applying these formulae to equation (97.4), we find, in the first place,

$$\frac{\partial T}{\partial \dot{y}} = \frac{M}{g}(\dot{y} + e\dot{\phi}\cos\phi),$$

whence
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = \frac{M}{g}(\ddot{y} + e\ddot{\phi}\cos\phi - e\dot{\phi}^2\sin\phi)$$

follows on differentiating with regard to the time, and we learn in this manner that the component of force

$$Q_y = \frac{M}{g}(\ddot{y} + e\ddot{\phi}\cos\phi - e\dot{\phi}^2\sin\phi), \quad . \quad . \quad . \quad (97.5)$$

since $\dfrac{\partial T}{\partial y} = 0$.

It likewise appears that, in the second place,

$$\frac{\partial T}{\partial \dot{z}} = \frac{M}{g}(\dot{z} - e\dot{\phi}\sin\phi),$$

and, therefore,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = \frac{M}{g}(\ddot{z} - e\ddot{\phi}\sin\phi - e\dot{\phi}^2\cos\phi),$$

so that the component of force

$$Q_z = \frac{M}{g}(\ddot{z} - e\ddot{\phi}\sin\phi - e\dot{\phi}^2\cos\phi), \quad . \quad . \quad . \quad (97.6)$$

since $\dfrac{\partial T}{\partial z} = 0$.

Similarly, in the formula for the component Q_ϕ we have

$$\frac{\partial T}{\partial \dot{\phi}} = \frac{M}{g}\{(k^2 + e^2)\dot{\phi} + e(\dot{y}\cos\phi - \dot{z}\sin\phi)\},$$

and so

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \frac{M}{g}\{(k^2 + e^2)\ddot{\phi} + e(\ddot{y}\cos\phi - \dot{y}\dot{\phi}\sin\phi - \ddot{z}\sin\phi - \dot{z}\dot{\phi}\cos\phi)\};$$

here there is also the relation

$$\frac{\partial T}{\partial \phi} = -\frac{M}{g}\{e\dot{\phi}(\dot{y}\sin\phi - \dot{z}\cos\phi)\}.$$

Hence $$Q_\phi = \frac{M}{g}\{(k^2 + e^2)\ddot{\phi} + e(\ddot{y}\cos\phi - \ddot{z}\sin\phi)\} \quad . \quad . \quad (97.7)$$

determines the third component of force.

Now suppose the system to be executing free vibrations, and let the inertia forces and couple implied in equations (97.5), (97.6), (97.7) be designated by Y, Z, $\mathfrak{T}$, respectively, so that

$$Q_y = -Y - M,\ Q_z = -Z,\ Q_\phi = -\mathfrak{T} + Me\cos\phi.$$

With these conventions we have, from the formulae (47.1) and (48.1), the inertia components.

$$\left.\begin{aligned}
Y &= -\frac{M}{g}(\ddot{y} + e\ddot{\phi}\cos\phi - e\dot{\phi}^2\sin\phi) - M,\\
Z &= -\frac{M}{g}(\ddot{z} - e\ddot{\phi}\sin\phi - e\dot{\phi}^2\cos\phi),\\
\mathfrak{T} &= -\frac{M}{g}\{(k^2 + e^2)\ddot{\phi} + e(\ddot{y}\cos\phi - \ddot{z}\sin\phi)\} + Me\cos\phi.
\end{aligned}\right\} \quad . \quad (97.8)$$

These equations exhibit an important characteristic, for they clearly indicate that the components of motion are not independent except in the extreme case where the eccentricity e is negligibly small. In other words, the presence of torsional vibrations will, in the given circumstances, imply transverse vibrations also, and *vice versa*, when e is not very small.

98. The eccentricity e must accordingly be very small compared with the other dimensions in the last set of equations if vibrations parallel to the y- and z-axes are to be investigated independently of the torsional motion. The process of tracing the vibratory motion will be understood from Art. 112, where the same analysis is applied to the analogous problem of shafts.

Before passing to other considerations, however, it is to be remarked that equations (97.8) relate, with evident restrictions, to the fuselage of an aeroplane. We may then have to study the more difficult question of a structure of non-circular cross-section, vibrating under the influence of the unbalanced components of force and torque of the engine. These unbalanced effects enter into the equations of motion through the symbols Y, Z, $\mathfrak{T}$.

99. Gyroscopic Forces. The disturbed motion of the important type of system just examined is not so simple as is suggested by the foregoing results. It is not difficult to discover the reason, for the analysis takes no account of the slope of the equivalent beam, and of certain constraints.

Simple experiments with rotating wheels and gyroscopes suffice to show that the vibrations are then affected by what is called the gyroscopic resistance of the rotating body.

If the motion of such a system be referred to rectangular axes Ox, Oy, Oz which rotate about Oz with angular velocity Ω, the equations for a component particle of mass m are

$$\frac{m}{g}(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x) = X,\ \frac{m}{g}(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y) = Y,\ \frac{m}{g}\ddot{z} = Z, \qquad (99.1)$$

where X, Y, Z denote the components of the impressed force.

We shall now assume that the relative co-ordinates x, y, z of each particle are expressed in terms of an appropriate number of generalized co-ordinates $q_1, q_2, \ldots, q_n$. On this supposition we may, with the help of Art. 19 (d), write for the complete system

$$2T' = \Sigma\frac{m}{g}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),\ 2T_0 = \Omega^2\Sigma\frac{m}{g}(x^2 + y^2), \quad . \quad . \quad (99.2)$$

where T' is the kinetic energy of the relative motion, expressed as a homogeneous quadratic function of the generalized velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$, with coefficients which are functions of the generalized co-ordinates $q_1, q_2, \ldots, q_n$; and T_0 is the kinetic energy of the

system when rotating, without relative motion, in the configuration $q_1, q_2, \ldots, q_n$.

If we further suppose the inevitable dissipative forces to be negligibly small, and let the generalized components of extraneous force be $Q_1, Q_2, \ldots, Q_n$, then in a virtual displacement

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = -\delta V + Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q_n\delta q_n, \quad (99.3)$$

where V is the potential energy of the system.

Thus, on multiplying equations (99.1) in turn by $\frac{\partial x}{\partial q_r}, \frac{\partial y}{\partial q_r}, \frac{\partial z}{\partial q_r}$, and adding, we obtain, after summing over all the particles and proceeding as in Art. 19, a set of n equations of the type

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{q}_r}\right)-\frac{\partial T'}{\partial q_r}+\beta_{r1}\dot{q}_1+\beta_{r2}\dot{q}_2+\ldots+\beta_{rn}\dot{q}_n+\frac{\partial}{\partial q_r}(V-T_0)=Q_r, \quad (99.4)$$

where $r = 1, 2, \ldots, n$, $\beta_{rs} = 2\Omega\sum\frac{m}{g}\frac{\partial(x, y)}{\partial(q_s, q_r)}$, and therefore $\beta_{rs} = -\beta_{sr}$, $\beta_{rr} = 0$.

It is evident that the equations which determine the vibrations of such a system will in general include *gyroscopic terms* of the type $q_r\dot{q}_s$, even when the vibrations take place about a position of relative equilibrium. A distinction must accordingly be drawn between vibrations about *equilibrium* and vibrations about *steady motion*, the reason being that vibrations about steady motion are the same as those about equilibrium of the *non-natural* system to which the problem is reduced by the operation known as *ignoration of co-ordinates*.[1] Terms of the type $q_r\dot{q}_s$ indicate the presence of what will be called *gyroscopic forces*, to use the nomenclature of Lord Kelvin and P. G. Tait,[2] who, together with E. J. Routh,[3] and other workers, developed the general theory of gyrostatic systems.

In the absence of disturbing forces the conditions for relative equilibrium are

$$\frac{\partial}{\partial q_r}(V - T_0) = 0, \quad \ldots \quad (99.5)$$

being the result of making $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n = 0$ in equation (99.4). Hence the equilibrium value of $V - T_0$ is 'stationary'.

Furthermore, since $\beta_{rs} = -\beta_{sr}$, we gather from a combination of equations (99.2) and (99.3) the relation

$$\frac{d}{dt}(T' + V - T_0) = Q_1\dot{q}_1 + Q_2\dot{q}_2 + \ldots + Q_n\dot{q}_n,$$

showing that, in the absence of extraneous forces,

$$T' + V - T_0 = \text{const.} \quad \ldots \quad (99.6)$$

[1] E. T. Whittaker, *Analytical Dynamics*, Art. 38, third edition.

[2] *Treatise on Natural Philosophy*, vol. 1, page 392 (new edition, 1912).

[3] *Advanced Rigid Dynamics.*

100. Let us now suppose the co-ordinates $q_1, q_2, \ldots q_n$ to be chosen so as to vanish in the state of rest. Confining ourselves to the case of *small* vibrations, we can, by virtue of results obtained in Arts. 46–49, then write

$$\left.\begin{aligned} 2T' &= a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots, \\ 2(V - T_0) &= c_{11}q_1^2 + c_{22}q_2^2 + \ldots + 2c_{12}q_1q_2 + \ldots, \end{aligned}\right\} \quad . \quad (100.1)$$

and treat the coefficients a_{rs}, c_{rs} as constants. Any terms of the first degree in $V - T_0$ would be irrelevant, by reason of the stationary property (99.5).

The equations may be further simplified, after the manner of Art. 53, by a linear transformation of co-ordinates which will reduce the functions T' and $V - T_0$ simultaneously to sums of squares, say

$$\left.\begin{aligned} 2T' &= a_1\dot{q}_1^2 + a_2\dot{q}_2^2 + \ldots + a_n\dot{q}_n^2, \\ 2(V - T_0) &= c_1q_1^2 + c_2q_2^2 + \ldots + c_nq_n^2. \end{aligned}\right\} \quad . \quad (100.2)$$

Although this operation is always practicable, it is to be pointed out that the transformation required for the purpose may vary with the values of the constant momenta, or constant velocity, involved in the rotation.

The quantities $q_1, q_2, \ldots q_n$ in the last set of equations may be called the *principal co-ordinates* of the system, on the understanding that they do not retain the mutually independent characteristic of ' normal co-ordinates ' in the ordinary theory of Art. 53. On this account we do not in general experience vibrations in which one principal co-ordinate varies alone. By analogy, the quantities $a_1, a_2, \ldots, a_n$ may be called the *principal coefficients of inertia*, and $c_1, c_2, \ldots, c_n$ the *principal coefficients of stability* or of *stiffness*. The latter would, by the argument of Art. 19 (*d*), be unchanged if we were to ignore the rotation, and to imagine centrifugal forces $\left(\frac{m}{g}\Omega^2 x, \frac{m}{g}\Omega^2 y, 0\right)$ as acting on each particle in the direction outwards from the axis of rotation.

The equations (99.4) now reduce to

$$\left.\begin{array}{l} a_1\ddot{q}_1 + c_1q_1 \qquad\qquad + \beta_{12}\dot{q}_2 + \beta_{13}\dot{q}_3 + \ldots + \beta_{1n}\dot{q}_n = Q_1, \\ a_2\ddot{q}_2 + c_2q_2 + \beta_{21}\dot{q}_1 \qquad\quad + \beta_{23}\dot{q}_3 + \ldots + \beta_{2n}\dot{q}_n = Q_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n\ddot{q}_n + c_nq_n + \beta_{n1}\dot{q}_n + \beta_{n2}\dot{q}_2 + \beta_{n3}\dot{q}_3 + \ldots\ldots = Q_n, \end{array}\right\} \quad (100.3)$$

where the coefficients β_{rs} may be treated as constants in the small vibrations under consideration.

It is worth while to point out that the gyroscopic terms are of the same type as would be introduced by constraints which vary with the time, for it is not always realized that the corresponding systems are in essentials the same.

The *free* motions may be investigated by making $Q_1, Q_2, \ldots, Q_n$ all equal to zero and, as usual, writing

$$q_1 = A_1 e^{\lambda t},\ q_2 = A_2 e^{\lambda t},\ \ldots,\ q_n = A_n e^{\lambda t},$$

the A-coefficients being arbitrary. Substituting in equations (100.3), and cancelling the common factor $e^{\lambda t}$, we find

$$\left.\begin{array}{l} (a_1\lambda^2 + c_1)A_1 + \beta_{12}\lambda A_2 + \ldots + \beta_{1n}\lambda A_n = 0, \\ \beta_{21}\lambda A_1 + (a_2\lambda^2 + c_2)A_2 + \ldots + \beta_{2n}\lambda A_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \beta_{n1}\lambda A_1 + \beta_{n2}\lambda A_2 + \ldots + (a_n\lambda^2 + c_n)A_n = 0. \end{array}\right\} \quad . \quad (100.4)$$

Elimination of the ratios $A_1 : A_2 : \ldots : A_n$ thus shows that the free vibrations are expressed by the determinantal equation

$$\begin{vmatrix} a_1\lambda^2 + c_1, & \beta_{12}\lambda, & \ldots, & \beta_{1n}\lambda \\ \beta_{21}\lambda, & a_2\lambda^2 + c_2, & \ldots, & \beta_{2n}\lambda \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{n1}\lambda, & \beta_{n2}\lambda, & \ldots, & a_n\lambda^2 + c_n \end{vmatrix} = 0, \quad . \quad . \quad . \quad (100.5)$$

which we shall briefly refer to as $\Delta(\lambda) = 0$.

For a given root λ the solution of equation (100.4) can now be written in the form

$$\frac{A_1}{\alpha_1} = \frac{A_2}{\alpha_2} = \ldots = \frac{A_n}{\alpha_n} = R, \quad . \quad . \quad . \quad (100.6)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the minors of any row in the determinantal equation (100.5), and the ratio R is arbitrary.

Since $\beta_{rs} = -\beta_{sr}$, $\Delta(\lambda)$ is a 'skew' determinant, so that if the sign of λ be reversed, the rows and columns are merely interchanged, and the value of $\Delta(\lambda)$ is therefore unaltered.[1] Hence the roots of equation (100.5) occur in pairs of the form

$$\lambda = \pm(\sigma + ip).$$

In order to secure stability in a state of relative equilibrium, however, it is essential that the values of σ shall all be zero. Were this condition not fulfilled, terms of the types $e^{\pm\sigma t}\cos pt$, $e^{\pm\sigma t}\sin pt$ would present themselves in the expression for any co-ordinate q_r when stated in terms of real quantities, and would, as we have seen, indicate instability in the form of a vibration having a continually increasing amplitude. Therefore we must have $\sigma = 0$ if the motion is to be strictly periodic, in which case the roots of equation (100.5) occur in pairs of the form

$$\lambda = \pm ip \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100.7)$$

The minors will in general be complex quantities, since odd as

[1] W. S. Burnside and A. W. Panton, *Theory of Equations*, vol. 2, page 45.

well as even powers of λ may occur in $\alpha_1, \alpha_2, \ldots, \alpha_n$. If, to take account of this fact, we write

$$\lambda = \pm ip, \; \alpha_r = \gamma_r + i\delta_r,$$

then

$$q_r = R(\gamma_r + i\delta_r)e^{ipt}.$$

Thus it appears, on taking the real part of the expression given by writing $De^{i\epsilon}$ for R, that the displacement

$$q_r = D\{\gamma_r \cos(pt + \epsilon) - \delta_r \sin(pt + \epsilon)\}, \quad . \quad . \quad (100.8)$$

where D, ϵ are arbitrary. This determines what may be regarded as a 'natural mode' of vibration, in so far as the number of such modes is equal to that of the degrees of freedom of the system. Any small vibration which the system may describe can be represented as the superposition of a number of such periodic motions.

A noteworthy characteristic ot the motion expressed by equation (100.8) is that the phase is different for different co-ordinates, in consequence of which the phase at any instant will not be uniform throughout the system.

101. To obtain the expressions for the *forced and undamped* vibrations produced by a periodic extraneous force, we proceed on the usual assumption that $Q_1, Q_2, \ldots, Q_n$ of equations (100.3) all vary as $e^{i\omega t}$, with a prescribed value for ω.

This leads, on omitting the common factor $e^{i\omega t}$, to a set of n equations of the form

$$\Delta(i\omega)q_r = \alpha_{r1}Q_1 + \alpha_{r2}Q_2 + \ldots + \alpha_{rn}Q_n, \quad . \quad . \quad (101.1)$$

where the α-coefficients represent the minors of the rth row in the determinant $\Delta(i\omega)$.

At this stage of the work we shall confine our attention to general inferences which may be drawn from the last equation, for an expression of the same form is examined in Ex. 2 of Art. 124. The motion in question evidently differs in one important particular from a 'normal mode' in the case where the gyroscopic terms are absent, since the displacement of any one type may be influenced by a disturbing force of any type. For example, the displacement q_1 may be affected by one or more of the forces $Q_2, Q_3, \ldots, Q_n$. As a consequence the motions of the component particles are in general elliptic harmonic, as we might have inferred from equation (100.8). The forced motion is, of course, more complicated than in the case of free vibrations; as a rule there will be differences of phase, variable with the period, between the displacements and the force of equations (101.1).

But the present motion resembles that discussed in Art. 58 when the frequency of the disturbing force is approximately equal to the natural frequency of the system, for then the determinant

$\Delta(i\omega)$ is very small and the displacement therefore tends to very large values.

These considerations have an obvious bearing on the kind of problem presented by the transverse vibrations of the wheel of a marine turbine when the vessel is executing 'yawing' or 'pitching' motion of very long period, remembering that the wheel is relatively thin in the fore-and-aft direction. Under these conditions, according to equations (101.1), the displacements will be sensibly the same as the 'equilibrium values'

$$q_1 = \frac{Q_1}{c_1},\ q_2 = \frac{Q_2}{c_2},\ \ldots,\ q_n = \frac{Q_n}{c_n} \quad . \quad . \quad . \quad (101.2)$$

when the impressed period is infinitely long, provided the coefficients $c_1, c_2, \ldots, c_n$ all differ from zero. If, to take the extreme case of an infinitely thin disc, we suppose that $c_1 = 0$, then the first row and column of the determinant $\Delta(\lambda)$ are both divisible by λ, hence the determinantal equation (100.5) has a pair of zero roots. The system may accordingly describe a free motion of infinitely long period, with the result that the coefficients $Q_2, Q_3, \ldots, Q_n$ in equations (101.1) become indeterminate for $\omega = 0$, and the displacements generally differ from the values given by equations (101.2).

If friction of the type implied in equations (59.1) be now introduced, small vibrations of the system will be expressed by equations of the same form as (99.4), for then the frictional and gyroscopic forces are both functions of the generalized components of velocity. By this means we could, with the prescribed type of friction, obtain the most general form of equations of a dynamical system capable of executing small vibrations, such as, for example, a ship fitted with a *gyroscopic stabilizer.*

It is conceivable, remarkable though it may seem, that the introduction of dissipative forces may bring about a kind of instability to which a special significance is attached in the case of aircraft. This peculiar motion arises partly from the slender form of various components of such systems, and partly from the manner in which the frictional forces may vary owing to agencies of aerodynamical origin. The most important instance of this is, perhaps, to be found in the torsional vibrations of the blades of a rotating airscrew.

A few simple examples will illustrate the above remarks, which have reference also to the applications of the theory given in Art. 124.

Ex. 1. Consider the ***undamped*** vibrations of a gyroscopic system with two degrees of freedom.

If the co-ordinates q_1, q_2 be assigned to the two degrees of freedom, according to equations (100.3) the motion is defined by

$$a_1\ddot{q}_1 + c_1 q_1 + \beta\dot{q}_2 = Q_1,\quad a_2\ddot{q}_2 + c_2 q_2 - \beta\dot{q}_1 = Q_2, \qquad (101.3)$$

where $\beta = \beta_{12} = -\beta_{21}$, from equation (99.4).

On the supposition that the displacements q_1, q_2 vary as $e^{\lambda t}$, we have the periods of the *free* vibrations determined by

$$a_1 a_2 \lambda^4 + (a_1 c_2 + a_2 c_1 + \beta^2)\lambda^2 + c_1 c_2 = 0 \quad . \quad . \quad (101.4)$$

Our previous treatment shows that stability in the ordinary sense will be secured if the roots of this quadratic in λ^2 are real and negative. It is a simple matter to prove that this condition is fulfilled if a_1, a_2, c_1, c_2 are all essentially positive. These terms will always be positive in mechanical systems of the kind considered in this work. Moreover, the condition for stability will still be satisfied even when c_1, c_2 are both negative, provided β^2 be sufficiently great.

If the system executes *forced* motion due to forces Q_1, Q_2 which vary as $e^{i\omega t}$, on making the substitutions in equations (101.3) and proceeding as previously, we find

$$(c_1 - \omega^2 a_1)q_1 + i\omega\beta q_2 = Q_1, \quad -i\omega\beta q_1 + (c_2 - \omega^2 a_2)q_2 = Q_2,$$

showing that the displacements are

$$\left.\begin{aligned} q_1 &= \frac{(c_2 - \omega^2 a_2)Q_1 - i\omega\beta Q_2}{(c_1 - \omega^2 a_1)(c_2 - \omega^2 a_2) - \omega^2\beta^2}, \\ q_2 &= \frac{i\omega\beta Q_1 + (c_1 - \omega^2 a_1)Q_2}{(c_1 - \omega^2 a_1)(c_2 - \omega^2 a_2) - \omega^2\beta^2}. \end{aligned}\right\} \quad . \quad . \quad (101.5)$$

To investigate the interesting case where c_2 is negligibly small compared with c_1, we may conveniently also assume that $Q_2 = 0$, for the first of these suppositions implies that a displacement of the type q_2 does not affect the value of $V - T_0$. Under these conditions, by equations (101.5),

$$q_1 = \frac{a_2}{a_2(c_1 - \omega^2 a_1) + \beta^2}Q_1, \quad \dot{q}_2 = \frac{\beta}{a_2(c_1 - \omega^2 a_1) + \beta^2}Q_1 \quad . \quad (101.6)$$

This result explains how a displacement of the type q_1 may be accompanied by a motion of the type q_2, *even when an impressed force of the type Q_2 is absent.* Again, if the impressed period is very long we may write $\omega = 0$ as a limiting value, and so deduce from the last set of equations

$$q_1 = \frac{1}{1 + \dfrac{\beta^2}{a_2 c_1}} \cdot \frac{Q_1}{c_1}, \quad \dot{q}_2 = \frac{\beta}{a_2 c_1 + \beta^2}Q_1,$$

which show that the displacement q_1 is then *less* than its equilibrium value $\dfrac{Q_1}{c_1}$, in the ratio of unity to $\left(1 + \dfrac{\beta^2}{a_2 c_1}\right)$.

But, by equations (99.2), Ω^2 is a factor in the expression for T_0, which means that in the general case Ω^2 will be a factor in the relation for c_2 when it tends to very small values as Ω becomes correspond-

ingly small. In this case the two roots of equation (101.4) are approximately given by

$$\lambda^2{}_1 = -\frac{c_1}{a_1},\ \lambda^2{}_2 = -\frac{c_2}{a_2}$$

when Ω is very small; and λ_2 becomes more and more proportional to Ω as $\Omega \to 0$. It is to be inferred from these results that types of circulatory motion, which are of infinitely long period in the case of no rotation, may be changed by a very small amount of rotation into oscillatory modes of periods comparable with that of the rotation.

Ex. 2. Investigate the motion which would result from the introduction in the previous problem of frictional forces proportional to the generalized components of velocity.

The damped motion may be represented by the equations

$$\left.\begin{aligned} a_1\ddot{q}_1 + b_{11}\dot{q}_1 + c_1q_1 + (b_{12} + \beta)\dot{q}_2 = Q_1,\\ a_2\ddot{q}_2 + b_{22}\dot{q}_2 + c_2q_2 + (b_{21} - \beta)\dot{q}_1 = Q_2,\end{aligned}\right\} \quad . \quad . \quad (101.7)$$

provided the 'frictional' coefficients b_{11}, b_{12}, b_{21}, b_{22} have the same meaning as in Art. 59.

As before, we find the *free* vibrations by making $Q_1 = 0$, $Q_2 = 0$, and assuming that the displacements q_1, q_2 vary as $e^{\lambda t}$. On this supposition it is readily seen that

$$\begin{aligned} a_1a_2\lambda^4 + (a_1b_{22} + a_2b_{11})\lambda^3 + (a_1c_2 + a_2c_1 + \beta^2 + b_{11}b_{22} - b^2{}_{12})\lambda^2 \\ + (b_{11}c_2 + b_{22}c_1)\lambda + c_1c_2 = 0 \quad . \quad . \quad . \quad (101.8)\end{aligned}$$

It can easily be proved, with the help of formulae developed by E. J. Routh,[1] that this biquadratic in λ should have the real parts of its roots all negative if ordinary stability is to be secured. Since in mechanical systems the coefficients a_1, a_2, c_1, c_2 are all essentially positive, it follows that stable motion will be maintained only so long as b_{11}, b_{22}, $b_{11}b_{22} - b^2{}_{12}$ remain positive.

Furthermore, if the *forced* vibrations be examined by assuming in equation (10.17) that Q_1, Q_2 vary as $e^{i\omega t}$, with a prescribed value for ω, it will be found that ordinary stability ultimately depends on the relative values of quantities which present themselves as ratios between ω and the coefficients a_1, a_2, b_{11}, b_{12}, b_{22}, c_1, c_2.

These conclusions have reference to the experimental study of vibrations in aircraft, since the 'frictional' coefficients may not all remain positive under actual conditions.

Mention may also be made of the blades of an airscrew running 'in reverse', as may happen with an airship. The effect of forces due to aerodynamic and gyroscopic agencies may then lead to 'flutter' of a blade that is free from this kind of disturbance when the propeller is running in 'forward' gear. This may be regarded as a case of the reversed forces discussed in Art. 19(*e*).

[1] *Advanced Rigid Dynamics*, Art. 287, sixth edition.

Ex. 3. Examine the *small* vibrations indicated by equations (101.3) when $q_1 = x$, $q_2 = y$, $a_1 = a_2 = a$, $c_1 = c_2 = c$, $Q_1 = 2cx$, $Q_2 = 2cy$.

It is to be understood that the extraneous forces Q_1, Q_2 are here restricted in magnitude so as not to violate our assumption as to slight displacements about a position of relative equilibrium.

The equations of the *forced* motion are, by (101.3),

$$a\ddot{x} - cx + \beta\dot{y} = 0,\ a\ddot{y} - cy - \beta\dot{x} = 0.$$

These expressions may conveniently be combined by multiplying the second throughout by $i(=\sqrt{-1})$, then the sum discloses the relation

$$\begin{aligned} a(\ddot{x} + i\ddot{y}) &= \beta(i\dot{x} - \dot{y}) + c(x + iy) \\ &= i\beta\left(\dot{x} - \frac{1}{i}\dot{y}\right) + c(x + iy) \\ &= i\beta(\dot{x} + i\dot{y}) + c(x + iy). \qquad (101.9) \end{aligned}$$

If, for brevity in working, we put

$$x + iy = \zeta, \text{ so that } \dot{x} + i\dot{y} = \dot{\zeta},\ \ddot{x} + i\ddot{y} = \ddot{\zeta},$$

equation (101.9) assumes the form

$$a\ddot{\zeta} - i\beta\dot{\zeta} - c\zeta = 0.$$

The solution, as may readily be verified, is

$$\zeta e^{-i\mu t} = Ge^{i\nu t} + He^{-i\nu t},$$

where $\mu = \frac{\beta}{2a}$, $\nu = \frac{(\beta^2 - 4ac)^{\frac{1}{2}}}{2a}$, and the 'complex' constants G, H are arbitrary.

If $\beta^2 > 4ac$, ν is real, and the motion described is epicyclic, consisting of two superposed circular vibrations of periods $\frac{2\pi}{\mu + \nu}$ and $\frac{2\pi}{-\mu\ \nu}$. The path is, therefore, the same as would be generated by an ellipse which revolves round a fixed origin with the angular velocity μ, the period in the ellipse being $\frac{2\pi}{\nu}$.

The epicyclic may, of course, exhibit an infinite variety of forms. For example, the curve will be of the 'direct' type, Fig. 158(*a*), if $|\mu| > \nu$; and of the 'retrograde' type, Fig. 158(*b*), if the sign of c be reversed.

It is not difficult to realize that the latter of these curves might well connote a serious kind of disturbance in structural systems, since a considerable increase of stress may arise from the sudden reversal of the motion. Special importance must therefore be attached to such systems as we shall have occasion to mention in

connection with motion of this retrograde epicyclic form, and particularly so when account is taken of the dissipative forces which actually operate.

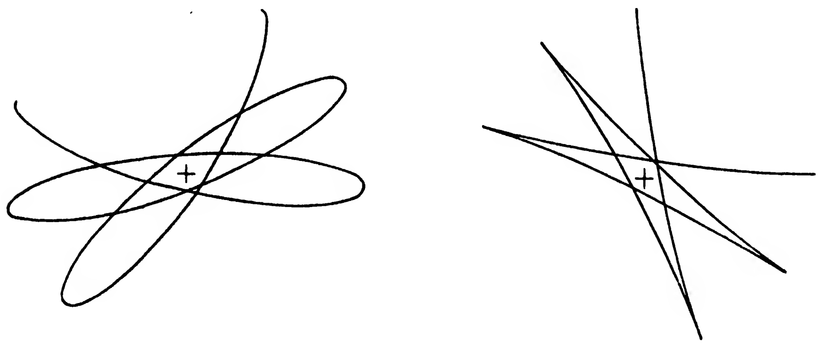


FIG. 158.

102. Aircraft. We may with advantage now discuss the possible causes and the general character of vibrations in the fuselage of aircraft, as a brief study of the matter offers a means of bringing into relation the significance of several results which have been disclosed in this and the preceding chapters. The vibration of the airscrew itself will be brought under review in Chapter VII, after the problem of rotating shafts and discs has been examined in Chapter VI, on the understanding that a specified aeroplane is ultimately treated as a unit.

Since the mechanical causes include both the engine and the airscrew, it will make for conciseness if we assign the term 'order' to the number of complete vibrations which occur in one revolution of the engine or of the airscrew. A vibration of the 'engine first-order' will accordingly have a frequency equal to that with which the crankshaft rotates; also vibrations of the engine first-order and of the airscrew second-order would have the same frequency if the gear reduction-ratio were 2 : 1. It follows from Art. 44 that the phenomenon of *interference* might well contribute to the vibratory motion in cases involving a two-blade airscrew and gearing with a reduction-ratio of nearly, but not exactly, 2 : 1.

For the sake of simplicity we shall, in the main, confine our attention to an aeroplane having one airscrew.

(*a*) Starting with the disturbing agencies which may be ascribed to the *engine*, we notice that the unbalanced primary and secondary effects of the rotating and reciprocating parts of radial engines may be adversely affected by the articulated type of connecting rod examined in Art. 16.

Of much greater importance is the *torque-reaction*, since the

torque generated by any one cylinder of an aero-engine usually includes harmonic components of both half and integral orders, and the components of low order are extremely sensitive to variations in the mean effective pressure of the different cylinders. As a rule, however, the engine half- and first-orders are the most troublesome of the unbalanced components of the torque-reaction. The former is generally the more important, but in geared engines of the radial type the first-order itself may cause appreciable vibration.

Reverting to equation (21.4), we at once realize that, with a particular engine running at a given speed, an experimental determination of the periodic changes in the couples identified with the torque-reaction ($\mathfrak{T}$) should comprise separate studies of the *throttle opening, altitude, and 'tune' or condition of the mechanism,* all of which factors enter into the equation of motion by way of the P-term. From the same expression we further learn that in different engines the *number and arrangement of the cylinders,* and the *firing order* influence the couples in question. If a record exhibits large harmonic components of a particular order, the shape of the torque-reaction graph may, therefore, be due partly to *coincidence in phase* of the components concerned, and partly to *faulty distribution of the fuel, and derangement of both the carburation and ignition in the different cylinders.* Several questions of design may thus present themselves; in radial engines the use of articulated connecting rods occasionally results in the ignition-advance being sensibly different for different cylinders, extending over the range of 43 ± 4 deg. in one instance. This augments the unbalanced harmonic components of low order, and notably the half-order, in the torque contributed by the individual cylinders, even when the distribution leaves little more to be desired. It may be noted, in passing, that the 'blurring' of aerial photographs is frequently an indication that the 'tune' of the engine is not as it should be, the 'roughness' in flight being due chiefly to the consequent increase in vibrations of the engine half-order.

While serious vibrations from harmonic components of the torque-reaction higher than the engine $1\frac{1}{2}$-order are rarely experienced in general practice, such disturbances sometimes lead to localized vibration in parts of the structural system which ultimately fail in 'fatigue'. The considerable difference between the impressed frequency and the natural frequency in torsional vibration of the fuselage is no doubt the chief reason why the high-order harmonic components are usually of secondary importance in the general problem.

On the average, then, the harmonic components of low order in the torque-reaction of an engine with articulated connecting rods

tend to larger values than in a similar engine fitted with forked connecting-rods. Furthermore, serious vibration may be initiated by the components of the engine half-order in the torque-reaction when the mechanism is out of 'tune'; and, in the same circumstance, troublesome vibrations of the engine first-order may occur. The remedial measures to be taken, apart from structural alterations, obviously include improvement of the distribution and the ignition.

(*b*) It is readily inferred from the results obtained in Art. 4 that the *airscrew* may be a contributory factor in this question of vibration. An airscrew may be *statically unbalanced* owing to its centre of gravity not lying on the axis of rotation, then we have an unbalanced force revolving in the plane and with the angular velocity of the blades. The consequences will be essentially the same if the airscrew is *out of alignment*, that is, if a line joining corresponding points on opposite blades does not pass through the axis of rotation. If the airscrew is *out of track*, that is, if corresponding points on the blades do not rotate in the same plane, we have a third kind of disturbing agency, consisting of a couple which acts about an axis perpendicular to that of rotation, and revolves with the same 'speed' as the blades.

Turning to the possible aerodynamic sources of disturbance which arise from this part of aircraft, vibrations may result from an airscrew that is *out of pitch*, which means, broadly speaking, that unequal thrusts are exerted by the separate blades. This introduces a disturbing couple which acts about an axis perpendicular to that of rotation, in the same sense and with the angular velocity of the blades. It is sometimes practicable to mitigate the consequences by giving the blades slightly different pitches, amounting to a difference of 1 deg. or 2 deg. in extreme cases. Moreover, periodic variations in the thrust may take place when the tip of a blade passes an obstruction such as the floats of a seaplane fitted with one engine.

In circumstances where the axis of a two-blade airscrew is changing in direction, as in a turn on the level, there is, besides the steady *gyroscopic couple*, an equal couple which revolves about an axis perpendicular to that of rotation, with twice the angular velocity and in the sense of the blades. We sometimes feel the consequences of this couple when an aeroplane is turning on the ground. It is plain that any vibrations which may thus be brought about will be of the airscrew second-order, and that this rotating couple will be zero if the airscrew has more than two blades.

It is natural to mention next the periodic forces and couples which operate when a two-blade airscrew is *inclined to the relative wind*, or, what amounts to the same thing, when the aeroplane is

pitching or yawing. These forces and couples revolve about an axis perpendicular to that of the shaft, at twice the angular velocity of the blades and in the same sense. While the forces are usually insignificant, considerable vibration may be initiated by the couples, since, at a given speed, they increase with the angle of incidence to the relative wind. An instructive investigation into the general problem of cross-wind has been carried out by H. Glauert.[1]

Vibration in straight and level flight may arise from other sources of aerodynamic origin, the principal ones of which are associated with *flutter of the airscrew, wings and tail,* together with the *effect of the slipstream, and buffeting.* Although readers who are interested in this aspect of the problem must be referred to treatises on aerodynamics, it should be remarked that a certain amount of vibration may be caused by the effect of the slipstream on the wings and tail-unit of aircraft. The reason for mentioning the point here is that these disturbances are sometimes characterized by long periods compared with that of the engine, so that the results of Exs. 1, 2, 3 of Art. 101 are involved, in so far as those problems relate to the vibratory motion of structural members with which is associated a relatively long period. A question of experimental interest is to be found in the extent to which the rotation of the slipstream can cause a disturbance of the 'buffeting' kind to originate only on one side of an aeroplane.

Thus, to summarize, we see that in straight flight at constant altitude serious vibration will not in general be initiated by the aerodynamic couples on the blades of an airscrew, except at speeds corresponding to the 'stalling point'.

(*c*) It is clear from our treatment of gyroscopic forces that additional disturbing agencies will be introduced when aircraft are *executing manœuvres,* but some will operate only in special cases. For example, aerodynamic couples of the airscrew second-order will probably occur only in a faulty banked-turn; and the increase in the torque-reaction which follows disturbed carburation will practically be restricted to the non-supercharged type of engine.

In all manœuvres, however, the incidence of the airscrew-shaft produces an aerodynamic couple which revolves about an axis perpendicular to that of the shaft, in the direction and at twice the angular velocity of the airscrew. This may conveniently be represented by a rotating vector. The accompanying gyroscopic couple is similar, and it also may be represented by a vector, rotating in the same sense as the blades. But, for a turn at constant altitude, we must refer the aerodynamic vector to axes fixed in the aeroplane, and the gyroscopic vector to axes fixed in space. The phase-difference between these vectors accordingly alters as the angle of

[1] *R. & M.*, No. 642 (1919).

'banking' is increased, in conformity with equation (100.8), and this alteration in phase must be taken into account in finding the resultant couple. Hence the related vibration may or may not increase as a consequence of the manœuvre, since any variation will depend on the phase-difference between the component couples. By this means we can, as the rate of turning is increased, trace the manner in which both the gyroscopic and the aerodynamic couples increase, the latter because of the increase in incidence of the airscrew-shaft.

In addition, there is, as already pointed out, the possibility of vibration due to disturbed carburation, the principal effect of which is to augment the harmonic components of the engine half- and first-orders in the torque-reaction. This is commonly indicated by 'spluttering' of the engine.

It is to be inferred from the preceding treatment that the vibration which takes place will in general be different for 'port' and 'starboard' turns of a given aeroplane, and that the magnitude and character of the disturbed motion will be influenced also by the number of airscrews and their relative directions of rotation.

(*d*) A troublesome case of vibration in a fuselage may, therefore, be produced by one or more of several unbalanced effects, associated with the secondary harmonic component of the unbalanced parts of the engine, increase in the engine first-order component of the torque-reaction, an imperfectly manufactured airscrew, periodic variations in the moment of inertia of a two-blade airscrew, as well as the effect of cross-wind and of the gyroscopic agencies associated with manœuvres.

So far little has been said about a geared drive. The torque-reaction of an ungeared engine is, of course, transmitted with but slight modification to the fuselage. With a geared engine, however, all the harmonic components of the torque are, as will be explained in Art. 116, magnified in proportion to the gear reduction-ratio. On account of this modification the torque-reaction of geared engines sometimes leads to a small increase in vibrations that arise from the torsional oscillation of the crankshaft. In extreme cases the crankshaft may thus induce severe vibration in the main bearings and so in the fuselage, even when the engine is of the ungeared type, for, according to Art. 97, the torsional vibration of any part of the system will generally be accompanied by transverse motion. Results to be obtained in the next chapter demonstrate that vibration of relatively small amplitude may be caused by imperfectly formed teeth on the gears, and by the reversal of the torque at the gearing when a crankshaft is passing through a state of resonance in torsional vibration.

In making a comparison between ungeared and geared engines

for a given aeroplane, account must be taken of the difference in size of the airscrews required for the two kinds of drive.

When any one of the forces or couples mentioned above operate with a frequency that approximates to one of the natural frequencies of a structural member, the part concerned is liable to serious vibration. The most severe stresses will generally be found to occur in a fast turn or a 'spin'.

If the relative positions of the nodes in a specified mode of vibration of a fuselage were known, we could estimate the natural frequency by the methods of Arts. 96 and 97, on the supposition that this part of the system is practically equivalent to a beam

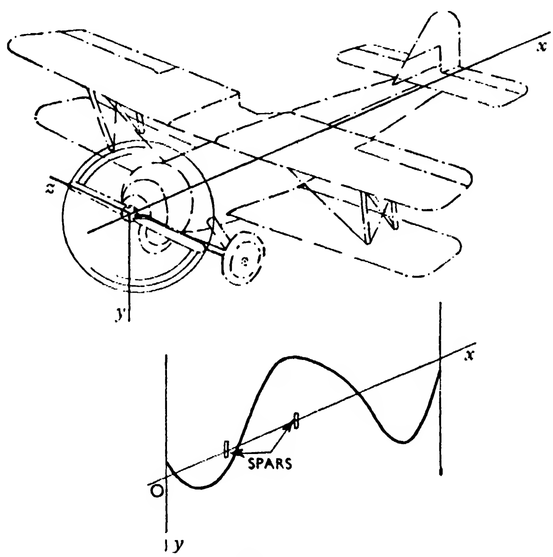


FIG. 159.

having non-uniform properties as to stiffness and loading per unit length along the principal axes of the structure. There is, however, no reason to suppose that the ends of a fuselage remain stationary throughout a disturbance about a position of relative equilibrium; as a matter of fact, the ends usually vibrate in all modes. But consideration of the distribution of the principal loads over the system suffices to indicate that in flexural vibrations the nodes will, on the average, be found near the engine- or back-plate, at a point situated between the spars, at another point some distance in front of the stern-post, and near the bay-points in the higher modes. Fig. 159 approximately represents the disposition of the stationary points in a certain fuselage when the aeroplane was vibrating in a mode containing three nodes, the deflection-curve in the (x,y) plane being greatly enlarged for purposes of illustration.

Considerable difficulties would, even so, attend an estimation of the frequency and the displacement in a transverse plane, as at present it is impracticable to draw general inferences from the available information on the values of the constraints which are actually exerted by the several connections between a fuselage and such components as its engine, wings, and tail-unit. The same may be said about the effective inertia and moment of inertia of the wings. These remarks apply to oscillations of both the flexural and torsional kinds, for, by Art. 97, the corresponding displacements are not in general independent. Hence it is not always permissible to treat the system as one capable of describing independent vibrations about the principal axes, and a full study cannot be effected without recourse to tests of either an aeroplane or its model.

If the analysis is extended to cover vibrations in a manœuvre, the solution is naturally rendered still more complicated by reason of the gyroscopic phenomena discussed in Arts. 99–101. Displacements of any one type are then no longer affected solely by forces of the same type; and the phase at any instant is in general not uniform throughout the system, being different for different co-ordinates. Taking another view of our results for flight under these conditions, we see that certain points on the system may describe a path which is elliptic harmonic; and the path may be an epicyclic, when a peculiar, and significant, kind of motion may present itself for consideration in connection with slender parts of the structural system. A reasonably complete investigation into the flexural vibration of a fuselage would evidently necessitate the use of at least three vibrographs for recording simultaneously fluctuations in the torque-reaction, and vibrations parallel to the two transverse axes of the structure.

At what may be called the 'cruising' speed we might well have the fuselage of a small aeroplane executing vertical vibrations in a mode with three nodes, as represented by Fig. 159. What actually happens when the speed of the engine is increased is much the same as if a fourth node were introduced at the airscrew and, simultaneously, the original three nodes were moved towards the stern-post. At still higher speeds of excitation this imaginary process is repeated when additional nodes enter the system. Disturbances of this kind may or may not be caused by the primary harmonic component of the engine, since the secondary component excites the transverse mode of vibration with five nodes which is described by the fuselage of certain types of aircraft.

The ratios between the natural frequencies in flexural vibration are, as might be anticipated, influenced by the size and class of the aeroplane concerned. Tests with a small number of comparable frames showed that the frequencies involved in modes containing

2, 3, 4, and 5 nodes were approximately related to each other by the ratios of the first four positive zeros of $J_0(x)$, i.e., 1 : 2·3 : 3·6 : 4·9. These ratios are merely given as a rough guide as to what may be experienced in particular cases, for it is impossible to generalize on this point without regard to the class of aeroplane under examination.

If an aeroplane is driven by more than one airscrew, an application of the principle of superposition clearly demonstrates that the resultant vibration parallel to a specified axis may vary over a considerable range, by reason of slight changes in the relative phases of the engines. But the chief sources of disturbances will in general be found to be the same as those for the single-airscrew system mentioned above, with one exception. The periodic variation in the moments of inertia of two-blade airscrews would not, with the customary arrangement of the engines, greatly affect the vibration of the fuselage of an aeroplane driven by an even number of airscrews.

103. Hamilton's Equations of Motion. It will no doubt have been noticed that although Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r \quad . \quad . \quad . \quad . \quad (103.1)$$

are general, in so far as the q's may be any independent variables $(q_1, q_2, \ldots, q_n)$ which serve to fix the position of a given system, the formula has the disadvantage of leading to differential equations of the second order, and not of the first order. The defect may be remedied by introducing, from equation (19.7), the generalized components of momentum

$$p_r = \frac{\partial T}{\partial \dot{q}_r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103.2)$$

where $r = 1, 2, \ldots, n$. Thus, on replacing the n Lagrangian equations by $2n$ differential equations of the *first order* in q_r, p_r as functions of the time t, we have

$$\dot{q}_r = f_r(q, p, t), \ \dot{p}_r = \frac{\partial T}{\partial q_r} + Q_r, \quad . \quad . \quad . \quad (103.3)$$

where $r = 1, 2, \ldots, n$. The first set of these expressions is obtained by solving equations (103.2) for the velocities $\dot{q}_r$ as functions of the co-ordinates q_r, the momenta p_r and the time t.

We may next introduce, after the manner of S. D. Poisson,[1] the function

$$K = \sum_{r=1}^{n} p_r \dot{q}_r - T, \quad . \quad . \quad . \quad . \quad . \quad . \quad (103.4)$$

[1] *Jour. de l'École Polyt.*, vol. 8, page 266 (1809).

with $\dot{q}_r$ expressed in terms of the co-ordinates, the momenta, and the time. Now

$$dK = \sum_{r=1}^{n}(p_r d\dot{q}_r + \dot{q}_r dp_r) - dT,$$

by equation (103.4), and

$$dT = \sum_{r=1}^{n}\left(\frac{\partial T}{\partial q_r}dq_r + \frac{\partial T}{\partial \dot{q}_r}d\dot{q}_r\right) + \frac{\partial T}{\partial t}dt$$

$$= \sum_{r=1}^{n}\left(\frac{\partial T}{\partial q_r}dq_r + p_r d\dot{q}_r\right) + \frac{\partial T}{\partial t}dt,$$

by equations (103.2). These results enable us to write the complete differential of K in the form

$$dK = \sum_{r=1}^{n}\left(\dot{q}_r dp_r - \frac{\partial T}{\partial q_r}dq_r\right) - \frac{\partial T}{\partial t}dt,$$

which is comprehended in the expressions

$$\frac{\partial K}{\partial q_r} = -\frac{\partial T}{\partial q_r}, \quad \frac{\partial K}{\partial p_r} = \dot{q}_r, \quad \frac{\partial K}{\partial t} = -\frac{\partial T}{\partial t},$$

where $r = 1, 2, \ldots, n$. In performing these partial differentiations, it is, of course, necessary to treat K as a function of the co-ordinates, the momenta, and the time; and T as a function of the co-ordinates, the velocities, and the time. Thus equations (103.3) become

$$\dot{q}_r = \frac{\partial K}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial K}{\partial q_r} + Q_r, \quad \ldots \quad (103.5)$$

where, as before, $r = 1, 2, \ldots, n$.

Let us now suppose the system to include generalized forces Q_r which are partial derivatives of a function of the corresponding co-ordinates, and possibly of the time. Writing $-V$ for this function, we have

$$Q_r = -\frac{\partial V}{\partial q_r},$$

with $r = 1, 2, \ldots, n$, and

$$-\left(\frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \ldots + \frac{\partial V}{\partial q_n}\delta q_n\right)$$

for the virtual work of the prescribed forces.

It is convenient to introduce the function

$$H = K + V, \quad \ldots \quad (103.6)$$

so that H is a function of the co-ordinates, the momenta, and the time, for then

$$\frac{\partial H}{\partial p_r} = \frac{\partial K}{\partial p_r}, \quad \frac{\partial H}{\partial q_r} = \frac{\partial K}{\partial q_r} - Q_r$$

in equations (103.5), whence

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial q_r}, \quad . \quad . \quad . \quad . \quad (103.7)$$

where $r = 1, 2, \ldots, n$.

The motion of the system is thus completely specified by a set of $2n$ differential equations of the *first order*, since they suffice to determine the n co-ordinates and the n momenta as functions of t. These $2n$ relations are known as *Hamilton's canonical equations of motion* for a system in which the number of degrees of freedom is equal to the number of distinct co-ordinates.[1] The significance of these equations is to be found in the fact that they constitute the basis of the most advanced theory of dynamics.

In many cases, however, the geometrical relations, which determine the absolute position of any given particle of the system in terms of the generalized co-ordinates, do not contain the time explicitly, and the kinetic energy T is a homogeneous quadratic function of the velocities. From equations (103.2) it follows that in this case $\sum_{r=1}^{n} p_r \dot{q}_r = 2T$, and, therefore, K as defined by equation (103.4) is equal to T. But these two quantities are expressed differently, K being a function of the co-ordinates and the momenta, whereas T is a function of the co-ordinates and the velocities. Hence we realize that H as given by equation (103.6) now has the value $T + V$, and it is called the *Hamiltonian function.* Moreover, T does not contain the time explicitly, nor do the equations (103.2) which enable us to state the velocities in terms of the momenta; therefore H does not involve the time explicitly, for V is supposed not to contain the time explicitly. This means, in other words, that H is fully determined by the phase of the system, and it is not essential to know the instant at which the given phase was assumed. Calculating the rate at which H changes with regard to the independent variable t, we find, with the aid of equations (103.7),

$$\frac{dH}{dt} = \sum_{r=1}^{n} \left(\frac{\partial H}{\partial q_r} \dot{q}_r + \frac{\partial H}{\partial p_r} \dot{p}_r \right) = 0, \quad . \quad . \quad . \quad (103.8)$$

whence it appears that H remains constant notwithstanding the varying phase. The Hamiltonian equations (103.7) consequently

[1] Sir W. R. Hamilton, *Brit. Assoc. Report for 1834*, page 513, and *Phil. Trans. Roy. Soc.*, vol. 125, page 95 (1835).

give an equation of energy, or of work, or of bending moment, in the case of *unvarying* relations.

Equations of the form (103.7) apply also in the case of *varying* relations, but a different meaning is then attached to H. To exhibit this difference we put

$$H = \sum_{r=1}^{n} p_r \dot{q}_r - T + V, \quad . \quad . \quad . \quad . \quad (103.9)$$

and treat H as a function of the co-ordinates, the momenta, and the time. Remembering that the *internal* forces of the system are supposed to be conservative, with a potential energy V which is a function of the co-ordinates only, we find, on performing a variation δ on both sides of equation (103.9),

$$\begin{aligned}\delta H &= \sum_{r=1}^{n}\left\{\dot{q}_r \delta p_r + \left(p_r - \frac{\partial T}{\partial \dot{q}_r}\right)\delta \dot{q}_r - \frac{\partial}{\partial q_r}(T - V)\delta q_r\right\} \\ &= \sum_{r=1}^{n}\{\dot{q}_r \delta p_r - \dot{p}_r \delta q_r\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103.10)\end{aligned}$$

The latter expression is often referred to as the *Hamiltonian form of the equation of virtual work.* In this case, then,

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \; \dot{p}_r = -\frac{\partial H}{\partial q_r},$$

as in equations (103.7), but instead of equation (103.8) we have

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial t} + \sum_{r=1}^{n}\left(\frac{\partial H}{\partial p_r}\dot{p}_r + \frac{\partial H}{\partial q_r}\dot{q}_r\right) \\ &= \frac{\partial H}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103.11)\end{aligned}$$

It is clear that the energy integral H will not be constant unless the generalized components of the force are of the form $Q_r = -\frac{\partial V}{\partial q_r}$, with V a function of the co-ordinates only, even when the geometrical relations do not involve the time explicitly. These conditions are not fulfilled in many engineering problems, as in, for example, the damped motion of Art. 59, where $Q_r = -\left(\frac{\partial V}{\partial q_r} + \frac{\partial F}{\partial \dot{q}_r}\right)$, F being a quadratic function of the generalized velocities. Under these conditions

$$\begin{aligned}\frac{dH}{dt} &= -\sum_{r=1}^{n}\frac{\partial H}{\partial p_r}\frac{\partial F}{\partial \dot{q}_r} = -\sum_{r=1}^{n}\frac{\partial F}{\partial \dot{q}_r}\dot{q}_r \\ &= -2F,\end{aligned}$$

connoting, as is to be expected from Art. 59, that $2F$ represents the rate at which the energy of the system is dissipated through

forces of a frictional kind. Denoting $-\dfrac{\partial F}{\partial \dot{q}_r}$ by G_r, and introducing the *Lagrangian function*

$$L = T - V,$$

we obtain equations (103.1) in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r}\right) = G_r \quad . \quad . \quad . \quad . \quad . \quad . \quad (103.12)$$

In view of equations (103.5) and (103.6), it is now seen that the 'canonical' equations for this type of damped motion are

$$\dot{q}_r = \frac{\partial H}{\partial p_r},\ \dot{p}_r = -\frac{\partial H}{\partial q_r} + G_r,$$

with $r = 1, 2, \ldots, n$.

Furthermore, the free undamped motion is, from equation (103.12), specified by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (103.13)$$

where $r = 1, 2, \ldots, n$. This form may obviously be used instead of (52.3) in the general theory of vibrations, when L is usually referred to as the *kinetic potential.*

At this stage of the work it is worth while to remark that in Hamilton's equations (103.7) the momenta, p_r, are variables which define the configuration of the system, and there is no reason why the momenta should not be used as co-ordinates, provided an appropriate meaning is given to the term 'co-ordinate'.

To throw light on this aspect of the matter, imagine, for a moment, that our object of study is the motion of the large suspension bridge indicated by Fig. 160. In a modern structure of this

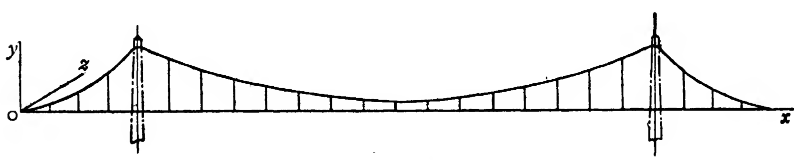


FIG. 160.

kind the 'unstiffened' trusses deform almost to the full extent and shape of the chains or cables when displaced from a position of rest, and to this degree the system is practically equivalent to a flexible cable of a corresponding cross-sectional area. A model could be constructed on this basis, with both the statical and dynamical loads represented, at least as a first approximation, by forces applied at each of the panel-points on the model. The motion of the model may, with the help of equations (103.7), be examined by assigning a pair of variables q_r and p_r to each panel-point. If the

motion were restricted to the plane of the paper, and there were m such panel-points, a complete specification of the *phase* at a given instant would contain $2m$ co-ordinates (q, p); and a full description of the *state* of the system would involve $2m + 1$ co-ordinates (q, p, t). It is not difficult, in a theoretical sense, to extend this idea so as to cover motion in the space formed by the axes Ox, Oy, Oz of Fig. 160.

Returning to the consideration of dynamical systems in general, we thus realize that there will be a gain in clearness if we avail ourselves of a geometrical representation of the problem in the space defined by the co-ordinates q, p, t, and in this way investigate the successive states of the system in relation to the *path* of a point in this hypothetical space. This path would be determined by the canonical equations (103.7) in the case of a *conservative system.* But it must be borne in mind that the convention is only one of convenience, and to emphasize this we shall use in the present connection the phrases 'phase-space', 'state-space', and 'state-co-ordinates'. It is natural to use also such geometrical terms as 'intersection' and 'adjacent' when speaking of these paths.

The introduction of a state-space of n dimensions would lead to still greater simplification of complex systems if we could discover a characteristic property of the paths, such as, for example, a 'stationary' property analogous to that found in Art. 55. Attention may on this account be drawn to the fact that equations (103.7) are both comprehended in equations (103.10), the reason being that the latter may be written in the more symmetrical form

$$\delta\left(\sum_{r=1}^{n} p_r dq_r - H dt\right) = d\left(\sum_{r=1}^{n} p_r \delta q_r - H\delta t\right).$$

A little consideration of this equality suggests that the differential form

$$\sum_{r=1}^{n} p_r dq_r - H dt$$

relates to a 'variational' property which may be worth further study.

The integral of this form is known as the *Action*,[1] and if it be denoted by A, then

$$A = \int \sum p_1 dq_1 + p_2 dq_2 + \ldots + p_n dq_n - H dt. \quad (103.14)$$

Hence, remembering that a path in the state-space implied in this expression for A is determined by equations (103.7), and that there is only one solution of a set of differential equations of the first

[1] See J. C. Maxwell, *Matter and Motion* (new edition, 1920) for a physical interpretation of Action.

order satisfying given initial conditions, we learn that only one path of the system can pass through any point in the state-space.

The value of the Hamiltonian method is now apparent, for with the aid of it we have reduced problems pertaining to the motion of any system of the prescribed type to the study of a line in the corresponding state-space, and so attained a position which presents a geometrical view of dynamics. It follows from a previous remark that the geometry in question does not conform with our notions of ordinary space, and no practical purpose would be served by attempting to define, for example, 'the distance between two points' in state-space. The really important thing to understand is that, as the state changes, we can in this manner obtain a curve that exhibits the 'history' of the system.

104. Principle of Least Action. Limiting ourselves to the case where the canonical equations can be put in the form (103.7), we shall now prove that the paths in state-space are the *extremal curves* of the *Action integral* (103.14). The procedure consists in taking two points on the same path and evaluating the Action integral along the path bounded by the given points, and in this manner demonstrating that the result is 'stationary' when compared with the value of the integral along adjacent curves that join the same two points.

It will simplify matters if we observe, from the preceding analysis, that the adjacent curves cannot themselves be paths, since two paths never intersect; and that, as a line integral defines the Action along any curve, whether a path or not, the Action is determinate when the curve is known, so that the Action may be treated as a function of the prescribed curve.

If, for the sake of symmetry, we consider a space of n dimensions in which the co-ordinates of any point are signified by $x_1, x_2, \ldots, x_n$, our problem becomes one of finding the extremal curves for a given line integral

$$I = \int C_1 dx_1 + C_2 dx_2 + \ldots + C_n dx_n, \quad . \quad . \quad (104.1)$$

where the C-coefficients are known functions of the x-co-ordinates. This expression is connected to equation (103.14) by the relations

$$\left.\begin{aligned} C_r &= p_r,\ C_{n+r} = 0,\ C_{2n+1} = -H, \\ x_r &= q_r,\ x_{n+r} = p_r,\ x_{2n+1} = t, \end{aligned}\right\} \quad . \quad . \quad (104.2)$$

where $r = 1, 2, \ldots, \frac{1}{2}(n-1)$. We note, for reasons which will appear presently, that an *odd* number $(2n+1)$ of dimensions is involved in this state-space.

Suppose the curve along which the integral (104.1) is to be evaluated is given by the equations

$$x_r = f(x_r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104.3)$$

where $r = 1, 2, \ldots, \frac{1}{2}(n-1)$, and f is an independent variable or parameter that enables us to pass from point to point along the curve of integration. If the curve (104.3) and the limits (0, 1) of integration are specified, I of equation (104.1) will be a fixed number and not a variable. Since we wish to vary I, it is therefore necessary to introduce a second independent variable or parameter, for the purpose of changing, when the value assigned to it is changed, the curve of integration. A change of the curve of integration will in general lead to a variation of the end-points. Writing k for the second parameter, equations (104.3) transform to

$$x_r = f, k(x_r), \qquad \ldots \ldots \quad (104.4)$$

where $r = 1, 2, \ldots, \frac{1}{2}(n-1)$, and k remains constant along any one curve of integration. That is to say, the variables f, k are independent, the purpose of the first being to distinguish points on a curve along which the second is constant, and that of the second parameter to distinguish the curves of integration.

The end-points of the curves may now be designated by the equations

$$x_r^0 = f^0, k(x_r), \; x_r' = f', k(x_r), \qquad \ldots \ldots \quad (104.5)$$

where $r = 1, 2, \ldots, \frac{1}{2}(n-1)$, and f^0, f' refer to certain constants which are independent of k. The variable k will not occur in the last set of equations if the end-points of all the curves of integration are the same. Stated otherwise, although the end-points of these curves may vary, the limits of the integral (104.1), when expressed as an ordinary definite integral with the variable f, do not vary from curve to curve, i.e. as the value of k is varied.

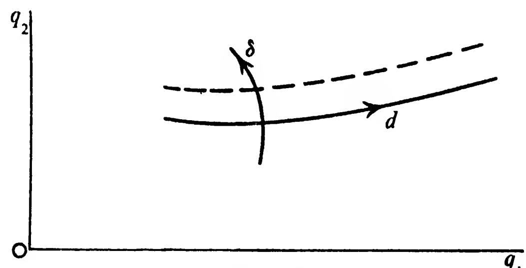


FIG. 161.

Since the x's of equations (104.4) are functions of two independent variables, it is necessary to discriminate between the two partial differentiations involved in any variation. The first, referring to Fig. 161, may be indicated by

$$dx_r = \frac{\partial x_r}{\partial f} df, \qquad \ldots \ldots \quad (104.6)$$

and the second by $$\delta x_r = \frac{\partial x_r}{\partial k} dk, \qquad \ldots \ldots \quad (104.7)$$

on the understanding that k is a constant in the former of these operations, and f is a constant in the latter.

Thus we find the differential of I by integrating with regard to f the partial differential of the integrand with regard to k, which leads to

$$\delta I = \int \sum (C_s \delta dx_s + \delta C_s dx_s) \quad . \quad . \quad . \quad (104.8)$$

But

$$\delta dx_s = \frac{\partial}{\partial k}\left(\frac{\partial x_s}{\partial f} df\right) dk$$

$$= \frac{\partial^2 x_s}{\partial k \partial f} df dk,$$

and this is equal to $d\delta x_s$, since the order of differentiation may be reversed in finding the second of these derivatives. The first group of terms of the summation in the integrand (104.8) may accordingly be integrated by parts, with the aid of

$$\int C_s d\delta x_s = C_s(\delta x_s)_0^1 - \int dC_s \delta x_s,$$

the limits of integration being (0, 1). Now the two partial differentials of C_s are easily seen to be

$$dC_s = \sum_r \frac{\partial C_s}{\partial x_r} dx_r, \quad \delta C_s = \sum_r \frac{\partial C_s}{\partial x_r} \delta x_r, \quad . \quad . \quad (104.9)$$

and these forms may be introduced in equation (104.8), whence we deduce

$$\delta I = \left(\sum_s C_s \delta x_s\right)_0^1 + \int \sum_{r,s} \left(\frac{\partial C_s}{\partial x_r} dx_s \delta x_r - \frac{\partial C_s}{\partial x_r} \delta x_s dx_r\right)$$

$$= \left(\sum_s C_s \delta x_s\right)_0^1 + \int \sum_{r,s} C_{rs} \delta x_s dx_r, \quad . \quad . \quad . \quad . \quad (104.10)$$

where $$C_{rs} = \frac{\partial C_r}{\partial x_s} - \frac{\partial C_s}{\partial x_r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104.11)$$

and $r, s = 1, 2, \ldots, (2n + 1)$. It is to be noticed, from the last relation, that $C_{rs} = - C_{sr}$.

If the end-points of the curves of integration be all the same, the first member on the right in equation (104.10) will vanish, for every δx_s is then zero at the upper and lower limits of integration. In this case the expression accordingly reduces to

$$\delta I = \int \sum_{r,s} C_{rs} \delta x_s dx_r, \quad . \quad . \quad . \quad . \quad . \quad (104.12)$$

C_{rs} being determined by equations (104.11) in terms of the given coefficients of I. In order that δI shall be zero for arbitrary values of the partial differentials δx_s, as is required, the necessary and

sufficient conditions are that the coefficients of each δx_s in th integrand (104.12) shall vanish separately. These considerations lead at once to the 2n + 1 equations

$$\sum_r C_{rs}dx_r = 0, \quad . \quad . \quad . \quad . \quad (104.13)$$

where $s = 1, 2, \ldots, (2n + 1)$. This set of expressions evidently defines the extremal curves for the specified integral I, seeing that the dx's are the partial differentials of x with regard to f, as we pass from point to point along a particular curve of integration. Since the procedure has led to $2n + 1$ linear homogeneous equations for the $2n + 1$ differentials dx_r corresponding to the values $r = 1, 2, \ldots, (2n + 1)$, it follows that there will be no solution other than the irrelevant one $dx_r = 0$ unless the determinant formed by the coefficients is zero. This indicates that special precautions must be taken to secure extremal curves for the integral under examination. But the relation $C_{rs} = - C_{sr}$, from equations (104.11), signifies that an interchange of rows and columns in the determinant of the coefficients merely results in the determinant being multiplied by $(-1)^{2n+1}$; hence the value of the determinant is unaffected by such an interchange. Therefore the determinant formed by the coefficients of equations (104.13) will always be zero provided the number of co-ordinates $x_1, x_2, x_3, \ldots$ is *odd*. The fact that this condition is here fulfilled, as pointed out in connection with equations (104.2), thus demonstrates that extremal curves for the integral (104.1) exist.

From the above results we gather that the extremal curves for the line integral

$$I = \int \sum_s C_s dx_s$$

are the solutions of the $2n + 1$ linear homogeneous differential equations

$$\sum_r C_{rs}dx_r = 0,$$

where $r = 1, 2, \ldots, (2n + 1)$, and the C_{rs}-coefficients are defined by equations (104.11). In applications of the theory, it is well to use the relation $C_{rs} = - C_{sr}$; and to take account of the obvious fact that the $2n + 1$ differential equations are not independent, for this means that one of them will be satisfied by a solution of the remaining $2n$ equations.

If the values (104.2) be introduced in the Action integral

$$A = \int \sum_s p_s dq_s - Hdt,$$

The foregoing theory demonstrates that

$C_{rs} = 0$ when $r, s = 1, 2, \ldots, \frac{1}{2}(n-1)$;
$C_{s, \frac{1}{2}(n-1)+s} = -1$ when $s = 1, 2, \ldots, \frac{1}{2}(n-1)$;
$C_{s, \frac{1}{2}(n-1)+r} = 0$ when $r, s = 1, 2, \ldots, \frac{1}{2}(n-1)$, and $s \neq r$;
$C_{s, n} = -\dfrac{\partial H}{\partial q_s}$ when $s = 1, 2, \ldots, \frac{1}{2}(n-1)$;
$C_{\frac{1}{2}(n-1)+s, n} = -\dfrac{\partial H}{\partial p_s}$ when $s = 1, 2, \ldots, \frac{1}{2}(n-1)$.

It may be recalled that in this work we effect the partial differentiations of C_{rs} in equations (104.11) by treating all the variables, i.e. (q, p, t), as independent, in consequence of which the partial derivative of any one of them with regard to any other is zero.

By equations (104.13), then, the differential expressions for the extremal curves of the Action integral are

$$dp_s + \frac{\partial H}{\partial q_s} = 0,$$

$$dq_s - \frac{\partial H}{\partial p_s}dt = 0,$$

$$\sum_s \left(\frac{\partial H}{\partial q_s}dq_s + \frac{\partial H}{\partial p_s}dp_s\right) = 0,$$

where $s = 1, 2, \ldots, \frac{1}{2}(n-1)$. Again, these $2n + 1$ equations are not independent, since the last one in the set is a linear combination of the remaining $2n$ relations. From the fact that the latter $2n$ expressions are the canonical equations (103.7) which determine the motion of the system, we conclude that *any dynamical system having a Hamiltonian function H will move so that a representative point on it describes in a state-space of $2n + 1$ dimensions an extremal curve of the Action integral*

$$A = \int p_1 dq_1 + p_2 dq_2 + \ldots + p_n dq_n - H dt.$$

We have in this manner proved that the Action along the actual path has a 'stationary' value compared with the Action along all other curves which join the initial and final states of the system. Although this merely establishes a 'stationary' property, it can be shown [1] that for small ranges the Action is a *minimum*, and to this extent the name *principle of least Action* is justified.

We notice, in passing, that this principle only gives us the *shape* of the path under consideration; it does not, for example, yield any information as to the velocities or the intervals of time involved in any change.

It is now possible to write down the equations of motion of the

[1] E. T. Whittaker, *Analytical Dynamics*, Art. 103, third edition.

system once we have any $2n + 1$ co-ordinates that define its state. For example, if it be required to express the equations in terms of any state-co-ordinates (θ, ϕ, τ) which are distinct functions of the q's, the p's, and t, we simply write the Action integral in terms of the given co-ordinates. This is effected by substituting for each p its value in the new co-ordinates, replacing each dq by

$$\sum_m \left(\frac{\partial q}{\partial \theta_m} d\theta_m + \frac{\partial q}{\partial \phi_m} d\phi_m\right) + \frac{\partial q}{\partial \tau} d\tau,$$

and likewise transforming H, dt. If the new form of the Action integral be

$$A = \int \sum_m (\Theta_m d\theta_m + \Phi d\phi_m) + M d\tau,$$

then the expressions given by putting $r = \theta$, $s = \phi$ in (104.13) are the required equations of motion.

105. The formulae just deduced refer to a system for which the *initial and final states are fixed.* In the more general case, where the initial and final states may be *varied*, the partial differential included in equations (104.10) has, instead of zero, the value

$$\delta A = \left(\sum_{m=1}^{n} p_m \delta q_m - H dt\right)_0^1 \quad . \quad . \quad . \quad . \quad (105.1)$$

when the curve of integration is an actual path of the system. Under these conditions the Action integral must be a function of $2n + 2$ independent variables, for only one of the final state-co-ordinates is at our disposal when the initial state is fixed, the other final state-co-ordinates being specified by this one.

Suppose we use the initial values (q^0, t^0) and the final values (q, t) as the independent variables in which to express the Action integral. This would enable us to replace equation (105.1) by the set of relations

$$\frac{\partial A}{\partial q_m} = p_m, \quad \frac{\partial A}{\partial q^0_m} = -p^0_m, \quad \frac{\partial A}{\partial t} = -H, \quad \frac{\partial A}{\partial t^0} = H^0, \quad . \quad . \quad (105.2)$$

where $m = 1, 2, \ldots, n$, and the zero suffix refers to initial values. Since H is known as a function of the q's, the p's, and t, we can, on the basis of the last set of equations, substitute $\dfrac{\partial A}{\partial q_r}$ for p_r, and so obtain an expression which may be represented symbolically by

$$\frac{\partial A}{\partial t} = -H\left(q, \frac{\partial A}{\partial q}, t\right). \quad . \quad . \quad . \quad . \quad (105.3)$$

In this way we have deduced a partial differential equation of the first order and second degree for the function A, containing $n + 1$ independent variables (q, t). Such an equation may, of course, have a number of solutions, but the Action integral, A, is

characterized by the circumstance that it includes certain arbitrary constants which exhibit the partial derivatives of A with respect to these constants as the initial momenta $p^0{}_m$, the arbitrary constants being the initial co-ordinates $q^0{}_m$. If it were possible to find the special solution of equation (105.3), or to express the Action along a path as a function of the initial and final values (q^0, t^0) and (q, t), respectively, the relations

$$\frac{\partial A}{\partial q_m} = p_m, \quad \frac{\partial A}{\partial q^0{}_m} = -p^0{}_m$$

of equations (105.2) would, *by algebraic methods alone*, afford means of writing down all but one of the final state-co-ordinates (θ, ϕ, τ) in terms of this one and the initial state-co-ordinates. We should then be in a position to determine the phase of the system at any future instant from a knowledge of the initial conditions of motion.

Sir W. R. Hamilton remarked that the Action integral, when expressed in terms of the initial and final states of the system, satisfies equation (105.3), but C. G. J. Jacobi [1] went further, and showed that it is unnecessary to restrict ourselves to the Action integral in order to determine the motion. Stated briefly, he demonstrated that any solution of equation (105.3) containing the proper number of arbitrary constants, provided they enter in the right way, will serve just as well as the Action integral, which is only a special solution of (105.3).

This short account of the principle of least Action must close here, as continued progress along the present lines would lead to *contact transformations*, an adequate discussion of which lies beyond the range of this book. For further information on the general subject the reader may be referred to Professor E. T. Whittaker's treatise,[2] and to the work of J. E. Wright,[3] and of E. Cartan. [4]

[1] *Vorlesungen über Dynamik*, No. XX, (Leipzig, 1866).

[2] *Analytical Dynamics*, Chaps. X–XII, third edition.

[3] *Invariants of Quadratic Differential Forms*, page 80 (No. 9 of Cambridge Tracts in Mathematics and Mathematical Physics, 1908).

[4] *Leçons sur les Invariants Intégraux* (Paris, 1922).

CHAPTER VI

ROTATING SHAFTS AND DISCS

106. Whirling of Shafts. If a rotating shaft be regarded as a beam, reference to Art. 86 at once discloses the fact that a number of natural modes of vibration will be associated with the shaft. The frequencies for the rotating shaft and a corresponding beam will, in a given mode of transverse vibration, actually differ only by the effect of constraints due to the torque, and to the rotation implied in the treatment of Art. 19(*d*). This analogy suggests an experimental method of determining the natural periods of a specified shaft, in which account is necessarily to be taken of the constraints exerted by the bearings, pulleys, 'shrunk on' rings, axial thrust (Art. 89), and so forth.

From the previous theory we further realize that if, for any reason, the centre of gravity of the rotating system does not lie on the axis of rotation, the phenomenon of resonance will occur when the shaft revolves with a periodicity that approximates to one of the natural periods of transverse vibration. Although an increase in the deflection will occur in this state of resonance, commonly known as *whirling* or *whipping* in the case of shafts, the most serious consequences are usually to be found in the damage thus done to the bearings. For this reason the 'critical' or whirling speeds of a shaft should be avoided as far as possible in the normal operation of machinery.

With a view to investigating the problem of slender shafts of circular cross-section, let Fig. 162 represent such a shaft when rotating with angular velocity ω about the x-axis. Here we suppose, for simplicity, that at all points along the shaft the centre of gravity and the axis of rotation are separated by a distance e when $\omega = 0$. The dimension e will be referred to as the 'eccentricity' of the shaft, and the frictional forces will be treated as negligibly small.

With the origin O at the mid-point of the span L, and y signifying the deflection of an element of the shaft at a point P distant x from O, it is evident that the unbalanced inertia effect will there be equivalent to a load $\dfrac{m\omega^2}{g}(y + e)$ per unit length, where the weight of the shaft per unit length is denoted by m. If the shearing force and bending moment on the left-hand face of the element of length δx are designated by S and M, respectively, then their

magnitudes for the right-hand face of the element will be $\left(S + \frac{dS}{dx}\delta x\right)$ and $\left(M + \frac{dM}{dx}\delta x\right)$.

From the theory of beams we can, with the usual conventions as to signs, thus write

$$\frac{dM}{dx} = -S, \quad \frac{dS}{dx} = \frac{m\omega^2}{g}(y+e),$$

which combine into

$$\frac{d^2M}{dx^2} = -\frac{m\omega^2}{g}(y+e) \quad . \quad . \quad . \quad . \quad . \quad (106.1)$$

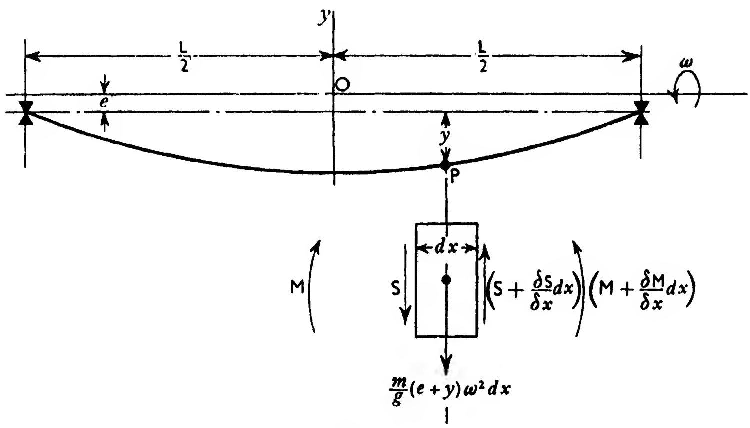


FIG. 162.

Now, with I and E written in succession for the proper moment of inertia of cross-section and the modulus of elasticity of the material, the well-known formula

$$M = -EI\frac{d^2y}{dx^2}$$

gives the additional information

$$\frac{d^2M}{dx^2} = -EI\frac{d^4y}{dx^4}.$$

Hence, by equation (106.1),

$$\frac{d^4y}{dx^4} = \frac{m\omega^2}{EgI}(y+e)$$

$$= \alpha^4(y+e), \quad . \quad . \quad . \quad . \quad . \quad . \quad (106.2)$$

where $\alpha^4 = \dfrac{m\omega^2}{EgI}$.

Arranging this in the form

$$\frac{d^4y}{dx^4} - \alpha^4 y = \alpha^4 e,$$

it follows from Art. 86 that in the present problem the solution is of the type

$$y = A\cos\alpha x + B\sin\alpha x + C\cosh\alpha x + D\sinh\alpha x - e, \quad . \quad (106.3)$$

where the constants A, B, C, D depend on the circumstances of the motion.

This solution is most conveniently examined by reference to the two possible conditions:

(i) y is an *even* function of x, implying that

$$y + e = A\cos\alpha x + C\cosh\alpha x\,;$$

(ii) y is an *odd* function of x, implying that

$$y + e = B\sin\alpha x + D\sinh\alpha x.$$

If, for definiteness, we take the shaft as supported by 'short' bearings, analogous to the case of a simply supported beam, on introducing in the equation

$$y + e = A\cos\alpha x + C\cosh\alpha x \quad . \quad . \quad . \quad (106.4)$$

the condition that the deflection y is zero at the ends $x = \pm\frac{1}{2}L$, we arrive at

$$e = A\cos\tfrac{1}{2}\alpha L + C\cosh\tfrac{1}{2}\alpha L \quad . \quad . \quad . \quad (106.5)$$

as the expression for the eccentricity. But the shaft must also fulfil the condition that the bending moment is zero at the ends. This will be secured if we put $\frac{d^2y}{dx^2} = 0$ when $x = \pm\frac{1}{2}L$ in equation (106.4), which is readily proved to mean that

$$0 = -A\cos\tfrac{1}{2}\alpha L + C\cosh\tfrac{1}{2}\alpha L \quad . \quad . \quad . \quad (106.6)$$

Since equations (106.5) and (106.6) give

$$A = \frac{e}{2\cos\frac{1}{2}\alpha L}$$

when subtracted, and

$$C = \frac{e}{2\cosh\frac{1}{2}\alpha L}$$

when added, equation (106.4) can be written in the more explicit form

$$y + e = \tfrac{1}{2}e\left(\frac{\cos\alpha x}{\cos\frac{1}{2}\alpha L} + \frac{\cosh\alpha x}{\cosh\frac{1}{2}\alpha L}\right).$$

In the prescribed circumstances the shaft will evidently whirl when the denominator of this expression becomes zero, which, as can easily be verified, occurs when

$$\alpha L = \pi,\ 3\pi,\ 5\pi,\ \ldots$$

It is a simple matter to prove that between these critical speeds are those associated with an odd function of x. Then, as already pointed out,

$$y + e = B \sin \alpha x + D \sinh \alpha x \quad . \quad . \quad . \quad (106.7)$$

the values of B, D being likewise evaluated by substituting the end-conditions

$$y = 0 \text{ and } \frac{d^2y}{dx^2} = 0 \text{ at the points } x = \pm \tfrac{1}{2}L.$$

Thus we find that in this case

$$y + e = \tfrac{1}{2}e\left(\frac{\sin \alpha x}{\sin \frac{1}{2}\alpha L} + \frac{\sinh \alpha x}{\sinh \frac{1}{2}\alpha L}\right),$$

the denominator of which becomes zero when

$$\alpha L = 2\pi,\ 4\pi,\ 6\pi,\ .\ .\ .\ ,$$

as was to be established.

Collecting the results obtained for even and odd functions of x, we gather that the shaft will whirl at speeds identified with the series of values

$$\alpha L = \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi,\ 6\pi,\ .\ .\ .\ ,$$

which correspond, in terms of equation (106.2), to the angular velocities

$$\frac{\pi^2}{L^2}\sqrt{\frac{EgI}{m}},\ \frac{2^2\pi^2}{L^2}\sqrt{\frac{EgI}{m}},\ \frac{3^2\pi^2}{L^2}\sqrt{\frac{EgI}{m}},\ .\ .\ .$$

These values are, as might be foreseen in view of the analogous case of beams, the same as those found for the beam of Ex. 1 in Art. 86, so that the curves (*a*), (*b*), (*c*) of Fig. 136 may be regarded as the configuration of the disturbed shaft in the first three critical speeds, the separate curves being constructed with the aid of the appropriate expression for $y + e$. Each mode must be examined on its merits, since for a specified system the several critical speeds are in general not of equal importance.

The constants A, B, C, D depend on the type of bearing, whether 'short' or 'long', in a manner which is best indicated by a few examples relating to slender shafts of uniform diameter.

Ex. 1. Calculate the whirling speeds of the shaft exhibited in Fig. 163, supported by long bearings so as to secure zero slope at both ends of the shaft.

Considering, first, the case where y is an even function of x in equation (106.3), we have, with the previous conventions,

$$y + e = A \cos \alpha x + C \cosh \alpha x \quad . \quad . \quad . \quad (106.8)$$

This leads to the relations

$$e = A \cos \tfrac{1}{2}\alpha L + C \cosh \tfrac{1}{2}\alpha L,$$
$$0 = -A \sin \tfrac{1}{2}\alpha L + C \sinh \tfrac{1}{2}\alpha L$$

when account is taken of the boundary conditions

$$y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ at the ends } x = \pm \tfrac{1}{2}L.$$

If these equations be separately multiplied by $\sinh \frac{1}{2}\alpha L$, $\cosh \frac{1}{2}\alpha L$, the difference of the resulting expressions shows that the constant

$$A = \frac{e \sinh \frac{1}{2}\alpha L}{\cos \frac{1}{2}\alpha L \sinh \frac{1}{2}\alpha L + \sin \frac{1}{2}\alpha L \cosh \frac{1}{2}\alpha L}.$$

On similarly multiplying the same equations by $\sin \frac{1}{2}\alpha L$, $\cos \frac{1}{2}\alpha L$, the sum of the resulting expressions shows that the constant

$$C = \frac{e \sin \frac{1}{2}\alpha L}{\cos \frac{1}{2}\alpha L \sinh \frac{1}{2}\alpha L + \sin \frac{1}{2}\alpha L \cosh \frac{1}{2}\alpha L}.$$

Thus, inserting these results in equation (106.8), we ascertain the deflection

$$y + e = e\left\{\frac{\cos \alpha x \sinh \frac{1}{2}\alpha L + \cosh \alpha x \sin \frac{1}{2}\alpha L}{\cos \frac{1}{2}\alpha L \sinh \frac{1}{2}\alpha L + \sin \frac{1}{2}\alpha L \cosh \frac{1}{2}\alpha L}\right\}$$

and so learn that whirling will take place when

$$\tan \tfrac{1}{2}\alpha L = -\tanh \tfrac{1}{2}\alpha L, \quad . \quad . \quad . \quad . \quad (106.9)$$

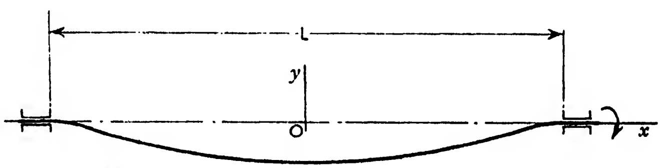


FIG. 163.

this being the condition under which the denominator of the expression for $y + e$ becomes zero. The roots of this equation can readily be determined by the graphical construction of Fig. 141, when it will be found that their approximate values are given by the series

$$\tfrac{1}{2}\alpha L = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$$

Turning next to the case where y is an odd function of x, in which

$$y + e = B \sin \alpha x + D \sinh \alpha x,$$

we arrive, by the same argument as before, at the equation

$$y + e = e\left\{\frac{\sin \alpha x \cosh \frac{1}{2}\alpha L - \sinh \alpha x \cos \frac{1}{2}\alpha L}{\sin \frac{1}{2}\alpha L \cosh \frac{1}{2}\alpha L - \cos \frac{1}{2}\alpha L \sinh \frac{1}{2}\alpha L}\right\},$$

and so conclude that whirling will occur when

$$\tan \tfrac{1}{2}\alpha L = \tanh \tfrac{1}{2}\alpha L. \quad . \quad . \quad . \quad . \quad (106.10)$$

The roots of this equation sensibly agree with the series of values

$$\tfrac{1}{2}\alpha L = \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots,$$

as can be verified by the graphical construction shown in Fig. 141.

Now, combining the series for both even and odd functions of x, it is plain that the complete range of whirling speeds is given by

$$\tfrac{1}{2}\alpha L = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots,$$

which means, in terms of equation (106.2), that the critical angular velocities are

$$\omega = \frac{3^2\pi^2}{4L^2}\sqrt{\frac{EgI}{m}}, \frac{5^2\pi^2}{4L^2}\sqrt{\frac{EgI}{m}}, \frac{7^2\pi^2}{4L^2}\sqrt{\frac{EgI}{m}}, \ldots$$

These values are, as is to be expected, the same as those of p in Ex. 3 of Art. 86.

In a given mode the configuration of the disturbed shaft may be traced, as in Fig. 136, by means of the proper expression for y, the positions of the nodes being determined by the roots of the relation for y equated to zero.

A comparison of equations (86.15) and (106.10) discloses the useful information that the case of y being an odd function of x relates to a slender beam fixed as to direction at one end and simply supported at the other, or, what is the same thing in this sense, a slender shaft supported by a 'long' bearing at one end and a 'short' bearing at the other.

The same method is applicable to a continuous shaft with any number of bearings, though the work of evaluating the constants is greatly simplified by an analytical device due to R. Macaulay.[1]

Ex. 2. Find an expression for the deflection, in a state of whirling, of a uniform shaft supported by any number of short bearings.

It will simplify matters if we suppose, for a moment, that there are only four bearings, as indicated in Fig. 164, where the reactions

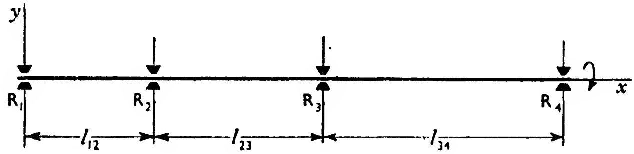


FIG. 164.

at the bearings are denoted by R_1, R_2, R_3, R_4. If, after the manner of Macaulay, in equation (106.3) we assign the symbols A, B, C, D to the constants for the span l_{12}, and A', B', C', D' to the correspond-

[1] *Mess. of Maths.*, vol. 48, page 129 (1919).

ing quantities for the span l_{23}, then considerations of continuity will enable us to pass along the complete shaft.

Taking the origin at the left-hand bearing, and introducing the condition that at the point $x = l_{12}$ the deflection is the same for the spans l_{12} and l_{23}, from equation (106.3) we derive the relation

$$A \cos \alpha l_{12} + B \sin \alpha l_{12} + C \cosh \alpha l_{12} + D \sinh \alpha l_{12}$$
$$= A' \cos \alpha l_{12} + B' \sin \alpha l_{12} + C' \cosh \alpha l_{12} + D' \sinh \alpha l_{12} \quad . \quad (106.11)$$

Since the bending moment M does not change abruptly at a bearing, $\frac{d^2y}{dx^2}$ must be continuous at the point $x = l_{12}$. From equation (106.3), then,

$$-(A \cos \alpha l_{12} + B \sin \alpha l_{12}) + C \cosh \alpha l_{12} + D \sinh \alpha l_{12}$$
$$= -(A' \cos \alpha l_{12}+B' \sin \alpha l_{12})+C' \cosh \alpha l_{12}+D' \sinh \alpha l_{12} \quad . \quad (106.12)$$

The condition that the slope $\frac{dy}{dx}$ is continuous at the point $x = l_{12}$ further leads, through equation (106.3), to

$$-A \sin \alpha l_{12} + B \cos \alpha l_{12} + C \sinh \alpha l_{12} + D \cosh \alpha l_{12}$$
$$= -A' \sin \alpha l_{12} + B' \cos \alpha l_{12} + C' \sinh \alpha l_{12} + D' \cosh \alpha l_{12} \quad . \quad (106.13)$$

Finally, taking account of the fact that $\frac{d^3y}{dx^3}$ increases by the amount $\frac{R_2}{2EgI}$ in passing the bearing at $x = l_{12}$, by equation (106.3) we obtain

$$A \sin \alpha l_{12}-B \cos \alpha l_{12}+C \sinh \alpha l_{12}+D \cosh \alpha l_{12}$$
$$=A' \sin \alpha l_{12}-B' \cos \alpha l_{12}+C' \sinh \alpha l_{12}+D' \cosh \alpha l_{12}+\frac{R_2}{2\alpha^3 EgI}. \quad (106.14)$$

Now it appears, on subtracting and adding the pairs of equations (106.11) and (106.12), (106.13) and (106.14), taken in order, that

$$\left.\begin{aligned}
&A \cos \alpha l_{12}+B \sin \alpha l_{12} =A' \cos \alpha l_{12}+B' \sin \alpha l_{12},\\
&C \cosh \alpha l_{12}+D \sinh \alpha l_{12}=C' \cosh \alpha l_{12}+D' \sinh \alpha l_{12},\\
&A \sin \alpha l_{12}-B \cos \alpha l_{12}=A' \sin \alpha l_{12}-B' \cos \alpha l_{12}+\frac{R_2}{2\alpha^3 EgI},\\
&C \sinh \alpha l_{12}+D \cosh \alpha l_{12}=C' \sinh \alpha l_{12}+D' \cosh \alpha l_{12}+\frac{R_2}{2\alpha^3 EgI}.
\end{aligned}\right\} \quad (106.15)$$

The constants for the span l_{23} are easily deduced from appropriate combinations of these equations; thus

$$A' = A - \frac{R_2 \sin \alpha l_{12}}{2\alpha^3 EgI}, \quad B' = B + \frac{R_2 \cos \alpha l_{12}}{2\alpha^3 EgI},$$

from the first and third, and

$$C' = C + \frac{R_2 \sinh \alpha l_{12}}{2\alpha^3 EgI}, \quad D' = D - \frac{R_2 \cosh \alpha l_{12}}{2\alpha^3 EgI},$$

from the second and fourth.

In this way it is seen, after making these substitutions in equation (106.3), that the deflection at any point x in the span l_{23} of Fig. 164 is defined by

$$y + e = A \cos \alpha x + B \sin \alpha x + C \cosh \alpha x + D \sinh \alpha x + \frac{R_2}{2\alpha^3 EgI}\{\sin \alpha(x - l_{12}) - \sinh \alpha(x - l_{12})\},$$

in terms of the constants for the span l_{12}.

We can now pass to the given problem, since this line of reasoning may be extended without difficulty to cover any number of spans, in virtue of the symmetrical form of the last equation. Hence, with any number of short bearings, the general expression for the deflection, in terms of the constants associated with the first span, is

$$\begin{aligned} y + e = {} & A \cos \alpha x + B \sin \alpha x + C \cosh \alpha x + D \sinh \alpha x \\ & + \frac{R_2}{2\alpha^3 EgI}\{\sin \alpha(x - l_{12}) - \sinh \alpha(x - l_{12})\} \\ & + \frac{R_3}{2\alpha^3 EgI}\{\sin \alpha(x - l_{13}) - \sinh \alpha(x - l_{13})\} \\ & + \dots \dots \dots \end{aligned} \qquad (106.16)$$

The following example will suffice to explain the manner in which the constants A, B, C, D are to be evaluated for a prescribed system.

Ex. 3. Apply the above results to the case of a similar shaft supported by three short bearings, arranged as in Fig. 165.

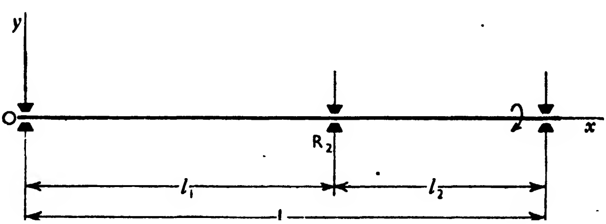


FIG. 165.

Writing m for the weight per unit length of shaft, and ω for its angular velocity, we have

$$\alpha^4 = \frac{m\omega^2}{EgI},$$

by equation (106.2), and

$$y + e = A \cos \alpha x + B \sin \alpha x + C \cosh \alpha x + D \sinh \alpha x + \frac{R_2}{2\alpha^3 EgI}\{\sin \alpha(x - l_1) - \sinh \alpha(x - l_1)\},$$

where R_2 signifies the inertia-reaction at the intermediate bearing and, as before, the origin coincides with O in the figure.

From the fact that the shaft is simply supported at the bearings it follows that the conditions

$$y = 0 \text{ and } \frac{d^2y}{dx^2} = 0 \text{ at the point } x = 0$$

must hold in the previous equation, and it is readily proved that these conditions will be fulfilled if

$$A + C = e, \quad -A + C = 0.$$

Hence

$$C = A = \tfrac{1}{2}e,$$

in consequence of which the deflection can be expressed in the form

$$y + e = \tfrac{1}{2}e(\cos \alpha x + \cosh \alpha x) + B \sin \alpha x + D \sinh \alpha x$$
$$+ \frac{R_2}{2\alpha^3 EgI}\{\sin \alpha(x - l_1) - \sinh \alpha(x - l_1)\} \quad . \quad (106.17)$$

This equation leads to

$$e = \tfrac{1}{2}e(\cos \alpha l_1 + \cosh \alpha l_1) + B \sin \alpha l_1 + D \sinh \alpha l_1,$$

i.e. $e(2 - \cos \alpha l_1 - \cosh \alpha l_1) = 2(B \sin \alpha l_1 + D \sinh \alpha l_1)$, . (106.18)

through the condition

$$y = 0 \text{ at the point } x = l_1;$$

also to

$$e = \tfrac{1}{2}e(\cos \alpha L + \cosh \alpha L) + B \sin \alpha L + D \sinh \alpha L$$
$$+ \frac{R_2}{2\alpha^3 EgI}(\sin \alpha l_2 - \sinh \alpha l_2),$$

and $0 = \tfrac{1}{2}e(-\cos \alpha L + \cosh \alpha L) - B \sin \alpha L + D \sinh \alpha L$

$$+ \frac{R_2}{2\alpha^3 EgI}(-\sin \alpha l_2 - \sinh \alpha l_2)$$

through the conditions

$$y = 0 \text{ and } \frac{d^2y}{dx^2} = 0 \text{ at the point } x = l_1 + l_2 = L,$$

taken in the same order.

The values of R_2, B, D can be derived from the last three equations. For our present purpose, however, it is sufficient to find the circumstances in which the denominators of the relations for B and D vanish simultaneously, since y then tends to very large values, in the implied absence of friction. This end is most easily achieved by writing the numerators as functions of α, L, l_1, l_2, and

straightforward eliminations show that we have then to consider expressions of the form

$$B = \frac{ef(\alpha,\ L,\ l_1,\ l_2)}{2(\sin\alpha L\sinh\alpha l_1\sinh\alpha l_2 - \sinh\alpha L\sin\alpha l_1\sin\alpha l_2)},$$

$$D = \frac{eF(\alpha,\ L,\ l_1,\ l_2)}{2(\sin\alpha L\sinh\alpha l_1\sinh\alpha l_2 - \sinh\alpha L\sin\alpha l_1\sin\alpha l_2)}.$$

These relations obviously mean that whirling will occur when

$$\sin\alpha L\sinh\alpha l_1\sinh\alpha l_2 = \sinh\alpha L\sin\alpha l_1\sin\alpha l_2,$$

i.e. when
$$\frac{\sin\alpha l_1\sin\alpha l_2}{\sin\alpha L} = \frac{\sinh\alpha l_1\sinh\alpha l_2}{\sinh\alpha L}.$$

The problem is completed by solving this equation, the usual graphical method being the most convenient in the general case.

In the special case where the intermediate bearing of Fig. 165 is situated at mid-span, so that $l_1 = l_2$, the critical speeds are the same as the second, fourth, sixth, . . . whirling speeds of a similar shaft twice the length and simply supported at its ends only. It is useful to remember that this relation is general and reciprocal.

Consistent units are, of course, to be used in numerical applications of the theory.

Ex. 4. Determine the first whirling speed of a hollow shaft, with external and internal diameters of 3 in. and 2 in., 10 ft. in length, and supported by two short bearings at the ends. The material is specified by its weight 0·28 lb. per cub. in., and direct modulus of elasticity 30,000,000 lb. per square inch.

It has been demonstrated that the first whirling speed

$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EgI}{m}}.$$

If, by way of illustration, we use pound-inch-second units, then $L=120$ in., $E=30\times10^6$ lb. per square inch, $I=\frac{\pi(81-16)}{64}=3{\cdot}19$ inch^4, $m=1{\cdot}10$ lb. per inch run of shaft, and g must be expressed as $(32{\cdot}2\times12)$ in. per sec. per sec. Hence the required critical speed

$$\omega_1 = \frac{\pi^2\times10^3}{144\times10^2}\sqrt{\frac{30\times32{\cdot}2\times12\times3{\cdot}19}{1{\cdot}1}}\ \text{rad. per sec.}$$

$$= 125{\cdot}1\ \text{rad. per sec.}$$

$$= 19{\cdot}9\ \text{rev. per sec.,}$$

nearly.

107. Strain Energy of a Shaft or Beam. At this stage of the work it will be understood that the general theory of a slender shaft and, consequently, of a similar beam, implies a knowledge of the potential or strain energy.

Let us, as in Chapter III, take two points r and s on the centre-line of such a system, and put

α_{rs} = the deflection at s due to a unit force at r,
β_{rs} = the rotation of the tangent at s due to a unit force at r,
γ_{rs} = the deflection at s due to a unit couple at r,
δ_{rs} = the rotation of the tangent at s due to a unit couple at r.

Here, remembering the reciprocal relations of Art. 49, $\alpha_{sr} = \alpha_{rs}$, $\beta_{rs} = \gamma_{sr}$, $\delta_{sr} = \delta_{rs}$, $\beta_{sr} \neq \beta_{rs}$, $\gamma_{sr} \neq \gamma_{rs}$.

A description of the way in which these coefficients enter the analysis will now be given with reference to a slender shaft of uniform diameter, supported by two bearings of a specified type.

In the simple system of Fig. 166 (a), which includes a vertical

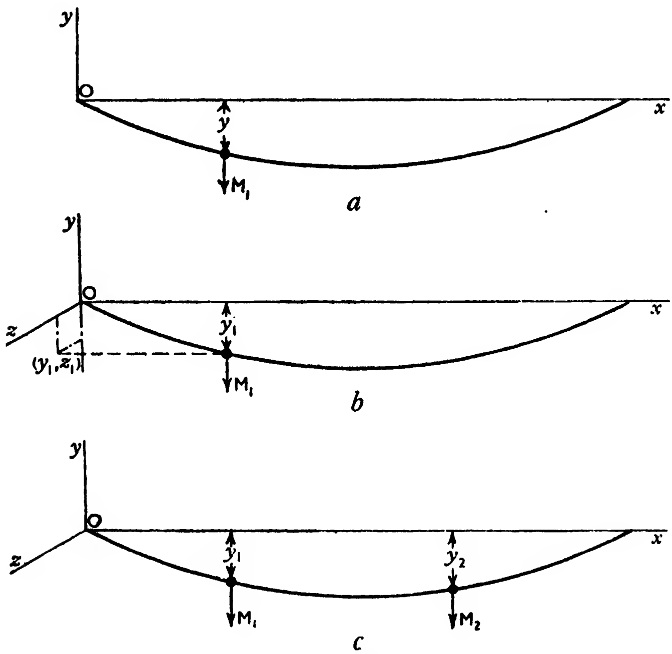


FIG. 166.

load M_1 applied at the point 1 when the deflected shaft lies in the plane of the paper, the deflection y and the load at that point are clearly connected by the relation

$$y = \alpha_{11}M_1.$$

We next take advantage of the circumstance that, in many practical problems of this kind, the extraneous loads are applied at a speed which is low compared with the velocity of wave-transmission through steel in particular, and metals in general. On the assumption that the dissipative forces in actual systems are sensibly

proportional to the velocity, we can on this account treat the frictional forces as negligibly small.

Thus it is feasible in the present problem to suppose that M_1 is applied gradually. The work done on the shaft will then amount to $\frac{1}{2}M_1y$, or $\frac{1}{2\alpha_{11}}y^2$, by the last equation.

If the deflected shaft lies in any plane, and (y_1, z_1) in Fig. 166 (*b*) are the co-ordinates of the point in question, it is readily proved that $\frac{1}{2\alpha_{11}}(y_1^2 + z_1^2)$ will represent the work done on the shaft. This is equivalent, in the assumed absence of friction, to the potential or strain energy of the shaft.

Extending this argument to the system of Fig. 166 (*c*), where the application of loads M_1, M_2 at the points *1*, *2* is supposed to cause those points to have co-ordinates $(y_1\ z_1)$, (y_2, z_2), respectively, we ascertain the potential energy V to be

$$2V = \frac{1}{\alpha_{11}}(y_1^2 + z_1^2) + \frac{2}{\alpha_{12}}(y_1y_2 + z_1z_2) + \frac{1}{\alpha_{22}}(y_2^2 + z_2^2).$$

If this shaft transmits a torque that induces angles of twist θ_1, θ_2 at the points *1*, *2*, taken in succession, the same line of reasoning leads to the result that the work done by the torque on the length, l_{12}, bounded by the given points is equal to $\frac{1}{2}c_{12}(\theta_2 - \theta_1)^2$, where c_{12} denotes the coefficient of stiffness with respect to torsion. This means that $c_{12} = \frac{NJ}{l_{12}}$, N being the modulus of rigidity of the material, and J the polar moment of inertia of the cross-section. Under combined bending and torsion, then,

$$2V=\frac{1}{\alpha_{11}}(y_1^2+z_1^2)+\frac{2}{\alpha_{12}}(y_1y_2+z_1z_2)+\frac{1}{\alpha_{22}}(y_2^2+z_2^2)+c_{12}(\theta_2-\theta_1)^2.$$

From the symmetrical properties of this expression it follows that in the general case, involving any number of such loads and torques,

$$\begin{aligned}2V=&\frac{1}{\alpha_{11}}(y_1^2+z_1^2)+\frac{1}{\alpha_{22}}(y_2^2+z_2^2)+\frac{1}{\alpha_{33}}(y_3^2+z_3^2)+\ \ldots\\&+2\left\{\frac{1}{\alpha_{12}}(y_1y_2+z_1z_2)+\frac{1}{\alpha_{23}}(y_2y_3+z_2z_3)+\frac{1}{\alpha_{34}}(y_3y_4+z_3z_4)+\ \ldots\right\}\\&+c_{12}(\theta_2-\theta_1)^2+c_{23}(\theta_3-\theta_2)^2+c_{34}(\theta_4-\theta_3)^2+\quad \ldots\ldots \qquad (107.1)\end{aligned}$$

To proceed further into the matter, consider the system of Fig. 167, in which a shaft, supported at its ends, is subjected at the point *1* to the action of a transverse load M_1 and couple $\mathfrak{T}_1$. If, for brevity in working, we take the complete system to lie in

the plane of the paper, and write β_1 for the slope of the shaft at the point 1, then in the present notation

$$\left.\begin{aligned} y_1 &= \alpha_{11}M_1 + \gamma_{11}\mathfrak{T}_1, \\ \beta_1 &= \beta_{11}M_1 + \delta_{11}\mathfrak{T}_1 \\ &= \gamma_{11}M_1 + \delta_{11}\mathfrak{T}_1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (107.2)$$

since $\beta_{11} = \gamma_{11}$.

An expression for the potential energy is most easily arrived at by supposing the disturbing force and couple to be applied gradually, one after the other. The application of M_1 alone will result in the work $\frac{1}{2}M_1y_1$ being done on the shaft, and this is obviously equal to $\frac{1}{2}\alpha_{11}M_1^2$. If the couple $\mathfrak{T}_1$ be now introduced, besides the work $\frac{1}{2}\delta_{11}\mathfrak{T}_1^2$, it will perform an amount $\gamma_{11}M_1\mathfrak{T}_1$ on the load M_1. Hence the potential energy of the shaft is determined by

$$\begin{aligned} 2V &= \alpha_{11}M_1^2 + 2\gamma_{11}M_1\mathfrak{T}_1 + \delta_{11}\mathfrak{T}_1^2 \\ &= \alpha_{11}M_1^2 + 2\beta_{11}M_1\mathfrak{T}_1 + \delta_{11}\mathfrak{T}_1^2, \end{aligned} \quad . \quad . \quad (107.3)$$

since $\gamma_{11} = \beta_{11}$, as before.

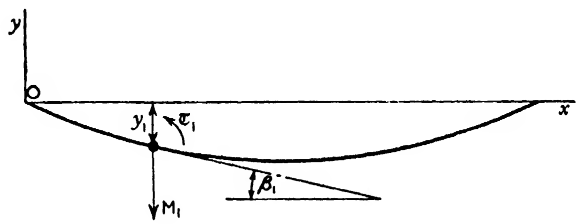


FIG. 167.

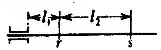
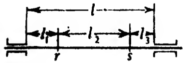
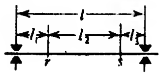
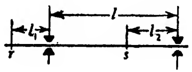
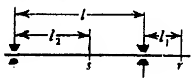
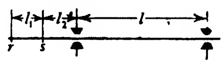
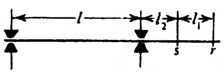
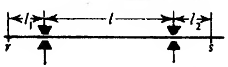
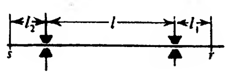
The same method may be used to find the strain energy of a shaft acted on by n loads and n torques of the types implied in equation (107.3).

We employ the theory of structures to evaluate the coefficients α_{rs}, β_{rs}, γ_{rs}, δ_{rs} for a prescribed system. The Table facing this page contains the values thus found for a slender shaft (or beam) of uniform diameter, loaded in various ways and supported by different types of bearings.

Ex. Investigate the inertia effect of the disc in the system of Fig. 168, comprising a thin disc, mounted at a distance $x = l_1$ measured from the origin O, on a light shaft supported by short bearings at its ends distant L apart.

Suppose, with the shaft rotating in the configuration shown, that the effect of the disc is a centrifugal force P and a couple $\mathfrak{T}$, referred to the point $x = l_1$. If y_1, β_1 respectively denote the deflection and the slope at that point, equations (107.2) give

$$\begin{aligned} y_1 &= \alpha_{11}P + \gamma_{11}\mathfrak{T}, \\ \beta_1 &= \gamma_{11}P + \delta_{11}\mathfrak{T}. \end{aligned}$$

Type of Bearing.		α_{rs}	$\beta_{rs} = \gamma_{sr}$	$\gamma_{rs} = \beta_{sr}$	δ_{rs}	c_{rs}
Long		$\frac{l^2_1(2l_1+3l_2)}{6EI}$	$\frac{l^2_1}{2EI}$	$\frac{l_1(l_1+2l_2)}{2EI}$	$\frac{l_1}{EI}$	$\frac{NJ}{l_{rs}}$
Long		$\frac{l^2_1l^2_3(3l_2l+2l_1l_3)}{6EIl^2}$	$-\frac{l^2_1l^2_3\{2l^2-(2l_1+3l_3)l+2l_1l_3\}}{2EIl^2}$	$\frac{l_1l^2_3\{2l^2-(3l_1+2l_3)l-l_1l_3\}}{2EIl^3}$	$-\frac{l_1l_3\{2l^2-(l_1+l_3)l+3l_1l_3\}}{EIl^3}$	$\frac{NJ}{l_{rs}}$
Short		$\frac{l_1l_3(l^2-l^2_1-l^2_3)}{6EIl}$	$\frac{l_1(l^2_1+3l^2_3-l^2)}{6EIl}$	$-\frac{l_3(3l^2_1+l^2_3-l^2)}{6EIl}$	$\frac{3(l^2_1+l^2_3)-l^2}{6EIl}$	$\frac{NJ}{l_{rs}}$
Short		$-\frac{l_1l_2(l^2-l^2_2)}{6EIl}$	$\frac{l_1(l^2-3l^2_2)}{6EIl}$	$\frac{l_2(l^2-l^2_2)}{6EIl}$	$\frac{l^2-3l^2_2}{6EIl}$	$\frac{NJ}{l_{rs}}$
Short		$-\frac{l_1l_2(l^2-l^2_2)}{6EIl}$	$\frac{l_1(l^2-3l^2_2)}{6EIl}$	$-\frac{l_2(l^2-l^2_2)}{6EIl}$	$-\frac{l^2-3l_2^2}{6EIl}$	$\frac{NJ}{l_{rs}}$
Short		$\frac{l_2\{3l_1l_2+2(l_1+l_2)l+2l^2_2\}}{6EI}$	$-\frac{3l_2(2l_1+l_2)+2(l_1+l_2)l}{6EI}$	$-\frac{l_2(3l_2+2l)}{6EI}$	$\frac{3l_2+l}{3EI}$	$\frac{NJ}{l_{rs}}$
Short		$\frac{l_2\{3l_1l_2+2l(l_1+l_2)+2l^2_2\}}{6EI}$	$\frac{3l_2(2l_1+l_2)+2l(l_1+l_2)}{6EI}$	$\frac{l_2(3l_2+2l)}{6EI}$	$\frac{3l_2+l}{3EI}$	$\frac{NJ}{l_{rs}}$
Short		$\frac{l_1l_2l}{6EI}$	$\frac{l_1l}{6EI}$	$-\frac{l_2l}{6EI}$	$-\frac{l}{6EI}$	$\frac{NJ}{l_{rs}}$
Short		$\frac{l_1l_2l}{6EI}$	$-\frac{l_1l}{6EI}$	$\frac{l_2l}{6EI}$	$-\frac{l}{6EI}$	$\frac{NJ}{l_{rs}}$

Thus, after eliminating in turn P and $\mathfrak{T}$ between this pair of equations, it appears that

$$\mathfrak{T} = \frac{\gamma_{11}y_1 - \alpha_{11}\beta_1}{\gamma_{11}^2 - \alpha_{11}\delta_{11}}$$
$$= ay_1 - b\beta_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (107.4)$$

where $a = \dfrac{\gamma_{11}}{\gamma_{11}^2 - \alpha_{11}\delta_{11}}$, $b = \dfrac{\alpha_{11}}{\gamma_{11}^2 - \alpha_{11}\delta_{11}}$; and

$$P = \frac{\delta_{11}y_1 - \gamma_{11}\beta_1}{\alpha_{11}\delta_{11} - \gamma_{11}^2}$$
$$= cy_1 + a\beta_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (107.5)$$

where $c = \dfrac{\delta_{11}}{\alpha_{11}\delta_{11} - \gamma_{11}^2}$.

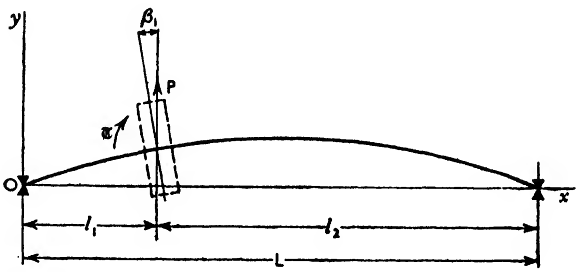


FIG. 168.

As to the values of these coefficients, we notice that the system in question will agree with the arrangement specified in the third row of the above Table if we there put $l_3 = l_2$. Hence, in the notation of Fig. 168,

$$\alpha_{11} = \frac{l_1 l_2 (L^2 - l_1^2 - l_2^2)}{6EIL}$$
$$= \frac{l_1^2 l_2^2}{3EIL},$$

since $l_1 + l_2 = L$; and

$$\gamma_{11} = \frac{l_1(l_1^2 + 3l_2^2 - L^2)}{6EIL}$$
$$= \frac{l_1 l_2 (l_2 - l_1)}{3EIL},$$
$$\delta_{11} = \frac{3(l_1^2 + l_2^2) - L^2}{6EIL}$$
$$= \frac{l_1^2 - l_1 l_2 + l_2^2}{3EIL}.$$

From these results we deduce

$$a = \frac{3EIL(l_1 - l_2)}{l_1^2 l_2^2},$$

$$b = -\frac{3EIL}{l_1 l_2},$$

$$c = \frac{3EIL(l_1^2 - l_1 l_2 + l_2^2)}{l_1^3 l_2^3},$$

and so conclude that the inertia force and couple associated with the disc are determined by

$$\left.\begin{aligned} P &= \frac{3EIL}{l_1^2 l_2^2}\left\{\left(\frac{l_1^2 - l_1 l_2 + l_2^2}{l_1 l_2}\right)y_1 + (l_1 - l_2)\beta_1\right\}, \\ \mathfrak{T} &= \frac{3EIL}{l_1 l_2}\left\{\left(\frac{l_1 - l_2}{l_1 l_2}\right)y_1 + \beta_1\right\}. \end{aligned}\right\} \quad . \quad . \quad (107.6)$$

Although this problem is further examined in Ex. 1 of Art. 124, we are now in a position to realize that the stress in a rotating shaft is affected by the gyroscopic couple thus identified with any discs or pulleys which may be included in the system.

108. Equivalent Length of a Non-Uniform Shaft. In many instances we have to consider shafts in which the cross-section varies from point to point in the axial direction, and it is then convenient to express the actual length in terms of an equivalent length of shaft of uniform cross-section. Experimental methods afford the most reliable standard of comparison for structural systems in general, since we have already demonstrated that the coefficients of stiffness depend on the dimensions, end conditions, relative positions of the loads, and elastic properties of the material concerned. With shafts, in particular, it may also be necessary to take account of the fact that, in certain circumstances, the constraints associated with the bearings vary with the state of lubrication, which means that these constraints are functions of the time for prescribed conditions of motion, especially in the initial stages of operation.

A noteworthy example is presented by the crankshaft of an engine which is executing torsional vibrations about a mean position. Experimental investigations into the twist of crankshafts have been undertaken by F. M. Lewis,[1] W. C. Stewart,[2] and G. H. Paulin.[3] Moreover, from tests covering a wide range of crankshafts B. C. Carter[4] deduced the empirical formula

[1] *Trans. Soc. Nav. Arch. and Mar. E.*, New York, vol. 33, page 109 (1925).
[2] *Jour. Amer. Soc. Nav. E.*, vol. 46, page 31 (1934).
[3] *Engineering*, vol. 144, page 711 (1937).
[4] *Engineering*, vol. 126, page 38 (1928).

$$L = (2f + 0{\cdot}8h) + \frac{3}{4}k\left(\frac{D^4 - d^4}{D_1^4 - d_1^4}\right) + \frac{3}{2}r\left(\frac{D^4 - d^4}{hw^3}\right)$$

for the equivalent length L, in terms of the diameter at the journals, of the crankshaft bounded by the centres of the main bearings implied in Fig. 169 (*a*). Here the coefficient of stiffness is $\frac{NJ}{L}$, where $J = \frac{\pi}{32}(D^4 - d^4)$.

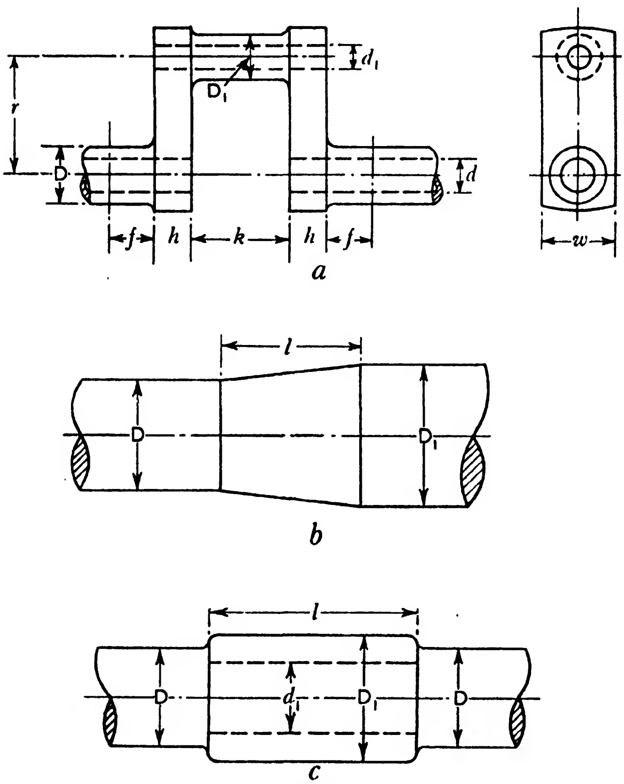


FIG. 169.

The infinitesimal calculus may be used to estimate the equivalent length of the tapered part of the shaft shown in Fig. 169 (*b*), by considering the hatched element, on the supposition that the angles of twist of the equivalent and the actual parts are the same for a given torque. In this way it will be seen that the tapered part is equivalent to a shaft of diameter D and length L, where

$$L = \frac{1}{3}l\left\{\frac{D}{D_1} + \left(\frac{D}{D_1}\right)^2 + \left(\frac{D}{D_1}\right)^3\right\},$$

provided always that no sharp change of section is involved.

It may likewise be proved that the enlarged part of the shaft represented by Fig. 169 (*c*) is equivalent to a shaft of diameter D and length L, where

$$L = \frac{D^4}{D_1^4 - d_1^4} l,$$

assuming easy curves at the enlargement.

Where a shaft is flanged, it is customary to assume that the equivalent shaft extends slightly into the coupling and that the bolt circle represents the effective diameter of the flange.

The work of estimating the 'critical' speeds in torsional vibration of an actual crankshaft is naturally simplified by effecting the modification just described.

Ex. 1. Examine the critical speeds in a normal mode of torsional vibration for the installation illustrated by Fig. 170, consisting of four similar marine oil-engines, each having eight cylinders,

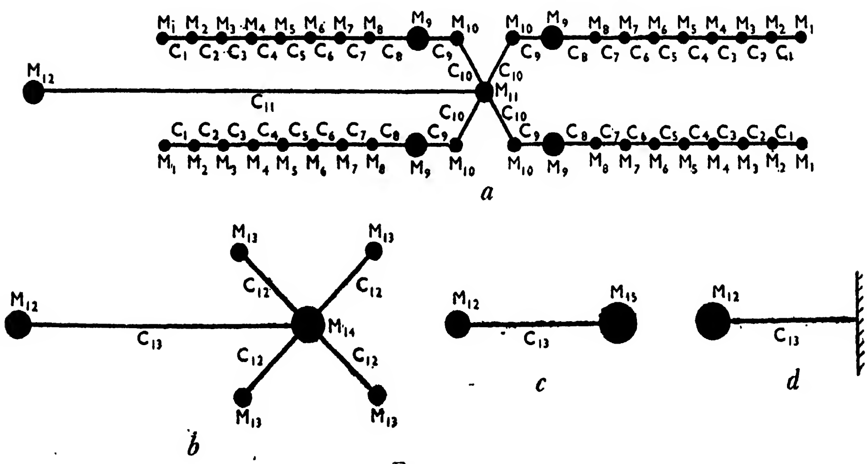


FIG. 170.

a flywheel, and a flexible coupling connected through a common set of gears to the propeller shaft.

It is not difficult to reduce the crankshaft to an equivalent length of shaft of uniform diameter and to refer the rotating masses to a common radius, say, that of a crank.

Suppose these modifications completed, and let:

$M_1, M_2, \ldots, M_8$ be the rotating masses of the respective cylinders of one engine, evaluated in accordance with Chapter I, and referred to the crank-throw;

M_9 be the combined mass of the flywheel and half that of a flexible coupling;

M_{10} be the mass of the other half of the coupling;

M_{11} be the mass of the common set of gears;
M_{12} be five-fourths the mass of the propeller, the 25 per cent. increase representing approximately the inertia effect of the water entrained in the propeller;
$c_1, c_2, \ldots, c_{11}$ be the coefficients of stiffness, in torsion, of the equivalent lengths of the shafts associated with the same suffixes;
$\theta_1, \theta_2, \ldots, \theta_{12}$ be the angular displacements of the masses associated with the same suffixes.

In this approximate method, it will be noticed, the mass of a shaft is neglected as of the second order compared with the mass of the rotating parts fixed to it. This matter will be looked into presently.

With only one of the engines operating, the vibratory motion in the specified mode is, to the prescribed degree of accuracy, defined by twelve equations of the type:

$$\begin{aligned}
&M_1\ddot{\theta}_1 + c_1(\theta_1 - \theta_2) = 0,\\
&M_2\ddot{\theta}_2 + c_1(\theta_2 - \theta_1) + c_2(\theta_2 - \theta_3) = 0,\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&M_{10}\ddot{\theta}_{10} + c_9(\theta_{10} - \theta_9) + c_{10}(\theta_{10} - \theta_{11}) = 0,\\
&M_{11}\ddot{\theta}_{11} + 4c_{10}(\theta_{11} - \theta_{10}) + c_{11}(\theta_{11} - \theta_{12}) = 0,\\
&M_{12}\ddot{\theta}_{12} + c_{11}(\theta_{12} - \theta_{11}) = 0.
\end{aligned}$$

To solve these equations for a normal mode of vibration, it is permissible to assume that

$$\begin{aligned}
\theta_1 &= A_1 \sin \omega t,\\
\theta_2 &= A_2 \sin \omega t,\\
&\cdot \quad \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot \quad \cdot \quad \cdot\\
\theta_{12} &= A_{12} \sin \omega t,
\end{aligned}$$

where ω relates to the angular velocity of the system, and A_m represents the maximum (angular) displacement of the mth mass. From these relations we thus derive, omitting a common factor,

$$\left.\begin{aligned}
&(c_1 - M_1\omega^2)A_1 - c_1A_2 = 0,\\
&-c_1A_1 + (c_1 + c_2 - M_2\omega^2)A_2 - c_2A_3 = 0,\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&-c_9A_9 + (c_9 + c_{10} - M_{10}\omega^2)A_{10} - c_{10}A_{11} = 0,\\
&-4c_{10}A_{10} + (4c_{10} + c_{11} - M_{11}\omega^2)A_{11} - c_{11}A_{12} = 0,\\
&-c_{11}A_{11} + (c_{11} - M_{12}\omega^2)A_{12} = 0.
\end{aligned}\right\} \quad (108.1)$$

If n of the engines are working, it is merely necessary to multiply the proper terms in these equations by n.

Proceeding on the supposition that only one of the engines is

operating, we eliminate the ratios $A_1 : A_2 : \ldots : A_{12}$ between these equations, and so obtain an expression in ω^2, the largest root of which gives the period $\frac{2\pi}{\omega}$ of the first critical speed in torsional vibration. The next largest root similarly refers to the second critical speed, and so on. Lengthy calculations would, however, be involved if this straightforward method of solution were followed in the present case. But it frequently happens in problems of this kind that a practical significance is restricted to the lower modes of vibration, and for many purposes a sufficiently accurate value of the critical speeds can then be found by reducing the actual installation to a simple system of masses and shafts of uniform diameter. The number of masses to be included in the modified system depends on the number of nodes in the highest possible mode of vibration.

If, for example, the first critical speed is the most important, we reduce the installation of Fig. 170 (*a*) to the three-mass system of Fig. 170 (*b*). Here the total mass of the rotating parts attached to each engine is designated by M_{13}, and treated as if concentrated at the mid-length of the crankshaft concerned; c_{12} denotes the coefficient of stiffness of the equivalent shaft connecting M_{13} and the flywheel; M_{14} denotes the combined mass of the flywheel, flexible coupling and gearing, the whole being regarded as concentrated at the flywheel; and, as previously, M_{12} denotes the effective mass of the propeller.

It is readily seen, with these conventions, that in this three-mass system the equations

$$\begin{aligned}
(c_{12} - M_{13}\omega^2)A_{12} - c_{12}A_{13} &= 0, \\
- c_{13}A_{12} - nc_{12}A_{13} + (nc_{12} + c_{13} - M_{14}\omega^2)A_{14} &= 0, \\
(c_{13} - M_{12}\omega^2)A_{12} - c_{13}A_{14} &= 0
\end{aligned}$$

hold when n of the four engines are working. Eliminating the ratios $A_{12} : A_{13} : A_{14}$ between these relations, we find

$$\omega^4 - \left\{\left(\frac{nM_{13} + M_{14}}{M_{13}M_{14}}\right)c_{12} + \left(\frac{M_{12} + M_{14}}{M_{12}M_{14}}\right)c_{13}\right\}\omega^2 + \left(\frac{M_{12} + nM_{13} + M_{14}}{M_{12}M_{13}M_{14}}\right)c_{12}c_{13} = 0 \quad . \quad . \quad (108{\cdot}2)$$

The larger root of this quadratic in ω^2 determines the free period of the first critical speed, with a degree of accuracy which does not differ by much more than 1 per cent. from the value given by the more elaborate solution of equations (108.1) when all the engines are operating. But the error may amount to about 6 per cent. when only one engine is working.

It is impossible to deduce the second critical speed from the

smaller root of the above quadratic in ω^2, since two nodes then occur, and our approximate solution is necessarily restricted to the case of one node.

The installation may further be reduced to the two-mass system of Fig. 170 (c), where M_{12} represents the effective mass of the propeller, and M_{15} the mass of the remaining parts of the mechanism, taken as concentrated at the flywheel. The symbol c_{13} relates to the same quantity as in Fig. 170 (b). This clearly means that $c_{12} = 0$ in equation (108.2), hence the free period $\frac{2\pi}{\omega}$ is now determined by

$$\omega^2 = \frac{M_{12} + M_{15}}{M_{12}M_{15}} c_{13}.$$

It may be remarked that this yields practically the same value, for the first critical speed, as equation (108.2), with similar combinations of the engines.

If, finally, the magnitude of M_{15} be regarded as infinitely great, we have the one-mass system of Fig. 170 (d), whose period is given by

$$\omega^2 = \frac{c_{13}}{M_{12}},$$

Although very rough, this approximation sometimes gives values which compare favourably, as regards percentage error, with the solutions of the two- and three-mass systems.

The configuration of the disturbed shaft and, consequently, the stress in the material can be calculated by inserting the appropriate value of ω in the equations corresponding to (108.1), the positions of the nodes being given by the roots of the expression obtained by making the angular displacement θ equal to zero.

This is perhaps a convenient place to discuss the effect on torsional vibrations of the weight of a given shaft, since the question has so far been passed over.

Ex. 2. Find the extent to which the torsional vibration of the uniform shaft of Fig. 171 is affected by the weight of the material, with the shaft rigidly fixed at the end O.

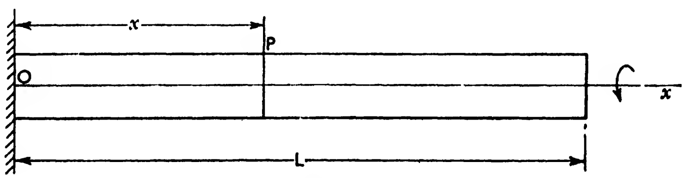


FIG. 171.

Writing a for the velocity with which the distortional type of wave is propagated in the material, and θ for the angular displace-

ment of an element of length at P distant x from O, from equation (75.3) we have

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2},$$

where $a = \sqrt{\frac{Ng}{\rho}}$, N being the modulus of rigidity of the material, and ρ the weight per unit length of shaft. It has been pointed out that a is approximately 10,700 ft. per sec. for steel.

Taking, as previously, the solution of this equation to be

$$\theta = \left(A \cos\frac{\omega}{a}x + B \sin\frac{\omega}{a}x\right) \cos \omega t,$$

with the free period of vibration about the x-axis denoted by $\frac{2\pi}{\omega}$, and introducing the boundary conditions, we thus deduce

$$A = 0$$

from the condition

$$\theta = 0 \text{ at the origin } x = 0 \text{ for all values of } t;$$

and

$$B \cos\frac{\omega L}{a} = 0$$

from the condition

$$\frac{\partial\theta}{\partial x} = 0 \text{ at } x = L \text{ for all values of } t,$$

since the torque is always zero at the free end $x = L$.

It is easily proved that these values for the constants A and B hold good only if

$$\frac{\omega L}{a} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Let us, for brevity in working, suppose the shaft to be of unit cross-sectional area, and confine our attention to the fundamental mode thus specified by

$$\omega_1^2 = \frac{\pi^2 a^2}{4L^2}.$$

Next, imagine the weight of the shaft as equivalent, in the present sense, to a load or disc having a polar moment of inertia I_0 concentrated at the free end $x = L$.

The period of this imaginary system will evidently be given by

$$\omega_0^2 = \frac{NgJ}{I_0 L}$$

$$= \frac{a^2\rho J}{I_0 L}.$$

Hence, since $\omega_1 = \omega_0$ by supposition,

$$\frac{\pi^2 a^2}{4L^2} = \frac{a^2 \rho J}{I_0 L},$$

i.e.,

$$I_0 = \frac{4\rho J L}{\pi^2}.$$

But ρL is the total weight, M, of the prescribed shaft, so that, in approximate numbers,

$$I_0 = 0{\cdot}406 MJ,$$

which means that the weight of the shaft itself can be taken into account by including in the system an imaginary load having a polar moment of inertia equal to about $\frac{2}{5}$ that of the shaft, and concentrated at the free end.

109. General Theory of Shafts. We may conveniently introduce the general problem of slender shafts with a study of the normal modes in transverse vibration of a circular shaft having a diameter that may vary in the axial direction, on the understanding that no sharp change of cross-section is involved, and the displacement about the equilibrium-position remains small throughout the motion.

Since our ultimate aim is that of employing Lagrange's formula, it will simplify matters if we notice at the outset the implications of the chief assumption on which the procedure is based. Mention has already been made of the reason why the generalized method can only be used in circumstances where it is feasible to divide a given system into a finite number of elements of length. On this account we shall treat a shaft as if it were composed of a finite number of rigid discs, each of which is capable of rotating about its own axes with the angular velocity imparted by the disturbing force. This is practically equivalent to a supposition that the actual shaft satisfies the theory of beams in which an initially plane section remains plane after bending.

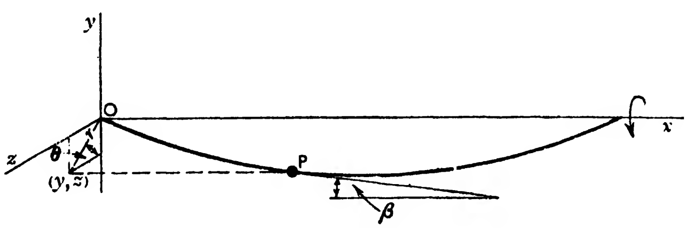


FIG. 172.

Suppose Fig. 172 to represent the centre-line of such a shaft when in a state of steady rotation, in bearings of a given type. The method of Art. 97 can, for many practical purposes, be used

to ascertain the motion of the complete system from considerations of an element of length identified with any point P, provided the diameter does not change abruptly at any place on the shaft. Here we may imagine P as the centre of gravity of the element.

Taking the bearing at O as the origin of the fixed rectangular axes Ox, Oy, Oz, and x, y as the co-ordinates of P referred to the plane yOz, so that

$$y = r \sin \theta, \; z = r \cos \theta,$$

with the angle θ as shown in the figure, we see, on differentiating with respect to the time, that the components of velocity are

$$\dot{y} = r\dot{\theta} \cos \theta, \; \dot{z} = -r\dot{\theta} \sin \theta,$$

since r is constant in a state of steady rotation. Stated otherwise,

$$r^2\dot{\theta}^2 = \dot{y}^2 + \dot{z}^2.$$

Further, let the cross-section at P have radius of gyration k_0 in relation to the axis of the shaft, and radius of gyration k in relation to a diameter. In this notation, with β written for the slope of the shaft at P,

$$k_0^2 \cos^2 \beta + k^2 \sin^2 \beta$$

is the square of the radius of gyration referred to the x-axis.

If M be the weight of the element, and T its kinetic energy about the axis of rotation, it follows that

$$\begin{aligned} 2T &= \frac{M}{g}\{\dot{y}^2 + \dot{z}^2 + (k_0^2 \cos^2 \beta + k^2 \sin^2 \beta)\dot{\theta}^2\} \\ &= \frac{M}{g}(r^2 + k_0^2 \cos^2 \beta + k^2 \sin^2 \beta)\dot{\theta}^2 \quad . \quad . \quad . \quad (109.1) \end{aligned}$$

Hence if T_0 is the kinetic energy of the element when it passes through the equilibrium-position corresponding to $r = 0$, $\beta = 0$,

$$2T_0 = \frac{Mk^2_0}{g}\dot{\theta}^2,$$

in virtue of which we can write

$$2(T - T_0) = \frac{M}{g}\{r^2 - k_0^2(1 - \cos^2 \beta) + k^2 \sin^2 \beta\}\dot{\theta}^2,$$

and this is obviously equal to twice the potential energy V, in the assumed absence of dissipative forces. Also, for a circular cross-section, the relation $k_0^2 = 2k^2$ will remain sensibly true so long as the slope β is small. Thus if the whirling speed implied in the figure occurs when $\dot{\theta} = \omega_1$, we have, after a little reduction,

$$2V = \frac{M}{g}\omega_1^2(r^2 - k^2 \sin^2 \beta) \quad . \quad . \quad . \quad (109.2)$$

Therefore

$$2V = \frac{M}{g}\omega_1^2(r^2 - k^2\beta^2)$$

if the slope is so small that $\sin\beta$ is approximately equal to β. To the same degree of accuracy, however,

$$2V = Mr$$

also holds, since the potential energy is equal to the work done on the element. Consequently,

$$\omega_1{}^2 = \frac{Mgr}{M(r^2 - k^2\beta^2)}.$$

With the shaft divided into n elements of length, we can now write, on summing, the whirling speed

$$\omega_1 = \sqrt{\frac{g\sum_{s=1}^{n} M_s r_s}{\sum_{s=1}^{n} M_s(r_s{}^2 - k_s{}^2\beta^2{}_s)}}, \quad . \quad . \quad . \quad (109.3)$$

in radians per second if, say, foot-pound-second units are used.

It is worth while to notice here, for reasons to be explained later, that terms in β relate to the gyroscopic effect of the mass of the shaft itself, and that that effect will operate even when the slope is small, according to the present treatment.

In a system where β is of the second order of small quantities, the last equation reduces to

$$\omega_1 = \sqrt{\frac{g\sum_{s=1}^{n} M_s r_s}{\sum_{s=1}^{n} M_s r_s{}^2}} \quad . \quad . \quad . \quad . \quad . \quad (109.4)$$

On the basis of Art. 55 we can also establish an analogous expression for the *second* whirling speed, involving one node, in the form

$$\omega_2 = \sqrt{\frac{g\sum_{s=1}^{n} M_s r_s x_s}{\sum_{s=1}^{n} M_s r_s{}^2 x_s}}, \quad . \quad . \quad . \quad . \quad . \quad (109.5)$$

where x_s represents the horizontal distance of the sth element reckoned from the node.

These formulae afford a ready means of calculating, with a fair degree of accuracy in favourable conditions, the stated whirling speeds of a slender shaft.

It is also instructive to apply the alternative method described in the example of Art. 52 to a uniform wire, and to compare the

fundamental frequencies of vibration thus obtained for different values of n with the corresponding frequency given by the more exact analysis of Art. 68. Some of the results of this comparison are exhibited in the following Table.

n	1	2	3	4	9	20
Ratio of frequency for $\dfrac{n \text{ elements}}{\text{continuous wire}}$	0·9003	0·9549	0·9745	0·9836	0·9959	0·9990

From these tabulated ratios it appears that the fundamental frequency of the wire can be calculated within 2 per cent. of the true value from experiments with a simple model, consisting of four concentrated masses attached to a light wire in such a manner that the model and the wire have similar mean line-densities. Furthermore, the error would not be greatly increased even if only three such masses were employed, as shown in Fig. 92. The practical significance of this procedure is to be found in the fact that a shaft may sometimes be regarded as equivalent, in the present connection, to a slender wire of the same line-density.

110. Equation (109.4) admits of a simple graphical solution which will be sufficiently explained if we consider the shaft of

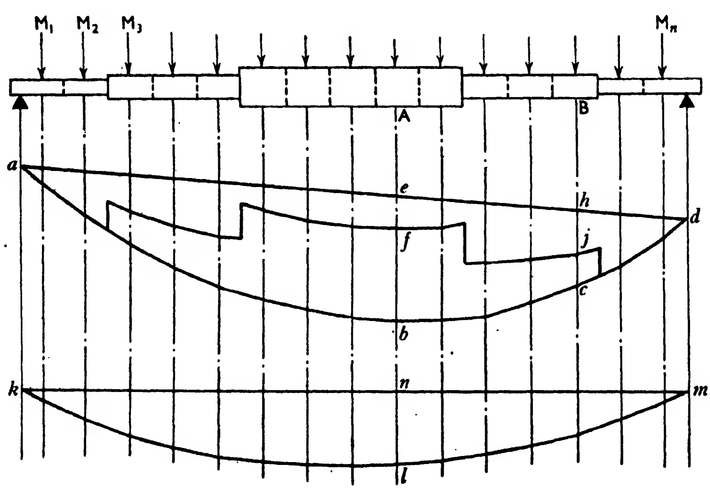


Fig. 173.

Fig. 173, on the assumption implied in Rayleigh's theorem, namely, that the displacement r_s is proportional to the displacement δ_s

which would be produced if the same load were gradually applied at the point s. Then

$$\omega_1 = \sqrt{\frac{g\sum_{s=1}^{n} M_s\delta_s}{\sum_{s=1}^{n} M_s\delta_s^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (110.1)$$

the symbols of summation being replaced by those of integration when the diameter is a continuous function of the x-co-ordinate of Fig. 172.

The first step consists in dividing the shaft into a convenient number n elements of length, the masses of which will be denoted by $M_1, M_2, \ldots, M_n$. Now the shaft may be treated as equivalent to the system formed by a light beam and a series of loads M_1, $M_2, \ldots, M_n$ concentrated at the centres of gravity of the corresponding elements, as indicated in the figure.

With the help of a movable polar diagram we next draw the bending moment diagram *abcdea*. Before proceeding to construct the deflection diagram, however, it is most convenient to modify the bending moment diagram so as to represent a shaft of uniform diameter, in accordance with Art. 108. For example, if R_1 be the least radius of the actual shaft, and R_2 that of the section at A, then the original ordinate be is reduced to fe in the ratio $\frac{fe}{be} = \frac{R_1^4}{R_2^4}$. Similarly, if R_3 be the radius of the section at B, the modified ordinate jh is derived from the original ordinate ch through the relation

$$\frac{jh}{ch} = \frac{R_1^4}{R_3^4}.$$

Finally, the deflection curve klm corresponding to the modified bending-moment diagram is traced with the aid of a polar diagram.

If nl be the deflection thus obtained for the section at A, on writing Δ for nl and δ for the deflection at that point on the actual shaft, it follows from the theory of these diagrams that

$$\delta = D\Delta,$$

$$\text{where } D = \frac{\beta^3 pquvw}{EI}, \text{ and}$$

β = length of shaft, in inches, represented by one inch on the diagram;

p = polar distance, in inches, of the first polar diagram;

q = polar distance, in inches, of the second polar diagram;

u = horizontal distance between the ordinates of the modified bending-moment diagram;

v = ratio of modified to original length of the ordinate on the bending-moment diagram if a change is made in the load-scale ;
w = pounds represented by one inch of the load-scale ;
E = direct modulus of elasticity of the material ;
I = moment of inertia of cross-section about a diameter.

Tests show that the foregoing procedure leads to fairly good results even when a shaft of the prescribed type is loaded, provided the system does not include relatively heavy external loads arranged unsymmetrically in the axial direction. This extension of the method is effected by making M_s in equation (110.1) represent the sum of the mass of the sth element and the load attached to it, so that if the load is due to a pulley, the length of the element must be sufficiently long to extend over the boss of the pulley.

An expression of the form (110.1) necessarily applies to a wide range of shafts that conform to standard proportions, such as are used, for example, in the construction of turbines. This is well illustrated in a paper by K. Baumann,[1] who found the formula

$$\omega_1^2 = \frac{(1{\cdot}06 \text{ to } 1{\cdot}07)g}{\delta}$$

to hold for a number of turbine-rotors, where δ is the deflection of the rotor under gravity. H. M. Martin [2] has examined this formula in a manner which deserves special mention.

111. More light will be thrown on to the same problem if we let Fig. 174 exhibit the centre-line of a uniform shaft in a state

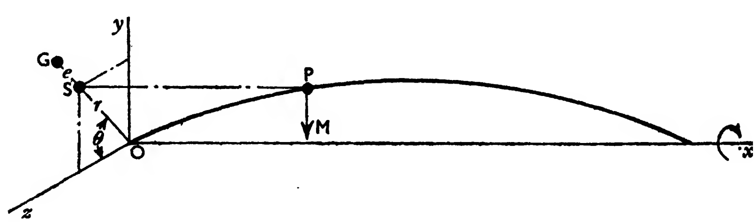


FIG. 174.

of steady rotation, in two bearings of a given type. Further, suppose the shaft to be slightly out of balance, to an extent which is specified, as before, by the eccentricity e.

It follows from considerations of continuity that the motion may be determined by projecting any point P on to the fixed plane yOz, as was done in Art. 97. For purposes of reference, let G be the centre of gravity and S the geometrical centre of the element of length associated with P, so that S coincides with the

[1] *Jour. Inst. E.E.*, vol. 48, page 807 (1911).
[2] *Engineering*, vol. 112, page 1 (1921).

origin O when the system is at rest. The points OSG will, to the implied degree of approximation, lie on a straight line, by reason of our assumption that any element of length is capable of rotating about its own axes.

Now the motion may be traced by writing (y_g, z_g) and (y, z) in turn for the co-ordinates of G and S referred to the plane yOz. In this notation

$$y_g = y + e \sin \theta, \; z_g = z + e \cos \theta,$$

e being the radial distance between S and G, and θ the angle subtended by the line OSG with the z-axis. Neglecting any change in e as of the second order of small quantities, we deduce, on differentiating with regard to the time,

$$\dot{y}_g = \dot{y} + e\dot{\theta} \cos \theta, \; \dot{z}_g = \dot{z} - e\dot{\theta} \sin \theta.$$

We shall, for simplicity, here neglect the effect of the slope of the shaft at P, since the matter has been discussed in Art. 109. If the element have polar moment of inertia J about an axis through S, and radius of gyration k about a parallel axis through G, then

$$J = \frac{M}{g}(k^2 + e^2),$$

where M relates to the mass concerned.

From these results we gather, on writing T for the kinetic energy of the element, that

$$2T = \frac{M}{g}(k^2\dot{\theta}^2 + \dot{y}_g{}^2 + \dot{z}_g{}^2)$$

refers to the centre G, and

$$2T = \frac{M}{g}\{(k^2 + e^2)\dot{\theta}^2 + \dot{y}^2 + \dot{z}^2 + 2e\dot{\theta}(\dot{y} \cos \theta - \dot{z} \sin \theta)\} \quad . \quad (111.1)$$

to the geometrical centre S.

Again, if the effect of a driving torque $\mathfrak{T}$ is a reaction with components Y, Z along the y- and z-axes of Fig. 174, respectively, the equations of motion can be obtained as in the analogous problem of Art. 97. Thus, replacing ϕ in equations (97.8) by θ, to take account of the present notation, it appears that the required expressions are

$$\left.\begin{aligned} Y &= -\frac{M}{g}(\ddot{y} + e\ddot{\theta} \cos \theta - e\dot{\theta}^2 \sin \theta) - M, \\ Z &= -\frac{M}{g}(\ddot{z} - e\ddot{\theta} \sin \theta - e\dot{\theta}^2 \cos \theta), \\ \mathfrak{T} &\doteq -\frac{M}{g}\{(k^2 + e^2)\ddot{\theta} + e(\ddot{y} \cos \theta - \ddot{z} \sin \theta\} + Me \cos \theta. \end{aligned}\right\} \quad (111.2)$$

These conclusions necessarily agree with those arrived at in Art. 97, since they show that the transverse and torsional displace-

ments may only be treated independently if the eccentricity e is very small compared with the other dimensions in equations (111.2).

112. To investigate the motion which will be initiated by a slight disturbance about the equilibrium-configuration, in the implied absence of friction, let us suppose the system to be then rotating with the uniform angular velocity defined by $\dot{\theta} = \omega$.

If, when so rotating, ω is increased by a small amount $\dot{\xi}$, the consequent variations in the inertia force can be found by putting

$$\dot{\theta} = \omega + \dot{\xi},$$

and therefore

$$\ddot{\theta} = \ddot{\xi},\ \dot{\theta}^2 = \omega^2 + 2\omega\dot{\xi} + \dot{\xi}^2.$$

Since the change of 'speed' is restricted to small values, however, the accuracy of the calculations will not be greatly affected if we neglect terms beyond the first power in $\dot{\xi}$, and so put

$$\cos(\omega t + \xi) = \cos \omega t,\ \sin(\omega t + \xi) = \sin \omega t.$$

Making these substitutions in equations (111.2), and cancelling terms that refer to the steady state, we have the disturbed motion determined by

$$\left.\begin{aligned} Y &= -\frac{M}{g}\{\ddot{y} + e\ddot{\xi}\cos\omega t - e(\omega^2 + 2\omega\dot{\xi})\sin\omega t\}, \\ Z &= -\frac{M}{g}\{\ddot{z} - e\ddot{\xi}\sin\omega t - e(\omega^2 + 2\omega\dot{\xi})\cos\omega t\}, \\ \mathfrak{T} &= -\frac{M}{g}\{(k^2 + e^2)\ddot{\xi} + e(\ddot{y}\cos\omega t - \ddot{z}\sin\omega t)\}. \end{aligned}\right\} \quad (112.1)$$

Here, by equation (48.1),

$$Y = \frac{\partial V}{\partial y},\ Z = \frac{\partial V}{\partial z},\ \mathfrak{T} = \frac{\partial V}{\partial \theta},$$

where the potential energy V is defined in equation (107.1), so that

$$Y = \frac{1}{\alpha_{11}},\ Z = \frac{1}{\alpha_{11}}$$

if the system is equally stiff in the y- and z-directions of Fig. 174.

From a combination of these results we infer, with the help of equations (47.1) and (48.1), that the vibrations are expressed by

$$\left.\begin{aligned} &\frac{M}{g}\ddot{y} + \frac{1}{\alpha_{11}}y + \frac{M}{g}e\{\ddot{\xi}\cos\omega t - (\omega^2 + 2\omega\dot{\xi})\sin\omega t\} = 0, \\ &\frac{M}{g}\ddot{z} + \frac{1}{\alpha_{11}}z - \frac{M}{g}e\{\ddot{\xi}\sin\omega t + (\omega^2 + 2\omega\dot{\xi})\cos\omega t\} = 0, \\ &\frac{M}{g}(k^2 + e^2)\ddot{\xi} + c\xi + \frac{M}{g}e(\ddot{y}\cos\omega t - \ddot{z}\sin\omega t) = 0, \end{aligned}\right\} \quad . \quad . \quad (112.2)$$

where c is the 'torsional' coefficient of stiffness of the shaft, as in equation (107.1).

It is to be noticed that different values would have been attached to the α_{rs}-coefficients in the expressions for Y and Z had the mounting of the bearings not been equally stiff along the y- and z-axes.

Also the corresponding equations for the case of a fuselage mentioned in Art. 98 may be derived from the foregoing results by annulling the rotation ω.

113. If, by way of illustration, we wish to study the transverse displacements alone, it is evidently necessary to introduce the additional supposition that $\dot{\xi}$ and $\ddot{\xi}$ are negligibly small in the first two of equations (112.2). To this order of approximation, we have, on adding a phase-term ε for purposes of generality,

$$\ddot{y} + \frac{g}{\alpha_{11}M}y = e\omega^2 \sin(\omega t + \varepsilon),$$

$$\ddot{z} + \frac{g}{\alpha_{11}M}z = e\omega^2 \cos(\omega t + \varepsilon).$$

By virtue of results obtained in Chapter III, however, we can put $\dfrac{g}{\alpha_{11}M} = p^2$, where $\dfrac{p}{2\pi}$ denotes the natural frequency of vibration. Hence the motion of the geometrical centre S is expressed by

$$\ddot{y} + p^2 y = e\omega^2 \sin(\omega t + \varepsilon),$$
$$\ddot{z} + p^2 z = e\omega^2 \cos(\omega t + \varepsilon),$$

the solutions of which disclose the component-displacements

$$\left.\begin{aligned} y &= A_1 \cos(pt + \alpha_1) + \frac{e\omega^2 \sin(\omega t + \varepsilon)}{p^2 - \omega^2},\\ z &= A_2 \cos(pt + \alpha_2) + \frac{e\omega^2 \cos(\omega t + \varepsilon)}{p^2 - \omega^2},\end{aligned}\right\} \quad . \quad . \quad (113.1)$$

where A_1, A_2, α_1, α_2 represent arbitrary constants.

Reference to Art. 39 now suffices to show that the path described by the centre S is the combination of:

(i) free vibrations signified by $A_1 \cos(pt + \alpha_1)$, $A_2 \cos(pt + \alpha_2)$, which together give an elliptical path; and

(ii) forced vibrations signified by $\dfrac{e\omega^2 \sin(\omega t+\varepsilon)}{p^2 - \omega^2}$, $\dfrac{e\omega^2 \cos(\omega t+\varepsilon)}{p^2 - \omega^2}$, which together give a circular path.

After the free motion has disappeared owing to the frictional agencies which always operate in actual systems of this kind, there will accordingly remain the forced motion

$$y = \frac{e\omega^2 \sin(\omega t + \varepsilon)}{p^2 - \omega^2},\quad z = \frac{e\omega^2 \cos(\omega t + \varepsilon)}{p^2 - \omega^2},$$

which, on writing $y^2 + z^2 = r^2$ as in Fig. 174, means that

$$r = \frac{e\omega^2}{p^2 - \omega^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (113.2)$$

This represents a circular path, the radius of which will tend to infinitely large values when the dissipative forces are negligibly small.

Our analysis further demonstrates that r will become negative when $\omega > p$, that is, when the shaft has passed through its first whirling speed. Then we may imagine G as lying between O and S in Fig. 174, and consider the system to be in a state of instability which will be examined more fully in Arts. 122–124, where account is taken of the gyroscopic action in systems of this kind.

An important consequence of this is that the sign of r will become negative when ω exceeds the value of p for the fundamental mode of vibration, and will, with increase in speed, continue so until ω becomes greater than the value of p for the second mode of vibration. This statement may be put in another way, and at the same time generalized, by saying that r is negative within the range of speeds bounded by the first and second whirling speeds, the third and fourth whirling speeds, and so on in the higher modes.

114. It is plain that the same method will apply without modification if we regard Fig. 174 as representing a light shaft with a thin rigid disc attached at the point P, provided always that the gyroscopic effect of the disc is of the second order of small quantities.

Moreover, it is not a difficult matter to extend the treatment so as to cover any number of such discs. For this purpose, consider the system of Fig. 175, comprising two bearings of a specified type,

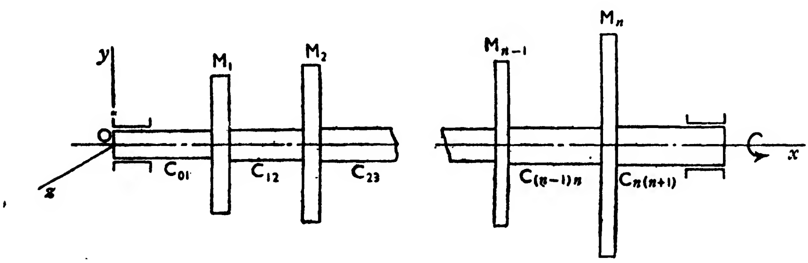


FIG. 175.

and n concentrated loads in the form of thin rigid discs fixed at points 1, 2, . . . , n on a light shaft.

Let $M_1, M_2, \ldots, M_n$ designate the masses of the several discs, and $c_{01}, c_{12}, \ldots, c_{n(n+1)}$ the proper coefficients of stiffness of the intermediate lengths of shaft, as indicated in the figure. The c-symbols may therefore relate either to transverse or to torsional displacements, depending on the kind of motion under examination. Account is taken of a particular type of bearing by introducing the appropriate α_{rs}-coefficients, from the Table of Art. 107.

The expression for the kinetic energy, T, of the complete system is, by equation (111.1),

$$2T = \sum_{s=1}^{n} \frac{M_s}{g} \{(k^2_s + e^2_s)\dot{\theta}^2_s + \dot{y}^2_s + \dot{z}^2_s + 2e_s\dot{\theta}_s(\dot{y}_s \cos\theta_s - \dot{z}_s \sin\theta_s)\} \quad (114.1)$$

these symbols having the same significance as before.

From the corresponding Lagrangian formulae we can now deduce the equations of motion, in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_s}\right) - \frac{\partial T}{\partial y_s} = -\frac{\partial V}{\partial y_s},$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}_s}\right) - \frac{\partial T}{\partial z_s} = -\frac{\partial V}{\partial z_s},$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_s}\right) - \frac{\partial T}{\partial \theta_s} = -\frac{\partial V}{\partial \theta_s},$$

since the potential energy V is defined by equation (107.1).

If the shaft be rotating with uniform angular velocity $\dot{\theta} = \omega$ just before a slight variation in the driving torque $\mathfrak{T}_s$ causes a correspondingly small change $\dot{\xi}_s$ in the angular velocity of the sth disc, then the relations

$$\dot{\theta}_s = \omega + \dot{\xi}_s, \; \ddot{\theta}_s = \ddot{\xi}_s, \; \dot{\theta}^2_s = \omega^2 + 2\omega\dot{\xi}_s$$

will hold so long as terms beyond the first power in $\dot{\xi}$ may be treated as insignificant.

Inserting these values in the above equations of motion and omitting, as previously, terms which refer to the steady state, we obtain $3n$ expressions of the type

$$\frac{M_1}{g}\ddot{y}_1 + \frac{1}{\alpha_{11}}y_1 + \frac{1}{\alpha_{12}}y_2 + \ldots$$
$$= \frac{M_1}{g}e_1\{(\omega^2 + 2\omega\dot{\xi}_1)\sin(\omega t + \varepsilon_1) - \ddot{\xi}_1\cos(\omega t + \varepsilon_1)\},$$

$$\frac{M_2}{g}\ddot{y}_2 + \frac{1}{\alpha_{21}}y_1 + \frac{1}{\alpha_{22}}y_2 + \ldots$$
$$= \frac{M_2}{g}e_2\{(\omega^2 + 2\omega\dot{\xi}_2)\sin(\omega t + \varepsilon_2) - \ddot{\xi}_2\cos(\omega t + \varepsilon_2)\},$$

. . . .

. . . .

$$\frac{M_1}{g}\ddot{z}_1 + \frac{1}{\alpha_{11}}z_1 + \frac{1}{\alpha_{12}}z_2 + \ldots$$
$$= \frac{M_1}{g}e_1\{(\omega^2 + 2\omega\dot{\xi}_1)\cos(\omega t + \varepsilon_1) + \ddot{\xi}_1\sin(\omega t + \varepsilon_1)\},$$

$$\frac{M_2}{g}\ddot{z}_2 + \frac{1}{\alpha_{21}}z_1 + \frac{1}{\alpha_{22}}z_2 + \quad . \quad . \quad . \quad .$$

$$= \frac{M_2}{g}e_2\{(\omega^2 + 2\omega\dot{\xi}_2)\cos(\omega t + \varepsilon_2) + \ddot{\xi}_2 \sin(\omega t + \varepsilon_2)\},$$

. . . .

. . . .

$$\frac{M_1}{g}(k^2_1 + e^2_1)\ddot{\xi}_1 - c_{12}(\xi_2 - \xi_1)$$

$$= \frac{M_1}{g}e_1\{\ddot{y}_1 \cos(\omega t + \varepsilon_1) - \ddot{z}_1 \sin(\omega t + \varepsilon_1)\} + \mathfrak{T}_1,$$

$$\frac{M_2}{g}(k^2_2 + e^2_2)\ddot{\xi}_2 - c_{12}(\xi_1 - \xi_2) - c_{23}(\xi_3 - \xi_2)$$

$$= \frac{M_2}{g}e_2\{\ddot{y}_2 \cos(\omega t + \varepsilon_2) - \ddot{z}_2 \sin(\omega t + \varepsilon_2) + \mathfrak{T}_2$$

. . . .

. . . .

Here ε_s, $\mathfrak{T}_s$ denote in turn the phase and the mean torque of the sth disc, so that these equations together determine, to the implied degree of accuracy, the general motion of the system.

It has already been demonstrated that these transverse and torsional vibrations can only be treated as independent phenomena under conditions in which small values are attached to the n eccentricities e_s, and values of the second order of small quantities are attached to the components of $\dot{\xi}$, $\ddot{\xi}$ in the motion parallel to the y- and z-axes. Then our equations reduce, on omitting terms in e^2 as insignificant, to

$$\frac{M_1}{g}\ddot{y}_1 + \frac{1}{\alpha_{11}}y_1 + \frac{1}{\alpha_{12}}y_2 + \frac{1}{\alpha_{13}}y_3 + \ . \ . \ . = \frac{M_1}{g}e_1\omega^2 \sin(\omega t + \varepsilon_1),$$

$$\frac{M_2}{g}\ddot{y}_2 + \frac{1}{\alpha_{21}}y_1 + \frac{1}{\alpha_{22}}y_2 + \frac{1}{\alpha_{23}}y_3 + \ . \ . \ . = \frac{M_2}{g}e_2\omega^2 \sin(\omega t + \varepsilon_2),$$

. . . .

. . . .

$$\frac{M_1}{g}\ddot{z}_1 + \frac{1}{\alpha_{11}}z_1 + \frac{1}{\alpha_{12}}z_2 + \frac{1}{\alpha_{13}}z_3 + \ . \ . \ . = \frac{M_1}{g}e_1\omega^2 \cos(\omega t + \varepsilon_1),$$

$$\frac{M_2}{g}\ddot{z}_2 + \frac{1}{\alpha_{21}}z_1 + \frac{1}{\alpha_{22}}z_2 + \frac{1}{\alpha_{23}}z_3 + \ . \ . \ . = \frac{M_2}{g}e_2\omega^2 \cos(\omega t + \varepsilon_2),$$

. . . .

. . . .

$$\frac{M_1}{g}k^2_1\ddot{\xi}_1 - c_{12}(\xi_2 - \xi_1) = \mathfrak{T}_1,$$

$$\frac{M_2}{g}k^2{}_2\ddot{\xi}_2 - c_{12}(\xi_1 - \xi_2) - c_{23}(\xi_3 - \xi_2) = \mathfrak{T}_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

A solution may be obtained by assuming, in accordance with the general theory of Chapter III, that $y_1, y_2, \ldots y_n, z_1, z_2, \ldots, z_n$ vary as e^{ipt}, and $\xi_1, \xi_2, \ldots, \xi_n$ vary as $e^{i\phi t}$.

At this stage of the work we shall draw a distinction between the corresponding coefficients of stiffness by putting

$$\frac{g}{\alpha_{rs}} = C_{rs}, \; gc_{rs} = \gamma_{rs}.$$

On making these substitutions, and eliminating the ratios

$$y_1 : y_2 : \ldots : y_n, z_1 : z_2 : \ldots : z_n \text{ and } \xi_1 : \xi_2 : \ldots : \xi_n,$$

it will be found that

$$\begin{vmatrix} C_{11}-M_1p^2, & C_{12}, & C_{13}, & \ldots, & C_{1n} \\ C_{21}, & C_{22}-M_2p^2, & C_{23}, & \ldots, & C_{2n} \\ C_{31}, & C_{32}, & C_{33}-M_3p^2, & \ldots, & C_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{n1}, & C_{n2}, & C_{n3}, & \ldots, & C_{nn}-M_np^2 \end{vmatrix} = 0, \quad . \quad (114.2)$$

$$\begin{vmatrix} \gamma_{12}-M_1k^2{}_1\phi^2, & \gamma_{12}, & 0, & \ldots, & 0 \\ -\gamma_{12}, & \gamma_{12}+\gamma_{23}-M_2k^2{}_2\phi^2, & 0, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & \gamma_{n(n+1)}-M_nk^2{}_n\phi^2 \end{vmatrix} = 0 \quad . \quad (114.3)$$

The n roots of these determinantal equations in p^2, ϕ^2 respectively define the 'critical' speeds of the shaft in transverse and in torsional vibration, the frequencies being $\frac{p}{2\pi}$ and $\frac{\phi}{2\pi}$. Each element of the shaft accordingly executes simple harmonic motion of the corresponding period, and all the discs pass simultaneously through the equilibrium-position.

Ex. 1. Find the natural frequencies in torsional vibration of the system shown in Fig. 176, consisting of a light shaft of circular

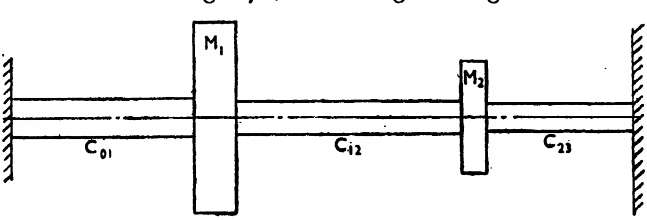


FIG. 176.

cross-section fixed at both ends, and two thin rigid discs specified, in the above notation, by $M_1k_1^2$, $M_2k_2^2$.

Suppose the discs M_1, M_2 to be slightly displaced from the position of rest by amounts ξ_1, ξ_2, taken in turn; and let c_{01}, c_{12}, c_{23} be appropriate coefficients of stiffness of the intermediate lengths of shaft, as indicated in the figure.

A repetition of the foregoing argument leads in a straightforward manner to

$$\frac{M_1}{g}k^2{}_1\ddot{\xi}_1 + c_{01}\xi_1 - c_{12}(\xi_2 - \xi_1) = 0,$$

$$\frac{M_2}{g}k^2{}_2\ddot{\xi}_2 + c_{21}(\xi_2 - \xi_1) + c_{23}\xi_2 = 0,$$

which may be rearranged in the form

$$\frac{M_1}{g}k^2{}_1\ddot{\xi}_1 + (c_{01} + c_{12})\xi_1 - c_{12}\xi_2 = 0,$$

$$- c_{21}\xi_1 + \frac{M_2}{g}k^2{}_2\ddot{\xi}_2 + (c_{21} + c_{23})\xi_2 = 0.$$

Introducing the assumption that ξ_1, ξ_2 vary as $e^{i\phi t}$, and eliminating the ratio $\xi_1 : \xi_2$ between the resulting expressions, we obtain, ultimately,

$$\phi^4 - g\left(\frac{c_{01} + c_{12}}{M_1k^2{}_1} + \frac{c_{12} + c_{23}}{M_2k^2{}_2}\right)\phi^2 + g^2\left(\frac{c_{01}c_{12} + c_{12}c_{23} + c_{23}c_{01}}{M_1M_2k^2{}_1k^2{}_2}\right) = 0,$$

since $c_{rs} = c_{sr}$.

If the roots of this quadratic in ϕ^2 be denoted by $\phi^2{}_1$, $\phi^2{}_2$, it follows that the required frequencies are $\frac{\phi_1}{2\pi}, \frac{\phi_2}{2\pi}$.

Ex. 2. Determine the natural frequencies in torsional vibration of a system formed by three thin rigid discs and a light shaft free to rotate in bearings situated at the ends.

Since the ends of the shaft are free, it is only necessary to use two coefficients of stiffness, say c_{12}, c_{23}, in the present notation. We may likewise write M_1, M_2, M_3 for the masses of the discs, and k_1, k_2, k_3 for their radii of gyration.

If we suppose that in the implied normal mode of vibration the masses M_1, M_2, M_3 separately undergo slight displacements ξ_1, ξ_2, ξ_3 about the position of rest, it is readily proved that the motion is expressed by

$$\frac{M_1}{g}k^2{}_1\ddot{\xi}_1 - c_{12}(\xi_2 - \xi_1) = 0,$$

$$\frac{M_2}{g}k^2{}_2\ddot{\xi}_2 + c_{12}(\xi_2 - \xi_1) - c_{23}(\xi_3 - \xi_2) = 0,$$

$$\frac{M_3}{g}k^2{}_3\ddot{\xi}_3 + c_{23}(\xi_3 - \xi_2) = 0.$$

Thus, with $\frac{1}{\mu_s}$ written for $\frac{M_s}{g}k^2_s$, it appears that

$$\begin{aligned}
&\ddot{\xi}_1 - c_{12}\mu_1(\xi_2 - \xi_1) = 0, \\
&\ddot{\xi}_2 + c_{12}\mu_2(\xi_2 - \xi_1) - c_{23}\mu_2(\xi_3 - \xi_2) = 0, \\
&\ddot{\xi}_3 + c_{23}\mu_3(\xi_3 - \xi_2) = 0,
\end{aligned}$$

which give, on subtracting the first from the second, and the second from the third,

$$\begin{aligned}
&\ddot{\xi}_2 - \ddot{\xi}_1 + c_{12}(\mu_1 + \mu_2)(\xi_2 - \xi_1) - c_{23}\mu_2(\xi_3 - \xi_2) = 0, \\
&\ddot{\xi}_3 - \ddot{\xi}_2 - c_{12}\mu_2(\xi_2 - \xi_1) + c_{23}(\mu_2 + \mu_3)(\xi_3 - \xi_2) = 0.
\end{aligned}$$

Introducing the usual assumption that ξ_1, ξ_2, ξ_3 vary as $e^{i\phi t}$, we find, after eliminating the ratios $\xi_1 : \xi_2 : \xi_3$ between the resulting equations,

$$\phi^4 - \{c_{12}(\mu_1 + \mu_2) + c_{23}(\mu_2 + \mu_3)\}\phi^2 + c_{12}c_{23}(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1) = 0,$$

so that if ϕ^2_1, ϕ^2_2 represent the roots of this quadratic in ϕ^2, then $\frac{\phi_1}{2\pi}$, $\frac{\phi_2}{2\pi}$ are the required frequencies.

The reader may profitably compare this result with that of the previous example.

115. This part of the work may be concluded by a study of the transverse vibration of a light shaft with any number of bearings whose elastic characteristics are symmetrical with respect to the axis of rotation, and with any number of thin rigid discs fixed to the shaft. In this extension of the analysis we avail ourselves of the fact that it is a matter of indifference whether the load at a given point is due to a bearing or to an extraneous force, provided the proper sign is attached to a reaction in the first place, and to an applied load in the second.

Since it is not always practicable to solve such a problem without recourse to the graphical method of Art. 91, the equations of motion will be arranged so as to exhibit the quantities which are to be evaluated by experimental means in the general case.

Our purpose will be best served by supposing that, with the same co-ordinates as before, the separate application of components of force $Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n$ at points $1, 2, \ldots, n$, cause component displacements $y_1, y_2, \ldots y_n, z_1, z_2, \ldots, z_n$ at the corresponding points on the shaft, measured from the equilibrium-position. It may help to fix ideas if we attribute the extraneous loads to thin rigid discs, and assume that their gyroscopic action is negligibly small, as will be the case if the slope of the shaft remains correspondingly small.

With the aid of equations (107.2) we can at once write $2n$ relations between the displacements and the forces, in the form

$$\left.\begin{array}{l} y_1 = \alpha_{11}Y_1 + \alpha_{12}Y_2 + \dots + \alpha_{1n}Y_n, \\ y_2 = \alpha_{21}Y_1 + \alpha_{22}Y_2 + \dots + \alpha_{2n}Y_n, \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \\ z_1 = \alpha_{11}Z_1 + \alpha_{12}Z_2 + \dots + \alpha_{1n}Z_n, \\ z_2 = \alpha_{21}Z_1 + \alpha_{22}Z_2 + \dots + \alpha_{2n}Z_n, \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \end{array}\right\} \quad . \quad . \quad (115.1)$$

where the α_{rs}–symbols are as defined in Art. 107.

Reference to equations (111.2) suffices to show that, with terms beyond the first power in ξ neglected as of the second order of small quantities, the forces are expressed by relations of the type

$$\left.\begin{array}{l} Y_1 = -\dfrac{M_1}{g}[\ddot{y}_1 + e_1\{\ddot{\xi}_1 \cos(\omega t+\varepsilon_1) - (\omega^2 + 2\omega\dot{\xi}_1)\sin(\omega t+\varepsilon_1)\}], \\ Z_1 = -\dfrac{M_1}{g}[\ddot{z}_1 - e_1\{\ddot{\xi}\sin(\omega t+\varepsilon_1) + (\omega^2+2\omega\dot{\xi}_1)\cos(\omega t+\varepsilon_1)\}], \end{array}\right\} \quad (115.2)$$

where M_s, e_s, ε_s refer in turn to the mass, eccentricity, and phase of the sth disc, and ω is the 'equilibrium' angular velocity of the system as a whole.

Combining equations (115.1) and (115.2), and taking account of the condition that the terms $\dot{\xi}$, $\ddot{\xi}$ must both be negligibly small in the expressions for Y, Z if the transverse vibration is to be examined alone, we thus deduce $2n$ relations of the type

$$y_1 = -\alpha_{11}\frac{M_1}{g}\{\ddot{y}_1 - e_1\omega^2\sin(\omega t+\varepsilon_1)\} - \alpha_{12}\frac{M_2}{g}\{\ddot{y}_2 - e_2\omega^2\sin(\omega t+\varepsilon_2)\}$$
$$-\alpha_{13}\frac{M_3}{g}\{\ddot{y}_3 - e_3\omega^2\sin(\omega t+\varepsilon_3)\} - \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$z_1 = -\alpha_{11}\frac{M_1}{g}\{\ddot{z}_1 - e_1\omega^2\cos(\omega t+\varepsilon_1)\} - \alpha_{12}\frac{M_2}{g}\{\ddot{z}_2 - e_2\omega^2\cos(\omega t+\varepsilon_2)\}$$
$$-\alpha_{13}\frac{M_3}{g}\{\ddot{z}_3 - e_3\omega^2\cos(\omega t+\varepsilon_3)\} - \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

These may easily be rearranged in the more convenient form

$$\left.\begin{aligned}
&\alpha_{11}M_1\ddot{y}_1 + gy_1 + \alpha_{12}M_2\ddot{y}_2 + \alpha_{13}M_3\ddot{y}_3 + \dots \\
&\qquad\qquad = g\omega^2 Q_1 \sin(\omega t + \zeta_1), \\
&\alpha_{21}M_1\ddot{y}_1 + \alpha_{22}M_2\ddot{y}_2 + gy_2 + \alpha_{23}M_3y_3 + \dots \\
&\qquad\qquad = g\omega^2 Q_2 \sin(\omega t + \zeta_2), \\
&\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots \\
&\alpha_{11}M_1\ddot{z}_1 + gz_1 + \alpha_{12}M_2\ddot{z}_2 + \alpha_{13}M_3\ddot{z}_3 + \dots \\
&\qquad\qquad = g\omega^2 R_1 \cos(\omega t + \eta_1), \\
&\alpha_{21}M_1\ddot{z}_1 + \alpha_{22}M_2\ddot{z}_2 + gz_2 + \alpha_{23}M_3\ddot{z}_3 + \dots \\
&\qquad\qquad = g\omega^2 R_2 \cos(\omega t + \eta_2), \\
&\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \qquad (115.3)$$

where, for example,

$$Q_1 = \frac{\alpha_{11}M_1e_1 \sin(\omega t + \varepsilon_1) + \alpha_{12}M_2e_2 \sin(\omega t + \varepsilon_2) + \dots}{g \sin(\omega t + \zeta_1)},$$

with $$\tan \zeta_1 = \frac{\alpha_{11}M_1e_1 \cos \varepsilon_1 + \alpha_{12}M_2e_2 \cos \varepsilon_2 + \dots}{\alpha_{11}M_1e_1 \sin \varepsilon_1 + \alpha_{12}M_2e_2 \sin \varepsilon_2 + \dots}.$$

This set of equations determines the forced vibrations which take place due to the $2n$ components of force Q_s, R_s, where $r, s = 1, 2, \dots, n$.

The most general law of variation with the time may, as already pointed out, be based on the supposition that the forces are of this simple-harmonic type. Analytically, it is simplest to assume that the forces vary as $e^{i\omega t}$, with a complex coefficient. Then equations (115.3) assume a form which may be solved by the method used in connection with equation (58.4). If the determinant of the coefficients in the resulting form be designated by $\Delta(\omega^2)$, from the same argument we infer that a condition of resonance will occur when ω^2 approximates to a root of

$$\Delta(\omega^2) = 0,$$

and therefore a root of

$$\begin{vmatrix}
\alpha_{11}M_1\omega^2 - g, & \alpha_{12}M_2\omega^2, & \alpha_{13}M_3\omega^2, & \dots, & \alpha_{1n}M_n\omega^2 \\
\alpha_{21}M_1\omega^2, & \alpha_{22}M_2\omega^2 - g, & \alpha_{23}M_3\omega^2, & \dots, & \alpha_{2n}M_n\omega^2 \\
\alpha_{31}M_1\omega^2, & \alpha_{32}M_2\omega^2, & \alpha_{33}M_3\omega^2 - g, & \dots, & \alpha_{3n}M_n\omega^2 \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
\alpha_{n1}M_1\omega^2, & \alpha_{n2}M_2\omega^2, & \alpha_{n3}M_3\omega^2, & \dots, & \alpha_{nn}M_n\omega^2 - g
\end{vmatrix} = 0 \qquad (115.4)$$

Hence if $\omega_1, \omega_2, \dots, \omega_n$ are the roots of this equation of degree n in ω^2, whirling will, in the implied absence of friction, occur when the shaft is rotating with periodicities $\frac{2\pi}{\omega_1}, \frac{2\pi}{\omega_2}, \dots, \frac{2\pi}{\omega_n}$. It

follows from a previous remark that the shaft will be stable at the intermediate speeds thus specified.

The present treatment is, of course, based on the supposition that the elastic characteristics of the bearings and of the shaft are symmetrical with respect to the axis of rotation. If, on the contrary, any one of the bearings is unequally stiff in the y- and z-directions of Fig. 174, the number of critical speeds for the system as a whole will thereby be increased, because the bearing concerned will then have two different coefficients of stiffness with regard to the fixed axes. This applies also in systems where the cross-section of the shaft is not circular, since the coefficient α_{rs} is then a function of the co-ordinate θ when referred to the y- and z-axes of the figure. But whirling speeds that arise from these causes are usually of secondary importance, and especially in cases where want of symmetry in this connection is due to a keyway of standard proportions. These points are, nevertheless, of practical interest because they relate to phenomena which are sometimes exhibited on records taken with instruments.

Furthermore, the vibration in any mode will be damped by the inevitable dissipative forces in an actual system. If we assume the friction to be proportional to the component-velocities $\dot{y}$, $\dot{z}$ as defined in equations (115.3), expressions for the damped motion follow on adding terms of the type $b\dot{y}$, $b\dot{z}$ to the right-hand sides of those equations, where b signifies a coefficient of friction to be found experimentally.

In the particular case of a relatively short shaft and a number of discs spaced closely together, the mass of the shaft itself may, as a first approximation, be included in the M-terms of our analysis by dividing the shaft into elements of length so that a disc is situated near the mid-point of each of the intermediate elements. It must, however, be conceded that this method is of limited value in all but favourable circumstances.

Ex. 1. Determine, on the foregoing assumptions, the whirling speeds of a system composed of two thin rigid discs, a light shaft and two bearings mounted so that they are equally stiff in relation to rectangular axes in the transverse plane.

It is clear that the whirling speed will be expressed by the first two rows and columns of the determinantal equation (115.4) if M_1, M_2, e_1, e_2 signify the masses and eccentricities of the discs. Thus we obtain

$$\begin{vmatrix} \alpha_{11}M_1\omega^2 - g, & \alpha_{12}M_2\omega^2 \\ \alpha_{21}M_1\omega^2, & \alpha_{22}M_2\omega^2 - g \end{vmatrix} = 0,$$

which, on expanding, gives

$$(\alpha_{11}\alpha_{22} - \alpha^2_{12})M_1M_2\omega^4 - g(\alpha_{11}M_1 + \alpha_{22}M_2)\omega^2 + g^2 = 0,$$

where the α_{rs}-symbols denote the coefficients of stiffness in question.

From this quadratic in ω^2 we infer that the 'critical' speeds in a normal mode of vibration are equal to

$$\left[g\frac{(\alpha_{11}M_1 + \alpha_{22}M_2) \pm \{(\alpha_{11}M_1 - \alpha_{22}M_2)^2 + 4\alpha^2{}_{12}M_1M_2\}^{\frac{1}{2}}}{2(\alpha_{11}\alpha_{22} - \alpha^2{}_{12})M_1M_2}\right]^{\frac{1}{2}},$$

in radians per second if pound-foot-second units are employed in the calculations. This form has been deduced by putting $\alpha_{21} = \alpha_{12}$, which expresses the fact that the bearings and the shaft are equally stiff about the axis of rotation.

Ex. 2. Find the first whirling speed of a slender shaft of equivalent length L, supported at its ends by 'short' bearings, and loaded with a concentrated mass M at its mid-length. The bearings are fitted in rigid mountings, and the shaft is heavy compared with M.

If, for a moment, we neglect the mass of the shaft, and let ω_1 be the whirling speed under consideration, then

$$\omega^2{}_1 = \frac{g}{\alpha_{11}M}.$$

Since the load is situated at mid-span, the value of α_{11} is given by putting $l_1 = l_2 = \frac{1}{2}l$ in the expression for α_{rs} in the third row of the Table of Art. 107. Hence, in the present notation,

$$\alpha_{11} = \frac{L^3}{48EI},$$

and therefore

$$\omega_1 = \sqrt{\frac{48EgI}{ML^3}},$$

where E, I refer in succession to the appropriate modulus of the material and moment of inertia of the cross-section.

With a view to estimating the whirling speed when account is taken of the mass of the shaft, write $\frac{\omega_0}{2\pi}$ for the fundamental frequency of the unloaded shaft. If we also let $\frac{\omega}{2\pi}$ be the gravest frequency when the given load and mass of the shaft are both included in the calculations, then in virtue of the stationary property of a normal mode of vibration (Art. 55) we can write

$$\frac{1}{\omega^2} = \frac{\mu}{\omega^2{}_0} + \frac{1}{\omega^2{}_1},$$

where μ is the measure of the constraint thus introduced. A knowledge of the magnitude of μ now suffices to complete the calculation, and it is therefore of interest to remark that S. Dunkerley [1] found,

[1] *Trans. Roy. Soc.*, vol. 185, page 279 (1894).

in a remarkable series of experiments, μ to be approximately equal to 0·88 for the present system.

He further established, on the basis of the tests undertaken, an approximate, but very useful, rule for n such loads, which in our notation is

$$\frac{1}{\omega^2} = \frac{1}{\omega^2_0} + \frac{1}{\omega^2_1} + \ldots + \frac{1}{\omega^2_n}.$$

The same problem has also been examined in an instructive manner by C. Chree.[1]

116. Gearing. At various places in the preceding treatment mention has been made of the fact that gearing is sometimes a contributory factor in the vibration of shafts. The following discussion will serve to elucidate the problem, since the consequences of a prescribed disturbance can be investigated by our usual method once the general motion of a given installation has been formulated.

Suppose the system to be as shown in Fig. 177, consisting of

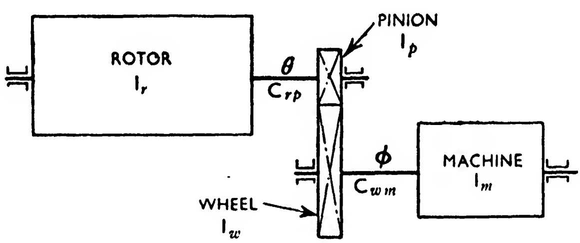


FIG. 177.

a 'machine' driven through a wheel and pinion by a 'rotor'. Typical examples are to be found on ships and aircraft, in which the term 'machine' may be used in reference to the propeller or airscrew, the 'rotor' being then associated with the engine.

The motion at any instant can be specified by assigning the angular co-ordinate

θ_r to the rotor,
θ_p to the pinion,
ϕ_w to the wheel,
ϕ_m to the machine, and putting

$n = \dfrac{\text{diameter of pinion}}{\text{diameter of wheel}} = -\dfrac{\partial\theta}{\partial\phi}$ if the rotor be taken as revolving in the positive direction.

Further, let I_r, I_p, I_w, I_m separately represent the moment of inertia, about the respective axes of rotation, of the rotor, pinion, wheel, and machine. The masses of the shafts will be omitted

[1] *Phil. Mag.*, vol. 7, page 504 (1904).

because they are most easily taken into consideration by the method described in Ex. 2 of Art. 108.

If, with these restrictions, the kinetic energy of the system be denoted by T, it is evident that

$$2T = I_r\dot{\theta}^2_r + I_p\dot{\theta}^2_p + I_w\dot{\phi}^2_w + I_m\dot{\phi}^2_m. \quad . \quad . \quad (116.1)$$

To write down the relation for the potential energy V, let c_{rp}, c_{wm} designate the coefficients of stiffness of the shafts between the pairs rotor–pinion, wheel–machine, respectively; also, let c_{pw} be the appropriate coefficient of stiffness of the pinion–wheel pair. Then, by equation (107.1),

$$2V = c_{rp}(\theta_r - \theta_p)^2 + c_{pw}(\Delta\phi)^2 + c_{wm}(\phi_w - \phi_m)^2, \quad (116.2)$$

where $\Delta\phi$ is the relative displacement between the pinion-wheel pair, expressed as a function of the ϕ-co-ordinate. All the displacements involved in this analysis necessarily relate to angular movements.

Now, in terms of *unit* torque, the displacement of the pinion alone may be signified by $\dfrac{1}{c_{pp}}$, and that of the wheel alone by $\dfrac{1}{c_{ww}}$, both of which can be evaluated experimentally. Furthermore,

$$\frac{1}{c_{wp}} = \frac{1}{c_{ww}} + \frac{1}{n^2c_{pp}},$$

from Art. 38, and

$$\theta_p = \phi_w\frac{\partial\theta}{\partial\phi} + (\Delta\phi + \Delta\phi_{pw})\frac{\partial\theta}{\partial\phi}$$
$$= -n(\phi_w + \Delta\phi + \Delta\phi_{pw}),$$

from the definition of the gear-ratio n. From the consequent relation

$$\Delta\phi = -\frac{1}{n}\theta_p - \phi_w - \Delta\phi_{pw}$$

it follows that equation (116.2) can be expressed in the more convenient form

$$2V = c_{rp}(\theta_r - \theta_p)^2 + c_{wp}\left(\frac{1}{n}\theta_p + \phi_w + \Delta\phi_{pw}\right)^2 + c_{wm}(\phi_w - \phi_m)^2. \quad . \quad (116.3)$$

The required equations of motion may now be derived from the foregoing information and Lagrange's formula

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q} + \mathfrak{T}_q, \quad . \quad . \quad . \quad . \quad (116{\cdot}4)$$

where $\mathfrak{T}_q$ is the torque associated with any co-ordinate q.

If, for example, we put $q = \theta_r$, equations (116.1) and (116.3) give

$$\frac{\partial T}{\partial \dot{\theta}_r} = I_r\dot{\theta}_r,\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_r}\right) = I_r\ddot{\theta}_r,\ \frac{\partial T}{\partial \theta_r} = 0,\ \frac{\partial V}{\partial \theta_r} = c_{rp}(\theta_r - \theta_p),$$

hence, by equation (116.4),

$$I_r\ddot{\theta}_r = -\,c_{rp}(\theta_r - \theta_p) + \mathfrak{T}_r,$$

$\mathfrak{T}_r$ being the driving torque concerned. Similarly, for the remaining co-ordinates,

$$I_p\ddot{\theta}_p = c_{rp}(\theta_r - \theta_p) - \frac{1}{n}c_{wp}\left(\frac{1}{n}\theta_p + \phi_w + \Delta\phi_{pw}\right),$$

$$I_w\ddot{\phi}_w = -\,c_{wp}\left(\frac{1}{n}\theta_p + \phi_w + \Delta\phi_{pw}\right) - c_{wm}(\phi_w - \phi_m),$$

$$I_m\ddot{\phi}_m = c_{wm}(\phi_w - \phi_m) - \mathfrak{T}_m,$$

where $\mathfrak{T}_m$ denotes the torque-reaction of the 'machine'.

Simple substitutions suffice to translate these four equations in terms of any one of the co-ordinates involved, since

$$\ddot{\theta}_r = -\,n\ddot{\phi}_r,\ \ddot{\theta}_p = -\,n\ddot{\phi}_p, \quad . \quad . \quad . \quad . \qquad (116.5)$$

which are derived from the obvious relations

$$\theta_r = -\,n\phi_r,\ \theta_p = -\,n\phi_p.$$

Taking ϕ as the most convenient co-ordinate of reference, we thus have the motion determined by

$$\left.\begin{aligned} I_r\ddot{\phi}_r + c_{rp}(\phi_r - \phi_p) &= -\frac{1}{n}\mathfrak{T}_r,\\ I_p\ddot{\phi}_p - c_{rp}(\phi_r - \phi_p) + \frac{1}{n^2}c_{pw}(\phi_w - \phi_p + \Delta\phi_{pw}) &= 0,\\ I_w\ddot{\phi}_w + c_{pw}(\phi_w - \phi_p + \Delta\phi_{pw}) + c_{wm}(\phi_w - \phi_m) &= 0,\\ I_m\ddot{\phi}_m - c_{wm}(\phi_w - \phi_m) &= -\,\mathfrak{T}_m. \end{aligned}\right\} \quad (116.6)$$

In the general case both $\mathfrak{T}_r$ and $\mathfrak{T}_m$ vary according to laws which should here be formulated, in a Fourier series, from a study of records taken with instruments, the proper signs being attached to the graphs when they are registered separately.

Imperfectly formed teeth on the gears occasionally constitute an additional source of disturbance. Suppose the defective member to have N teeth that vary from the true shape according to a simple-harmonic law. If ω be the steady angular velocity of the shaft, the motion will on this account be modified to an extent which is determined by putting

$$\Delta\phi_{pw} = A\cos N\omega t,$$

in the above expressions, A being a determinate constant.

It is to be remembered, in conclusion, that perfectly rigid bearings are implied in the above analysis, for the vibration set up by gearing is sometimes magnified owing to the elastic nature of actual bearings and their mountings. The effect of such elastic supports may readily be investigated by the graphical method of Art. 91 (*b*).

117. The Wheel and Blading of a Turbine. All the natural

modes of vibration for the rotating part of a steam turbine could be determined, to the implied degree of accuracy, from the results of Art. 115 if the wheels and blading were perfectly rigid. To bring conclusions thus arrived at into agreement with fact, we must therefore include the vibrations which actual wheels and blading can execute due to the elastic nature of the constituent materials. It may then be necessary to study a number of modes for a given wheel, involving nodal circles and nodal diameters, with the central part more or less rigidly fixed to the shaft. In work of this kind we are guided by the fact that a practical significance is only attached to modes with frequencies that come within the range of working speeds; the mode defined by one nodal diameter, for example, is unimportant because it is not excited under ordinary conditions. It is to be remarked in this connection that the inertia of the blading, which will be considered in Art. 121, influences the positions of the nodal circles in relation to the periphery of the wheel proper.

Such vibrations may be produced by a number of causes, the chief of which have reference to the horizontal joint, the stationary blading, and partial admission, taking for granted the high degree of balance of modern installations.

Experience confirms our expectation of a considerable increase in the alternating stress in certain parts of the rotating system when any combination of the disturbing agencies operate in a state of resonance. Serious failures have indeed been traced to this source, and it is therefore common practice to arrange matters so that the impressed frequency does not come within a range of about 30 per cent. of the natural frequency of the principal parts of a specified machine. This rule applies in particular to the rotor, and especially one of the 'disc' type, seeing that the critical speed exceeds the running speed by a safe margin in such cases as usually occur with the 'drum' type.

To formulate the stress-wave implied in any disturbed motion of a specified wheel, let us confine our attention to vibrations in the axial direction and, for a moment, imagine the wheel to be without rotation. If r, θ be the co-ordinates of any point on the wheel, and $\dfrac{p}{2\pi}$ its natural frequency in a mode characterized by m nodal diameters, the introduction in equation (63.4) of the fact that the velocity of propagation

$$a = \text{wave-length} \times \text{frequency}$$

allows us to put the transverse or axial displacement

$$u = f(m\theta - pt) + f(m\theta + pt), \quad . \quad . \quad . \quad (117.1)$$

assuming the material to be uniform in all essentials throughout the

region covered by the analysis. We thus learn, on taking the right-hand members of this equation in succession, that stress-waves will travel in the positive and negative directions of rotation, with angular velocity $\frac{p}{m}$ relative to the wheel, provided the material is homogeneous and isotropic throughout. These waves will continue indefinitely with the time so long as the dissipative forces remain negligibly small, as we shall suppose to be so for the present.

It has further been demonstrated that the displacement may be represented by a relation of the type

$$u = U \sin (m\theta \pm pt) \quad . \quad . \quad . \quad . \quad (117.2)$$

when the extraneous force is of the periodic kind, and U denotes a function that satisfies the prescribed conditions of motion.

Provided an appropriate expression for U is known, the above information will suffice for the purpose of estimating, by the method exemplified in Art. 93, the frequency, deflection and nodal pattern when the wheel is not rotating. In the general case, however, it is not always easy to write down a reasonably accurate relation for U without recourse to data of an experimental origin.

118. When the wheel is revolving with a velocity of the customary magnitude, it is essential to take account of the effect of rotation disclosed in Art. 19 (*d*).

A combination of the methods of Arts. 55 and 94 obviously affords a means of adapting the foregoing treatment to the new circumstances, without greatly affecting the degree of approximation involved.

To fix ideas on the point, let the wheel and the part of the blading that matters be as shown in Fig. 178, where the radius R_2 is supposed to be known, and the vibration takes place in a direction parallel to the x-axis. If the material have density ρ, and thickness $2h$ at radius r, it is readily seen from considerations of an annular element, indicated by the hatched cross-section, that the kinetic energy, T, of the complete wheel is expressed by

$$T = \frac{2\pi\rho}{g}\int_{R_1}^{R_2} hr\left(\frac{\partial u}{\partial t}\right)^2 dr, \quad . \quad . \quad . \quad . \quad (118.1)$$

h being in general a function of r.

The fact that the displacement u is usually a function of r and θ may, with the same conventions, be formulated by putting

$$u = U \sin m\theta \cos pt,$$

where U is a function of r. On this understanding, it follows from equation (118.1) that the maximum kinetic energy

$$T_{\text{max.}} = \frac{2\pi\rho p^2}{g}\int_{R_1}^{R_2} hU^2 r dr, \quad . \quad . \quad . \quad . \quad (118.2)$$

with, as previously, the frequency of vibration designated by $\frac{p}{2\pi}$.

A corresponding estimate of the maximum potential energy, $V_{\text{max.}}$, for the *non-rotating* wheel can be derived without much difficulty from equation (94.11), though, on the present suppositions, it is necessary to treat the symbol h^3 of equation (94.6) as a function of r, and therefore place it under the sign of integration.

This is perhaps the most suitable place to repeat a previous remark, to the effect that the relation for the potential energy V will be free from terms in $\frac{\partial u}{\partial \theta}$ and $\frac{\partial^2 u}{\partial \theta^2}$ if nodal circles only are involved.

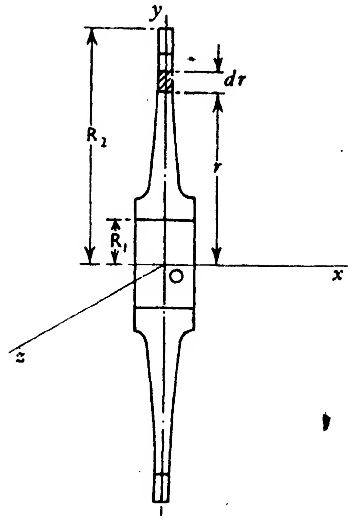


FIG. 178.

There remains to be found a term corresponding to W of equation (19.11). As the mass of the annular element is

$$\frac{4\pi}{g}\rho h r\,dr,$$

its centrifugal effect will be

$$\frac{4\pi\omega^2}{g}\rho h r^2 dr$$

when the wheel is rotating with angular velocity ω. Moreover, if dr of Fig. 178 becomes ds in the disturbed configuration of the wheel, for small displacements we have

$$ds = \{(dr)^2 + (du)^2\}^{\frac{1}{2}}$$

$$= dr\left\{1 + \left(\frac{du}{dr}\right)^2\right\}^{\frac{1}{2}}$$

which gives, on expanding and neglecting terms of the second order of small quantities,

$$ds - dr = \tfrac{1}{2}\left(\frac{du}{dr}\right)^2.$$

Therefore, with the present definition of U and the positive direction taken radially outwards,

$$-\frac{1}{2}\int_{R_1}^{r}\left(\frac{dU}{dr}\right)^2 dr$$

represents the radial component of the displacement. Multiplying

this by the above expression for the centrifugal force on the element, it appears that the work done on this account is

$$-\frac{2\pi\omega^2}{g}\rho h r^2 dr\int_{R_1}^{r}\left(\frac{dU}{dr}\right)^2 dr.$$

Thus, on integrating over the complete system, we find

$$W_{\text{max.}} = \frac{2\pi\rho\omega^2}{g}\int_{R_1}^{R_2} hr^2dr\int_{R_1}^{r}\left(\frac{dU}{dr}\right)^2 dr, \quad . \quad . \quad . \quad (118.3)$$

where the variables have their maximum values.

The next step is based on the implied assumption that

$$T_{\text{max.}} = V_{\text{max.}} + W_{\text{max.}},$$

by virtue of which we can combine the foregoing results in the symbolic form

$$\frac{2\pi\rho p^2}{g}\int_{R_1}^{R_2} hU^2rdr = V_{\text{max.}} + W_{\text{max.}},$$

i.e.

$$p^2 = \frac{g}{2\pi\rho}\,\frac{V_{\text{max.}} + W_{\text{max.}}}{\int_{R_1}^{R_2} hU^2rdr}. \quad . \quad . \quad . \quad . \quad (118.4)$$

Finally, to evaluate p we make the right-hand member of this equation a minimum, in accordance with the stationary property of normal modes (Art. 55). Although this work can be carried out as described in Ex. 1 of Art. 94 in the case of a simple system, the calculations are often rendered less laborious by introducing graphical methods in the process of deducing the requisite derivatives from an assumed curve for U.

It must, nevertheless, be admitted that the foregoing procedure is only to be recommended in circumstances where the frequency of vibration cannot be found experimentally, since it is not always a light task to adjust the variables so as to comply with the minimal and the boundary conditions.

Let us, in view of this restriction, separate the quantities on the right of equation (118.4), and write $\frac{p_1}{2\pi}$ for the frequency when $\omega = 0$, $\frac{p_2}{2\pi}$ for the change brought about in the frequency by the centrifugal action alone, so that p_2 relates to the rotating wheel when its flexural rigidity is ignored. The frequency, $\frac{p}{2\pi}$, of the rotating wheel is then given by

$$p^2 = p_1^2 + p_2^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118.5)$$

where

$$p^2_1 = \frac{g}{2\pi\rho} \cdot \frac{V_{\text{max.}}}{\int_{R_1}^{R_2} hU^2 r dr}, \quad p^2_2 = \frac{g}{2\pi\rho} \cdot \frac{W_{\text{max.}}}{\int_{R_1}^{R_2} hU^2 r dr}.$$

We notice that this is of the same form as (57.7), and that of the quantities denoted by p_1 and p_2, the latter is the more difficult to determine.

In the special case of a wheel which practically conforms with a uniform disc of the kind implied in Art. 94, of thickness $2h$ and radius R, it is now feasible, by virtue of equations (94.20) and (19.11), to obtain a first approximation on the supposition that

$$p_1 = \frac{A}{R^2}\sqrt{\frac{gD}{\rho h}}, \quad p_2 = B\omega, \quad . \quad . \quad . \quad . \quad (118.6)$$

where the coefficients A, B necessarily vary with the mode of vibration and, as usual, D is the flexural rigidity of the equivalent disc.

Professor R. V. Southwell[1] has calculated the values of these coefficients for such a disc when vibrating in a few modes characterized by n nodal circles and m nodal diameters, with Poisson's ratio taken as 0.3, and his results may be tabulated as follows:

VALUES OF A.

n \ m	0	1	2	3
0	3·755	0	5·385	12·49
1	20·93	20·54	34·78	53·30

VALUES OF B.

n \ m	0	1	2	3
0	0	1	1·533	2·013
1	1·817	2·440	2·990	3·506

These Tables are of interest as showing the relative order of magnitude of A and B in the higher modes of vibration.

119. We can approach the experimental side of the subject still closer by combining equations (118.5) and (118.6), in the form

$$p^2 = p^2_1 + C\omega^2, \quad . \quad . \quad . \quad . \quad . \quad (119.1)$$

where the coefficient C is to be derived from tests.

The evaluation of $\dfrac{p}{2\pi}$ for a particular mode will now depend solely on a sufficient knowledge of C, assuming the appropriate value of p_1 has been found for the case of no rotation.

An understanding of the practical problem to be solved will therefore be acquired if we summarize the preceding theory with reference to the constituent factors of C in the last equation. Thus we gather that this coefficient in general depends on the nodal pattern and profile of the wheel, as well as on the constraints introduced through the rotation, and through the degree of fit

[1] *Proc. Roy. Soc.*, vol. 101, page 133 (1922).

between such pairs as the blading-wheel, wheel-shaft, shaft-bearings, and so forth. The constraints exerted by the bearings will be somewhat affected by the state of lubrication, but this is an irrelevant point with modern installations. Of much greater significance is the tendency of C to increase with ω up to a certain velocity and then to decrease in consequence of the effect of centrifugal action on the degree of fit.

Furthermore, reverting to the Table on page 209, C is a function of the distribution of temperature over the wheel. This deserves mention for two reasons ; the temperature may, in certain circumstances, vary considerably between the centre and circumference of a wheel, and the temperatures met with in practice may cause the degree of fit between a wheel and the adjacent parts to undergo variations which cannot be ignored.

Although separate examination of these factors is the most convenient procedure to follow, we now realize that they may collectively vary in a complicated manner, and this is confirmed by accounts of tests and failures with turbine-wheels.[1]

120. If, for definiteness, we regard the direction of rotation as positive, the terms 'forward' and 'backward' may be ascribed to the two systems of waves implied in equation (117.2).

A point (r, θ) on the vibrating part of a wheel will necessarily be stationary in space when the 'forward' velocity ω is equal to the velocity $\frac{p}{m}$ of the 'backward' wave, and the wheel is then liable to the consequences of synchronism between the impressed force and the natural vibrations. This is usually called a *major critical speed*, to distinguish it from a *minor critical speed* which may occur when the wave is not travelling backwards as fast as the wheel is moving forwards. Hence it appears that there may be two series of critical speeds for a specified machine, but the former is the more important as regards displacement in a state of resonance.

By reason of the fact that a turbine-wheel very rarely complies in essentials with our specification of a slender disc, and that the source of disturbance usually extends over a finite area of actual wheels, we may anticipate a difficulty arising in the work of interpreting records that exhibit the wave-motion, owing to the 'overlapping' phenomenon mentioned in Art. 88. And if we combine this with the related question of skin-friction, it is conceivable that

[1] A. Stodola, *Schweiz. Bauzeit.*, vol. 63, page 112 (1914) ; K. Baumann, *Jour. Inst. E.E.*, vol. 59, page 565 (1921) ; W. Campbell, *Trans. Amer. Soc. Mech. E.*, vol. 46, page 31 (1924) ; J. von Freudenreich, *Engineering*, vol. 119, page 2 (1925) ; E. Oehler, *Zeits. V. d. I.*, vol. 69, page 335 (1925) ; W. Hort, *Zeits. V. d. I.*, vol. 70, page 1375 (1925) ; R. H. Collingham, *The Engineer*, vol. 132, page 370 (1931) ; B. Pochobradsky, L. B. W. Jolley and J. S. Thompson, *Engineering*, vol. 132, page 541 (1931).

the effect of the two phenomena will, under certain conditions, cause the rate of dissipation of energy to be different for the 'forward' and 'backward' waves, being the greater with the former wave. The greater amplitude will, on the average, accordingly be associated with the waves that travel backwards in the present sense, as proved to be the case in some of the tests undertaken by W. Campbell. This point, nevertheless, merits further attention on the part of readers engaged on experimental research into the subject.

It may be observed that the friction of rotating discs is chiefly affected by quantities which are included in the formulae put forward by G. G. McDonald.[1]

121. The natural frequencies of vibration for the blading of a turbine-wheel evidently constitute another series of speeds to be avoided as far as may be under working conditions. It is equally clear that the disturbance may arise from one of the sources already referred to in connection with the wheel itself; the blading may vibrate due to partial admission in the first few stages of an impulse turbine, or to the effect of the horizontal joint in a reaction turbine, to mention but two examples.

A detached blade may be regarded as a beam of non-uniform cross-section, fixed at the root and free at the tip. Several analytical methods of finding the natural frequency of wedge-shaped forms have been developed for the case of no rotation, notably by J. W. Nicholson,[2] and by D. M. Wrinch.[3] But it is customary to determine the natural period of such a blade by experimental methods, as indicated in the literature cited in the footnote to Art. 119.

Actual systems, however, differ from this simple type, in so far as the blades are connected by means of shrouding and, sometimes, intermediate bracing. The group thus formed will vibrate as a unit provided equal and sufficient degrees of stiffness are separately associated with the shrouding and the joints. If, as will be assumed in what follows, these conditions are fulfilled, in a given mode the vibratory motion of such a length of blading can be determined from considerations of any one of the constituent blades.

This is perhaps an appropriate place to point out, from Art. 55, that the effect of this kind of bracing is to raise the natural frequency of the blading, as the remark suggests a practical method of modifying this characteristic for blading of specified dimensions. A consequence of this is that if the shrouding becomes loose in service, as sometimes happens, the frequency of vibration will

[1] *The Engineer*, vol. 165, page 248 (1938).

[2] *Proc. Roy. Soc.*, vol. 93, page 506 (1917), and vol. 97, page 172 (1920).

[3] *Proc. Roy. Soc.*, vol. 101, page 493 (1922); *Phil. Mag.*, vol. 46, page 273 (1923).

thereby be lowered, and this is a matter of primary importance when the change tends to a state of resonance.

Anything approaching an exact solution to the problem is rendered extremely difficult by the combined effect of the shrouding, the rotation and the non-uniform cross-section of actual blading. In order to simplify an analytical investigation without sacrificing any of the essential points, we shall here assume: (i) the number of blades to be very great, which enables us to treat any pair of them as being sensibly parallel to one another; (ii) the blading and the shrouding to be separately of uniform cross-section throughout, since a non-uniform cross-section may, to our degree of approximation, readily be treated by the methods of J. W. Nicholson, D. M. Wrinch, and others; (iii) the vibration to be in the direction of the least moment of inertia of cross-section of the blading.

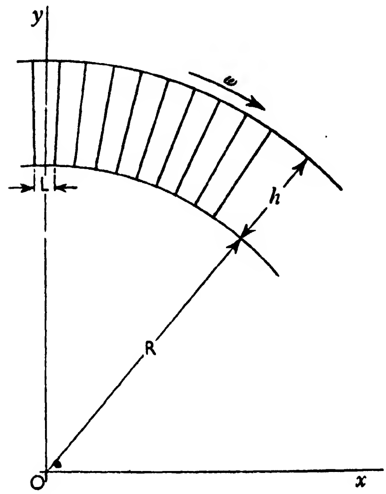


FIG. 179.

Suppose Fig. 179 to represent a system that complies with the above specification, consisting of a disc-like wheel of radius R, to the periphery of which are rigidly fixed, at an effective distance L apart, blades of a constant cross-section throughout their height h. The wheel rotates, as indicated in the figure, with angular velocity ω.

In this way we have reduced the problem to one which is most easily examined by the approximate method due to Lord Rayleigh when account is taken of the implication of Art. 19 (*d*). In order to proceed with a previous notation, let us write $\frac{p_1}{2\pi}$ for the natural frequency of the blading and shrouding when $\omega = 0$, $\frac{p_2}{2\pi}$ for the modification introduced in that frequency by the centrifugal action alone when the flexural rigidity of the system is ignored, and $\frac{p}{2\pi}$ for the natural frequency when the system is rotating with angular velocity ω. Thus the required value of p is to be calculated from the relation

$$p^2 = p^2_1 + p^2_2, \quad . \quad . \quad . \quad . \quad . \qquad (121.1)$$

which is of the same form as equation (118.5).

The only unknown quantity in this expression for p is p_2, for p_1 can, to the same degree of accuracy, be derived from the analysis of Art. 92, after substituting the terms 'blade' and 'shrouding' in turn for 'column' and 'beam', since any two adjacent blades are to be regarded as approximately parallel to each other.

To formulate p_2 in the symbolism of Art. 92, let m_1, I_1, E_1 signify in succession the mass per unit length of a blade, the proper moment of inertia of its cross-section, and the direct modulus of elasticity of the material. The corresponding characteristics of the shrouding will be denoted by m_2, I_2, E_2.

The matter will be greatly simplified if we next imagine the shrouding-strip as severed at the mid-point of every pair of blades and, further, its mass as concentrated at the tip of each blade. It is now only necessary to consider a blade rigidly fixed at its root and free at its tip, on the basis of our assumption that all the blades move in phase throughout the motion in question.

We can proceed in the manner of Art. 92 by supposing the x- and y-axes of Fig. 179 to rotate with angular velocity ω. A similar application of the theory of beams to a given blade with the prescribed load at its tip shows, on taking the origin at the root, that the vibratory motion is determined by

$$m_1\frac{\partial^2 x}{\partial t^2}+E_1 gI_1\frac{\partial^4 x}{\partial y^4}-\omega^2\frac{\partial}{\partial y}\left[\left\{m_1\int_y^h(R+y)dy+m_2L(R+h)\right\}\frac{\partial x}{\partial y}\right]=0, \quad (121.2)$$

provided the dimension L is measured in the direction of the vibration. For this reason L is in general slightly less than the true pitch of the blades, by an amount that depends on the form and the angle of a given blade.

But the flexural rigidity included in the term $E_1 gI_1\dfrac{\partial^4 x}{\partial y^4}$ can be neglected in finding p_2 by the present method, and the equation thus reduced to

$$m_1\frac{\partial^2 x}{\partial t^2} - \omega^2\frac{\partial}{\partial y}\left[\left\{m_1\int_y^h (R + y)dy + m_2L(R + h)\right\}\frac{dx}{\partial y}\right] = 0 \quad . \quad (121.3)$$

If u be the displacement about the equilibrium-position in a direction nearly parallel to the x-axis of Fig. 179, and the assumed vibration in a normal mode be defined by

$$u = U\cos p_2 t, \quad . \quad . \quad . \quad . \quad . \quad (121.4)$$

where U is a function of y alone, it appears on making this substitution in equation (121.3), that

$$\left\{\frac{m_2L}{m_1}(R+h)+R(h-y)+\tfrac{1}{2}(h^2-y^2)\right\}\frac{d^2U}{dy^2}-(R+y)\frac{dU}{dy}+\frac{p^2_2}{\omega^2}U=0.$$

A little rearrangement of the terms gives

$$\tfrac{1}{2}\left\{2\frac{m_2L}{m_1}(R+h)+(R+h)^2-(R+y)^2\right\}\frac{d^2U}{dy^2}-(R+y)\frac{dU}{dy}+\frac{p^2{}_2}{\omega^2}U=0,$$

which, on dividing throughout by $(R+h)^2$, reduces to

$$\left\{\frac{2m_2L}{m_1(R+h)}+1-\left(\frac{R+y}{R+h}\right)^2\right\}\frac{d^2U}{dy^2}-2\frac{R+y}{(R+h)^2}\frac{dU}{dy}+2\frac{p^2{}_2}{\omega^2(R+h)^2}U$$
$$=0. \quad (121\cdot5)$$

The next step consists in combining these terms so as to form a constant quantity on the one hand, and a function of the significant variable on the other, with the aim of identifying, if possible, (121.5) with one of the principal equations of physics whose solution is known.

A few trials suffice to demonstrate, having recourse to those fundamental equations, that the most suitable combination is given by putting

$$\frac{2m_2L}{m_1(R+h)}+1=b^2,\ \frac{R+y}{R+h}=\phi, \quad . \quad . \quad . \quad (121.6)$$

so that b is a constant and ϕ a variable for the specified system. Thus we have

$$\frac{dU}{dy}=\frac{1}{R+h}\frac{dU}{d\phi},\ \frac{d^2U}{dy^2}=\frac{1}{(R+h)^2}\frac{d^2U}{d\phi^2}$$

in equation (121.5) and, therefore, the vibrations expressed by

$$(b^2-\phi^2)\frac{d^2U}{d\phi^2}-2\phi\frac{dU}{d\phi}+2\frac{p^2{}_2}{\omega^2}U=0.$$

This will be exhibited in a more obvious form if we next put

$$\frac{\phi}{b}=\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121.7)$$

for the procedure leads to

$$(1-\psi^2)\frac{d^2U}{d\psi^2}-2\psi\frac{dU}{d\psi}+2\frac{p^2{}_2}{\omega^2}U=0,$$

i.e. $$(1-\psi^2)\frac{d^2U}{d\psi^2}-2\psi\frac{dU}{d\psi}+n(n+1)U=0 \quad . \quad . \quad (121.8)$$

provided $n(n+1)=2\frac{p^2{}_2}{\omega^2}$. The quantity denoted by p_2 is accordingly fixed by the value of n, with a given angular velocity ω.

We are now in a position to recognize (121.8) as *Legendre's equation of degree n*,[1] the solution of which is

$$U=c_1P_n(\psi)+c_2Q_n(\psi), \quad . \quad . \quad . \quad . \quad (121.9)$$

where c_1, c_2 represent arbitrary constants, and $P_n(\psi)$, $Q_n(\psi)$ are

[1] E. T. Whittaker and G. N. Watson, *A Course on Modern Analysis*, page 298, second edition.

called *Legendre's functions of degree n of the first and second kinds*, respectively. This equation, of the second order, has two independent particular solutions, and every other particular solution can be expressed in terms of these two.

In numerical calculations use may be made of tabulated values of these functions,[1] or the formulae

$$P_n(\psi) = \frac{1}{2^n \,.\, n!} \frac{d^n}{d\psi^n}(\psi^2 - 1)^n, \quad Q_n(\psi) = \tfrac{1}{2}P_n(\psi) \log \frac{1+\psi}{1-\psi} - Z_n,$$

with $Z_n = \dfrac{2n-1}{1\,.\,n}P_{n-1} + \dfrac{2n-5}{3(n-1)}P_{n-3} + \ldots$ Hence it appears that the forms of these functions are as follows:

$$\begin{aligned}
P_0(\psi) &= 1,\\
P_1(\psi) &= \psi,\\
P_2(\psi) &= \tfrac{1}{2}(3\psi^2 - 1),\\
P_3(\psi) &= \tfrac{1}{2}(5\psi^3 - 3\psi),\\
P_4(\psi) &= \tfrac{1}{8}(35\psi^4 - 30\psi^2 + 3),\\
P_5(\psi) &= \tfrac{1}{8}(63\psi^5 - 70\psi^3 + 15\psi),\\
P_6(\psi) &= \tfrac{1}{16}(231\psi^6 - 315\psi^4 + 105\psi^2 - 5),\\
& \cdots\cdots\cdots\cdots\cdots\\
& \cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$\begin{aligned}
Q_0(\psi) &= \tfrac{1}{2} \log \frac{1+\psi}{1-\psi},\\
Q_1(\psi) &= \tfrac{1}{2}\psi \log \frac{1+\psi}{1-\psi} - 1,\\
Q_2(\psi) &= \tfrac{1}{4}(3\psi^2 - 1) \log \frac{1+\psi}{1-\psi} - \tfrac{3}{2}\psi,\\
Q_3(\psi) &= \tfrac{1}{4}(5\psi^2 - 3\psi) \log \frac{1+\psi}{1-\psi} - \tfrac{5}{2}\psi^2 + \tfrac{2}{3},
\end{aligned}$$

and so on.

Now, to continue with the main problem, a combination of equations (121.4) and (121.9) discloses the fact that the displacement

$$u = \{c_1P_n(\psi) + c_2Q_n(\psi)\} \cos p_2t. \quad . \quad . \quad . \quad (121.10)$$

This yields the necessary pair of solutions.

The reader must consult treatises on the subject for information about the conditions under which the series for $P_n(\psi)$ and $Q_n(\psi)$ converge; it is here sufficient to observe that $P_n(\psi)$ is the more important of the Legendre functions when $|\,\psi\,| < 1$, and $Q_n(\psi)$ when $|\,\psi\,| > 1$.

[1] W. E. Byerly, *Fourier Series and Spherical, Cylindrical and Elliptical Harmonics* (Boston, U.S.A., 1893).

The practical significance of this remark will be understood if we state ψ in terms of the original symbols and thus write

$$\psi = \frac{\phi}{b} = \frac{R+y}{R+h}\left(1 + \frac{2m_2L}{m_1(R+h)}\right)^{-\frac{1}{2}}, \quad . \quad . \quad . \quad (121.11)$$

whence it is manifest that $|\psi| < 1$ in the present problem, since h is the maximum value of y. If we avail ourselves of this circumstance by ignoring the part $Q_n(\psi)$ of the general solution (121.10), it follows that, to the implied order of approximation,

$$u = c_1 P_n(\psi) \cos p_2 t. \quad . \quad . \quad . \quad . \quad (121.12)$$

There remain to be considered the boundary conditions. At the root of the blade, where $x = 0$,

$$y = 0 \text{ for all values of } t,$$

hence we obtain

$$\psi = \frac{R}{R+h}\left(1 + \frac{2m_2L}{m_1(R+h)}\right)^{-\frac{1}{2}},$$

on taking account of the substitutions (121.6) and (121.7). At the same point, however,

$$u = 0 \text{ for all values of } t,$$

whence we infer that equation (121.12) will only be satisfied if

$$P_n\left\{\frac{R}{R+h}\left(\frac{m_1(R+h)}{m_1(R+h) + 2m_2L}\right)^{\frac{1}{2}}\right\} = 0, \quad . \quad (121.13)$$

where, to repeat, $n(n+1) = 2\dfrac{p_2^2}{\omega^2}$.

The conditions at the assumed free end of the blade are already secured, by reason of the fact that equation (121.3) holds good even when the blade is free at its tip, where $y = h$.

Therefore, with blading of specified dimensions, the required value of p_2, i.e. $\{\frac{1}{2}n(n+1)\}^{\frac{1}{2}}\omega$, is given by the solution to equation (121.13). It will be found that this solution must in general be arrived at by trial, with the aid of graphs that exhibit the relation $\dfrac{m_2L}{m_1h}$ = const. when different values are assigned to this 'constant' in the process of plotting $\dfrac{p_2}{\omega}$ against $\dfrac{h}{R}$.

Once these arithmetical and graphical operations are effected, the corresponding natural frequency, $\dfrac{p}{2\pi}$, of the rotating blading can be determined by equation (121.1).

We might anticipate, from Art. 55, that a different value for p_2 will be given by calculations based on the more exact solution (121.10), due to the partial constraint which must be introduced in order to render valid our supposition that $Q_n(\psi)$ is insignificant.

But this discrepancy is usually small, and not of a magnitude such as to justify additional work in what is but a first approximation to the frequency in question.

A similar method can be employed to investigate the disturbed motion which may be executed in a direction almost at right-angles to the plane of the paper in Fig. 179, though vibrations in this sense are, in many cases, of secondary importance with shrouded blading.

122. Gyroscopic Action of Rotating Bodies. In what remains of this chapter we shall take account of the obvious fact that the system of Art. 109 will, strictly speaking, become one of the gyroscopic type discussed in Arts. 99–101 when we discard the assumption that the slope of the shaft remains negligibly small throughout slight disturbances about a position of equilibrium. Under these conditions we have to consider a couple which might well be much greater than is commonly supposed to occur even with engines, being sufficient, in one test undertaken by the author, to cause a shaft to 'float' in the clearance of a bearing and to rotate in this manner through a fraction of a revolution without actually touching the bearing. It is, indeed, possible that this was a contributory cause of the 'unexplained secondary disturbances' discovered by Professor A. Stodola [1] in some experiments with whirling shafts.

Another illustration is offered by the shaft and airscrew of an aeroplane when it is executing manœuvres in general, and a 'spin' in particular. The significance of the matter in this connection will be appreciated if we notice that, with a rigid body, the gyroscopic couple $\mathfrak{T}$ is expressed by

$$\mathfrak{T} = I\omega\omega',$$

where

I = polar moment of inertia of the airscrew,
ω = angular velocity of rotation of the shaft about its own axis,
ω' = angular velocity of spin about a fixed axis in space.

Hence, to exemplify by reference to what is nearly the limiting 'speed' of spin for a certain aeroplane, if $I = 200$ lb.-ft.^{2}, $\omega = 100\pi$ radians a second, and $\omega' = 3{\cdot}5$ radians a second, then $\mathfrak{T} = 6{,}830$ lb.-ft. Since the related modulus of section of the shaft was $1{\cdot}172$ inch units, it follows that this gyroscopic couple alone induced in the shaft a stress of

$$\frac{6{,}830 \times 12}{1{\cdot}172} \text{ lb. per square inch,}$$

say, 70,000 lb. per square inch. This calculation illustrates but one aspect of the danger which attends a fast spin, for there are

[1] *Steam and Gas Turbines*, vol. 2, page 1128 (London, 1927).

equally serious effects on the airscrew. The consequences in the case of large systems become manifest when it is remembered that the moment of inertia in question increases approximately as the fifth power of the diameter of an airscrew.

It is perhaps not so widely recognized that a disturbance from a similar source may give rise to considerable stresses in a wheel of a marine turbine when the vessel is 'pitching' or 'yawing', as will be explained in Ex. 2 of Art. 124.

From these preliminary remarks it will be understood that the rotating body may be an airscrew or a turbine-wheel, to mention only two important examples, but the term 'body' will be employed as a brief description of the rotating mass under examination.

We might anticipate that with gyroscopic systems the vibrations are of a complex character, as was pointed out in a previous remark, to the effect that there are differences of phase, variable with the frequency, between the displacements and the force.

123. The peculiar characteristics of the motion may, nevertheless, be gathered from considerations of the simple system shown in Fig. 180, comprising a rigid body fixed to an elastic shaft that rotates in bearings of a given type. Stated briefly,

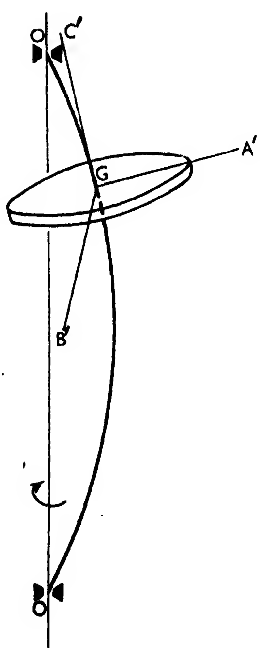


FIG. 180.

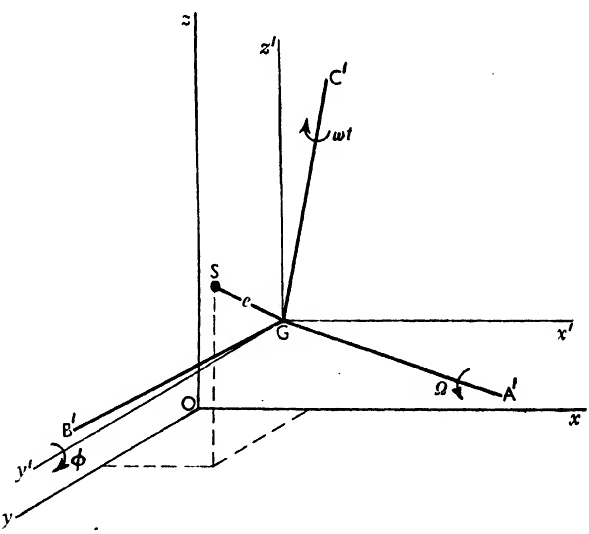


FIG. 181.

the problem is to trace, under prescribed conditions, the motion of the centre of gravity, G, of the body whose principal axes of symmetry are indicated by GA', GB', GC'.

It will be assumed, for reasons already mentioned, that the displacements about a position of equilibrium are always small;

and, for simplicity, that the mass of the shaft may be ignored in comparison with that of the body, whose mass is further supposed to be concentrated in a short axial length of the shaft.

With the origin at the lower bearing O, let Ox, Oy, Oz in Fig. 181 be rectangular axes fixed in space; and let Gx', Gy', Gz' represent a parallel set of axes drawn through G.

We shall take the initial conditions of motion to be such that GA', GB', GC' coincide in order with Gx', Gy', Gz' at time $t = 0$, and suppose the movement of the body in the succeeding interval of time t to be equivalent to (small) angular displacements ϕ, Ω, ωt described in turn about the axes Gy', GA', GC' shown in the figure.

Hence if ω_1, ω_2, ω_3 of Fig. 182 signify the instantaneous angular

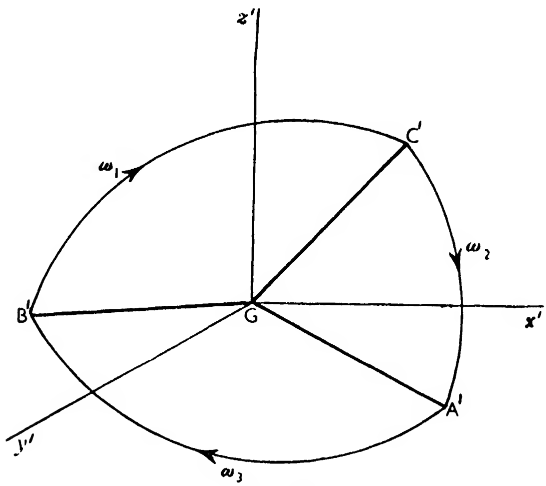


FIG. 182.

velocities of the axes GA', GB', GC', it is easily proved[1] that

$$\left.\begin{aligned} \omega_1 &= \dot{\phi} \sin \omega t - \dot{\Omega} \cos \omega t, \\ \omega_2 &= \dot{\phi} \cos \omega t + \dot{\Omega} \sin \omega t, \\ \omega_3 &= \omega. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (123.1)$$

The 'eccentricity' e as defined in Art. 109 will be introduced if we regard S in Fig. 181 as the geometrical centre of that part of the shaft covered by the body. Thus $SG = e$, where e is, again, assumed to be small.

If, with these conventions, the centres S, G have the co-ordinates (x, y), (x_g, y_g) with respect to the fixed axes Ox, Oy, then

$$\left.\begin{aligned} x_g &= x + e \cos (\omega t + \varepsilon), \\ y_g &= y + e \sin (\omega t + \varepsilon), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (123.2)$$

[1] W. H. Besant and A. S. Ramsey, *A Treatise on Dynamics*, Chap. XIV, fifth edition.

where the angle subtended by the lines Gx' and GS in Fig. 181 is denoted by $(\omega t + \varepsilon)$.

Next, suppose the body to be specified by moments of inertia A, B, C about its principal axes GA', GB', GC', respectively; and let its angular momenta about the axes Gx', Gy' be designated in succession by h_x, h_y. Then Euler's dynamical equations[1] give, in our notation,

$$h_x = \frac{A}{g}\omega_1 \cos \omega t - \frac{B}{g}\omega_2 \sin \omega t + \frac{C}{g}\omega_3 \phi,$$

$$h_y = \frac{B}{g}\omega_2 \cos \omega t + \frac{C}{g}\omega_3 \Omega + \frac{A}{g}\omega_1 \sin \omega t,$$

or, on writing a for $\frac{A}{g}$, b for $\frac{B}{g}$, c for $\frac{C}{g}$,

$$\begin{aligned} h_x &= a\omega_1 \cos \omega t - b\omega_2 \sin \omega t + c\omega_3 \phi, \\ h_y &= b\omega_2 \cos \omega t + c\omega_3 \Omega + a\omega_1 \sin \omega t. \end{aligned}$$

These results mean, having regard to equations (123.1), that

$$\begin{aligned} h_x = a(\dot\phi \sin \omega t - \dot\Omega \cos \omega t) \cos \omega t \\ - b(\dot\phi \cos \omega t + \dot\Omega \sin \omega t) \sin \omega t + c\omega\phi, \end{aligned}$$

$$\begin{aligned} h_y = b(\dot\phi \cos \omega t + \dot\Omega \sin \omega t) \cos \omega t \\ + c\omega\Omega + a(\dot\phi \sin \omega t - \dot\Omega \cos \omega t) \sin \omega t. \end{aligned}$$

This information will be exhibited in a more convenient form if we put a' for $\frac{1}{2}(a + b)$, and b' for $\frac{1}{2}(a - b)$. Then the angular momenta are defined by

$$\left.\begin{aligned} h_x &= - a'\dot\Omega + b'(\dot\phi \sin 2\omega t - \dot\Omega \cos 2\omega t) + c\omega\phi, \\ h_y &= - b'(\dot\phi \cos 2\omega t + \dot\Omega \sin 2\omega t) + c\omega\Omega + a'\dot\phi \end{aligned}\right\} \quad (123.3)$$

Thus if M denotes the mass of the body, and μ be written for $\frac{M}{g}$, equations (123.2) and (123.3) are seen to imply that the shaft at S is subject:

(i) in the x-direction, to a force $-\mu\ddot{x}_g$, i.e. $-\mu\{\ddot{x} - e\omega^2 \cos(\omega t + \varepsilon)\}$, together with a couple $-\frac{d}{dt}(h_y)$ about an axis through S and parallel to Gy';

(ii) in the y-direction, to a force $-\mu\ddot{y}_g$, i.e. $-\mu\{\ddot{y} - e\omega^2 \sin(\omega t + \varepsilon)\}$, together with a couple $\frac{d}{dt}(h_x)$ about an axis through S and parallel to Gx'.

The bearings will be sufficiently specified by the symbols α_{11}, γ_{11}, δ_{11} of Art. 107.

In virtue of the fact that the product of mass and generalized

[1] W. H. Besant and A. S. Ramsey, *A Treatise on Dynamics*, page 351.

acceleration is equal to the generalized force we can now write down the equations of motion

$$\left.\begin{aligned}
&\alpha_{11}\mu\ddot{x} + x + \gamma_{11}a'\ddot{\phi} + \gamma_{11}c\omega\dot{\Omega} \\
&\quad = \gamma_{11}b'\frac{d}{dt}(\dot{\phi}\cos 2\omega t + \dot{\Omega}\sin 2\omega t) + \alpha_{11}\mu e\omega^2\cos(\omega t + \varepsilon), \\
&\alpha_{11}\mu\ddot{y} + y + \gamma_{11}a'\ddot{\Omega} - \gamma_{11}c\omega\dot{\phi} \\
&\quad = \gamma_{11}b'\frac{d}{dt}(\dot{\phi}\sin 2\omega t - \dot{\Omega}\cos 2\omega t) + \alpha_{11}\mu e\omega^2\sin(\omega t + \varepsilon), \\
&\delta_{11}a'\ddot{\phi} + \phi + \gamma_{11}\mu\ddot{x} + \delta_{11}c\omega\dot{\Omega} \\
&\quad = \delta_{11}b'\frac{d}{dt}(\dot{\phi}\cos 2\omega t + \dot{\Omega}\sin 2\omega t) + \gamma_{11}\mu e\omega^2\cos(\omega t + \varepsilon), \\
&\delta_{11}a'\ddot{\Omega} + \Omega + \gamma_{11}\mu\ddot{y} - \delta_{11}c\omega\dot{\phi} \\
&\quad = \delta_{11}b'\frac{d}{dt}(\dot{\phi}\sin 2\omega t - \dot{\Omega}\cos 2\omega t) + \gamma_{11}\mu e\omega^2\sin(\omega t + \varepsilon).
\end{aligned}\right\} \quad (123.4)$$

These expressions disclose a noteworthy characteristic of the motion, since they show that the gyroscopic terms will remain even when the eccentricity e is zero and the system is, therefore, balanced in the ordinary sense. On this account we may have to consider vibrations brought about by the gyroscopic agencies only, as already pointed out.

It is easily verified that a 'particular' solution of equations (123.4) will result from the substitutions

$$\left.\begin{aligned}
x &= x_1\cos(\omega t + \varepsilon) + x_2\cos(\omega t - \varepsilon), \\
y &= x_1\sin(\omega t + \varepsilon) + x_2\sin(\omega t - \varepsilon), \\
\phi &= \phi_1\cos(\omega t + \varepsilon) + \phi_2\cos(\omega t - \varepsilon), \\
\Omega &= \phi_1\sin(\omega t + \varepsilon) + \phi_2\sin(\omega t - \varepsilon),
\end{aligned}\right\} \quad . \quad . \quad (123.5)$$

where x_1, x_2, ϕ_1, ϕ_2 relate to arbitrary constants.

These substitutions give

$$\begin{aligned}
&(1 - \alpha_{11}\mu\omega^2)x_1 + \gamma_{11}(c - a')\omega^2\phi_1 - \gamma_{11}b'\omega^2\phi_2 = \alpha_{11}\mu e\omega^2, \\
&(1 - \alpha_{11}\mu\omega^2)x_2 + \gamma_{11}(c - a')\omega^2\phi_2 - \gamma_{11}b'\omega^2\phi_1 = 0, \\
&- \gamma_{11}\mu\omega^2 x_1 + \{1 + \delta_{11}(c - a')\omega^2\}\phi_1 - \delta_{11}b'\omega^2\phi_2 = \gamma_{11}\mu e\omega^2, \\
&- \gamma_{11}\mu\omega^2 x_2 + \{1 + \delta_{11}(c - a')\omega^2\}\phi_2 - \delta_{11}b'\omega^2\phi_1 = 0.
\end{aligned}$$

Thus, from the sums and the differences of these relations when taken in pairs, we deduce

$$\begin{aligned}
(1 - \alpha_{11}\mu\omega^2)(x_1 + x_2) + \gamma_{11}(c - a)\omega^2(\phi_1 + \phi_2) &= \alpha_{11}\mu e\omega^2, \\
- \gamma_{11}\mu\omega^2(x_1 + x_2) + \{1 + \delta_{11}(c - a)\omega^2\}(\phi_1 + \phi_2) &= \gamma_{11}\mu e\omega^2,
\end{aligned}$$

because $(c - b' - a') = (c - a)$; and

$$\begin{aligned}
(1 - \alpha_{11}\mu\omega^2)(x_1 - x_2) + \gamma_{11}(c - b)\omega^2(\phi_1 - \phi_2) &= \alpha_{11}\mu e\omega^2, \\
- \gamma_{11}\mu\omega^2(x_1 - x_2) + \{1 + \delta_{11}(c - b)\omega^2\}(\phi_1 - \phi_2) &= \gamma_{11}\mu e\omega^2,
\end{aligned}$$

because $(c + b' - a') = (c - b)$.

A rearrangement of the terms at once shows that the maximum displacements are determined by

$$x_1 + x_2 = \frac{\mu e\omega^2\{(c-a)(\alpha_{11}\delta_{11} - \gamma^2{}_{11})\omega^2 + \alpha_{11}\}}{-\Delta_1},$$

$$\phi_1 + \phi_2 = \frac{\gamma_{11}\mu e\omega^2}{-\Delta_1},$$

$$x_1 - x_2 = \frac{\mu e\omega^2\{(c-b)(\alpha_{11}\delta_{11} - \gamma^2{}_{11})\omega^2 + \alpha_{11}\}}{-\Delta_2},$$

$$\phi_1 - \phi_2 = \frac{\gamma_{11}\mu e\omega^2}{-\Delta_2},$$

provided we let

$$\Delta_1 = \mu(c-a)(\alpha_{11}\delta_{11} - \gamma^2{}_{11})\omega^4 - \{\delta_{11}(c-a) - \alpha_{11}\mu\}\omega^2 - 1, \quad . \quad . \quad (123.6)$$

$$\Delta_2 = \mu(c-b)(\alpha_{11}\delta_{11} - \gamma^2{}_{11})\omega^4 - \{\delta_{11}(c-b) - \alpha_{11}\mu\}\omega^2 - 1. \quad . \quad . \quad (123.7)$$

The Δ-symbols are used to emphasize the fact that such expressions are to be derived from determinantal forms in the general case.

As the displacements tend, in the assumed absence of friction, to infinitely large values when

$$\Delta_1 = 0, \ \Delta_2 = 0,$$

we realize that the phenomenon of whirling will occur at speeds given by the roots of the resulting quadratics in ω^2.

Of the conclusions to be drawn from these quadratics, we notice the following:

(i) By the equation $\Delta_1 = 0$, the first whirling speed, ω_1, of the body in question is the same as that for a symmetrical body whose principal moments of inertia are A, A, C, as indicated by the absence of terms in b;

(ii) By the equation $\Delta_2 = 0$, the second whirling speed, ω_2, of the body in question is the same as that for a symmetrical body whose principal moments of inertia are B, B, C, as indicated by the absence of terms in a.

(iii) $\omega_1 < \omega_2$ when $A > B$.

(iv) In the special case of a symmetrical body with A, A, C as its principal moments of inertia and such that $C > A$, the equation $\Delta_1 = 0$ will have but one real root and, in consequence, there will be only one speed of whirling. These particular conditions deserve mention because they have reference to a large number of actual systems.

124. The path described by the specified system is easily traced with the help of the foregoing results, remembering that the forced and the free vibrations are represented in turn by the 'particular integral' and the 'complementary function' of the equations of motion.

From the fact that the first pair of equations (123.5) give

$$x^2 + y^2 = x^2_1 + 2x_1x_2 \cos 2\varepsilon + x^2_2$$
$$= r^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124.1)$$

say, we learn that in forced vibrations the centre S will describe a circle of radius r. It is further seen that r depends on ω, and tends, in the absence of friction, to infinitely large values only when the system is rotating at the two speeds which we have identified with whirling. Therefore the possible states of instability in the forced vibration are restricted to those two speeds.

The free vibrations are, as can readily be verified, found by assuming that

$$\left.\begin{aligned} x &= x_1 \cos \{(\omega + \eta)t + \zeta\} + x_2 \cos \{(\omega - \eta)t - \zeta\}, \\ y &= x_1 \sin \{(\omega + \eta)t + \zeta\} + x_2 \sin \{(\omega - \eta)t - \zeta\}, \\ \phi &= \phi_1 \cos \{(\omega + \eta)t + \zeta\} + \phi_2 \cos \{(\omega - \eta)t - \zeta\}, \\ \Omega &= \phi_1 \sin \{(\omega + \eta)t + \zeta\} + \phi_2 \sin \{(\omega - \eta)t - \zeta\} \end{aligned}\right\} \quad . \quad . \quad (124.2)$$

hold good in equations (123.4), with x_1, x_2, ϕ_1, ϕ_2, ζ symbolizing arbitrary constants.

In order to exhibit the path described by the centre S, put

$$x \cos \omega t + y \sin \omega t = X, \quad - x \sin \omega t + y \cos \omega t = Y,$$

in equations (124.2), and so transform the first pair of expressions to

$$X = (x_1 + x_2) \cos (\eta t + \zeta), \quad Y = (x_1 - x_2) \sin (\eta t + \zeta),$$

i.e.
$$\frac{X^2}{(x_1 + x_2)^2} + \frac{Y^2}{(x_1 - x_2)^2} = 1 \quad . \quad . \quad . \quad (124.3)$$

In free vibrations the path of S is, therefore, an ellipse which moves about G with the steady angular velocity, ω, of the shaft. This path may be described otherwise as an epicyclic in space (Fig. 158), a result which agrees, as it should, with the conclusions arrived at in Ex. 3 of Art. 101. The epicyclic will be retrograde if the supposed positive sign of the eccentricity e is reversed, as may happen according to equation (113.2).

At this stage it is useful to notice that the condition $\eta = 0$ corresponds to a state of resonance between the free and forced motions, for then both X and Y represent constants and, therefore, the path becomes a circle.

The effect of the gyroscopic agencies alone is most easily traced, in outline, by putting $e = 0$ in the process of introducing in equations (123.4) the relations (124.2). These operations yield expressions between which the constants x_1, x_2, ϕ_1, ϕ_2 can be eliminated. If the final form thus arrived at be treated as an equation in η^2, the states of instability will be determined by the conditions under which η changes sign, from positive to negative, then negative to positive, and so forth.

This work involves lengthy calculations which will be rendered

unnecessary if we avail ourselves of the circumstance that, for practical purposes in the present kind of problem, the physical significance of a retrograde epicyclic may be interpreted as a peculiar type of instability which arises from sources of gyroscopic origin. In this sense an unsymmetrical body will, on the implied assumptions, become unstable when the system is rotating with an angular velocity coming within the range bounded by the whirling speeds ω_1, ω_2.

For example, a two-blade airscrew will be unstable at speeds within the limits thus defined, even when the airscrew is balanced, but stability will be secured at all other intermediate speeds *so long as the dissipative forces remain negligibly small.*

The above statements apply to *whirling* in general, with a shaft and any number n unsymmetrical bodies arranged so that their principal axes are parallel. Such a system will accordingly attain a state of instability when the angular velocity lies between the first and second whirling speeds, the third and fourth whirling speeds, and so on, depending upon the value of n.

It is scarcely necessary to say that a complete specification of the *free* vibration of such a system cannot be written down without recourse to generalized forms of the preceding equations. This extension of the analysis is, however, hardly justified in engineering practice where we are chiefly concerned with what takes place when a shaft is actually whirling.

Ex. 1. Find the whirling speed of the system formed, as in Fig. 168, by a thin circular disc, a slender shaft of circular cross-section, and two 'short' bearings. It may be supposed that the disc is rigid and that the shaft is elastic.

Writing I for the moment of inertia about a diameter of the cross-section of shaft, we have, from the example of Art. 107,

$$\alpha_{11} = \frac{l^2_1 l^2_2}{3EI(l_1+l_2)},\quad \gamma_{11} = \frac{l_1 l_2(l_2 - l_1)}{3EI(l_1+l_2)},\quad \delta_{11} = \frac{l^2_1 - l_1 l_2 + l^2_2}{3EI(l_1+l_2)},$$

in the dimensions of Fig. 168.

Further, if A, C represent in turn the moments of inertia of the disc about a diameter and about the axis of the shaft, then $C = 2A$.

Introducing these data in equation (123.6), and designating the mass of the disc by M, we see that the required whirling speed will be given by the roots of

$$\omega^4 - \frac{3EgI(l_1+l_2)}{MAl^3_1 l^3_2}\{A(l^2_1 - l_1 l_2 + l^2_2) - Ml^2_1 l^2_2\}\omega^2 - \frac{9E^2g^2I^2(l_1+l_2)^2}{M^2Al^3_1 l^3_2} = 0$$

when treated as a quadratic in ω^2. It has already been pointed out that there will be but one real root in this case.

When the deflection, γ_{11}, produced by the couple is negligibly small, the positive real root is to be derived from

$$\frac{Ml^2{}_1l^2{}_2}{3EgI(l_1 + l_2)}\omega^2 - 1 = 0,$$

whence we conclude that the whirling speed is then independent of the moments of inertia of the disc, a result which conforms, as it should, with the analysis of Art. 109.

Moreover, the common point of the disc and shaft is subject to a force P and couple $\mathfrak{T}$ which can be evaluated from equations (107.6) for known values of the deflections y_1, β_1 of Fig. 168.

It is, of course, necessary to discard our assumption that the disc is rigid before we can estimate the stress in that part of the system.

Ex. 2. Investigate the effect on the wheel of a marine turbine which will be caused by 'pitching' or 'yawing' motion of the vessel when the wheel is rotating about its own axis with angular velocity ω.

In order to render the problem amenable to mathematical treatment we shall assume: first, the displacements to be always small and proportional to the disturbing force; secondly, the precession about the vertical axis to involve a constant angular velocity of magnitude Ω; thirdly, the wheel to be equivalent to a thin circular disc of uniform thickness $2h$, rigidly fixed at its inner radius a and free at its outer radius b.

The equations of motion are most easily formulated from considerations of a unit volume of the material, of density ρ, whose

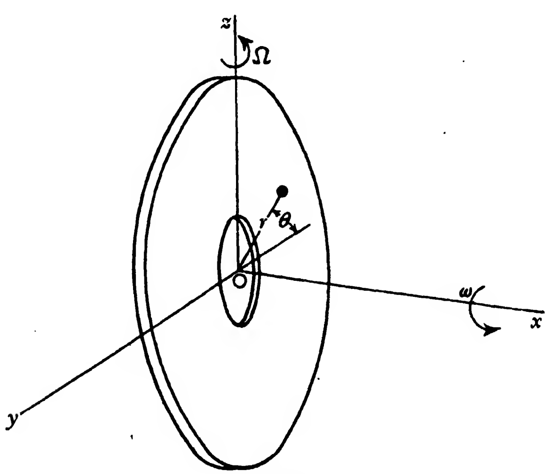


Fig. 183.

position is specified by the co-ordinates r, θ of Fig. 183, where we imagine the rectangular axes Ox, Oy, Oz as rotating with the disc, the origin O being taken at the centre of the disc.

In the notation of Fig. 184, then, $\theta = \omega t$, $\phi = \Omega t$, and

$$x = r \cos \theta \cos \phi,$$

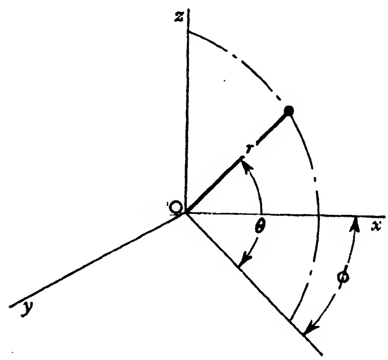


FIG. 184.

whence, differentiating with regard to the time,

$$\ddot{x} = 2r\dot{\theta}\dot{\phi} \sin \theta \sin \phi - r(\dot{\theta}^2 + \dot{\phi}^2) \cos \theta \cos \phi.$$

In the position $\phi = \frac{\pi}{2}$, therefore,

$$\begin{aligned} \ddot{x} &= 2r\dot{\theta}\dot{\phi} \sin \theta \\ &= 2r\omega\Omega \sin \theta. \end{aligned}$$

Hence at any instant t the element of mass ρ is subjected to a force of gyroscopic origin amounting to

$$2\frac{\rho}{g} r\omega\Omega \sin (\omega t + \theta),$$

being the product of the mass and acceleration under consideration.

If such a disturbing force causes the element to undergo a displacement u in the x-direction of Fig. 183 we can, on the assumption that the displacement is proportional to the force, write

$$u = U \sin (\omega t + \theta), \quad . \quad . \quad . \quad . \quad (124.4)$$

where U is a function of the radius r.

At this stage of the work we may profitably neglect, for a moment, the effect of the centrifugal action on the stiffness of the constituent material, and so utilize the preceding theory of thin flat plates.

Thus, equating the elastic forces already found for a thin circular plate to the disturbing force above mentioned, we obtain

$$\frac{m^2Egh^2}{3(m^2-1)}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)^2+\rho\frac{\partial^2 u}{\partial t^2}=2\rho r\omega\Omega\sin(\omega t+\theta),$$

i.e.

$$\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)^2+\frac{3(m^2-1)\rho}{m^2Egh^2}\frac{\partial^2 u}{\partial t^2}=\frac{6(m^2-1)\rho\omega\Omega}{m^2Egh^2}r\sin(\omega t+\theta),$$

where Poisson's ratio for the material is, as usual, represented by $\frac{1}{m}$.

By equation (124.4), however,

$$\frac{\partial u}{\partial r}=\sin(\omega t+\theta)\frac{dU}{dr},\ \frac{\partial^2 u}{\partial r^2}=\sin(\omega t+\theta)\frac{d^2U}{dr^2},\ \frac{\partial^2 u}{\partial t^2}=-U\omega^2\sin(\omega t+\theta),$$

$$\frac{\partial^2 u}{\partial \theta^2}=-U\sin(\omega t+\theta).$$

Therefore the motion is expressed by

$$\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{1}{r^2}U\right)^2\sin(\omega t+\theta)-\frac{3(m^2-1)\rho\omega^2}{m^2Egh^2}U\sin(\omega t+\theta)$$
$$=\frac{6(m^2-1)\rho\omega\Omega}{m^2Egh^2}r\sin(\omega t+\theta),$$

or, after cancelling the common factor, and putting

$$k^4 \text{ for } \frac{3(m^2-1)\rho\omega^2}{m^2Egh^2},\quad n^2 \text{ for } \frac{6(m^2-1)\rho\omega\Omega}{m^2Egh^2},$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)^2U-k^4U=n^2r\ . \quad . \quad . \quad (124.5)$$

This will be more easily identified with one of the principal equations of physics if we multiply throughout by r^4, and so state the result in the form

$$\left(r^2\frac{d^2}{dr^2}+r\frac{d}{dr}-1\right)^2U-k^4r^4U=n^2r^5,$$

or, more fully,

$$\left\{r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+(k^2r^2-1)\right\}\left\{r^2\frac{d^2}{dr^2}+r\frac{d}{dr}-(k^2r^2+1)\right\}U=n^2r^5 \quad . \quad (124.6)$$

It follows, on putting the right-hand side of this relation equal to zero, that, with the centrifugal action ignored, the free vibrations are determined by

$$\left\{r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+(k^2r^2-1)\right\}\left\{r^2\frac{d^2}{dr^2}+r\frac{d}{dr}-(k^2r^2+1)\right\}U=0,$$

i.e.

$$r^2\frac{d^2U}{dr^2}+r\frac{dU}{dr}+(k^2r^2-1)U=0,\quad r^2\frac{d^2U}{dr^2}+r\frac{dU}{dr}-(k^2r^2+1)U=0.$$

These are forms of *Bessel's equation*, as reference to treatises [1] on the subject shows, whence it appears that, with the customary notation, their solutions are

$$U_1 = A_1 J_1(kr) + A_2 Y_1(kr), \quad U_2 = A_3 I_1(kr) + A_4 K_1(kr),$$

where the A's refer to arbitrary constants, and J_1, Y_1, I_1, K_1 to *Bessel's functions and their modified forms of the first order.*

In the prescribed free vibration, therefore, U of equation (124.4) is defined by

$$U = U_1 + U_2 = A_1 J_1(kr) + A_2 Y_1(kr) + A_3 I_1(kr) + A_4 K_1(kr).$$

Again, it is easily verified that $-\dfrac{n^2}{k^4}$ is a 'particular integral' of equation (124.6).

Since a combination of these results shows the complete solution to be

$$U = A_1 J_1(kr) + A_2 Y_1(kr) + A_3 I_1(kr) + A_4 K_1(kr) - \frac{n^2}{k^4}r,$$

we realize, having regard to equation (124.4), that at time t and position r, θ the axial displacement

$$u = \left\{A_1 J_1(kr) + A_2 Y_1(kr) + A_3 I_1(kr) + A_4 K_1(kr) - \frac{n^2}{k^4}r\right\} \sin(\omega t + \theta), \quad (124.7)$$

provided the centrifugal effect on the stiffness is neglected, and k, n are as specified in equation (124.5).

The constants A_1, A_2, A_3, A_4 must, of course, be evaluated so as to satisfy the boundary conditions:

(i) bending moment is zero at the outer radius b;
(ii) shearing force is zero at the outer radius b;
(iii) deflection u is zero at the inner radius a;
(iv) slope $\dfrac{\partial u}{\partial r}$ is zero at the inner radius a.

To proceed, we remark that our degree of accuracy will not be sensibly affected if we assume the bending moment on the disc to be the same as would be produced if the extraneous force were gradually applied. With this bending moment signified by $\mathfrak{T}$, it is known that [2]

$$\mathfrak{T} = D\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{m}\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)\right\}, \quad . \quad . \quad . \quad (124.8)$$

where, as previously, the flexural rigidity $D = \dfrac{2m^2Eh^3}{3(m^2 - 1)}$.

[1] e.g., A. R. Forsyth, *Differential Equations*, page 176, fourth edition.
[2] A. E. H. Love, *Mathematical Theory of Elasticity*, page 488, fourth edition.

Now, the boundary conditions may be described otherwise as :

(i) $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{m}\left(\dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}\right) = 0$ at radius $r = b$,

(ii) $\dfrac{\partial}{\partial r}\nabla^2 u + \dfrac{(m-1)}{mr^2}\dfrac{\partial^2}{\partial \theta^2}\left(\dfrac{\partial u}{\partial r} - \dfrac{u}{r}\right) = 0$ at radius $r = b$,

(iii) $u = 0$ at radius $r = a$,

(iv) $\dfrac{\partial u}{\partial r} = 0$ at radius $r = a$,

where ∇^2 is written for $\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$.

Imposing these restrictions on equation (124.7) and, for brevity, putting f for $\dfrac{m-1}{mkb}$, we obtain

$$\begin{aligned}
&A_1\{J_1(kb) - fJ_2(kb)\} + A_2\{Y_1(kb) - fY_2(kb)\} \\
&\qquad - A_3\{I_1(kb) - fI_2(kb)\} - A_4\{K_1(kb) + fK_2(kb)\} = 0,\\
&A_1J_2(kb) + A_2Y_2(kb) + A_3I_2(kb) - A_4K_2(kb) = 0,\\
&A_1J_1(ka) + A_2Y_1(ka) + A_3I_1(ka) + A_4K_1(ka) = \frac{n^2}{k^4}a,\\
&A_1J_0(ka) + A_2Y_0(ka) + A_3I_0(ka) - A_4K_0(ka) = \frac{2n^2}{k^5}.
\end{aligned}$$

If Δ represents the determinant formed by the coefficients of the A's, then

$$\Delta = \begin{vmatrix}
J_1(kb)-fJ_2(kb), & Y_1(kb)-fY_2(kb), & -I_1(kb)+fI_2(kb), & -K_1(kb)-fK_2(kb)\\
J_2(kb) & , Y_2(kb) & , I_2(kb) & , -K_2(kb)\\
J_1(ka) & , Y_1(ka) & , I_1(ka) & , K_1(ka)\\
J_0(ka) & , Y_0(ka) & , I_0(ka) & , -K_0(ka)
\end{vmatrix}$$

and the usual treatment of the above equations gives

$$A_1 = \frac{n^2}{k^4}\,\frac{\begin{vmatrix}
0, & Y_1(kb)-fY_2(kb), & -I_1(kb)+fI_2(kb), & -K_1(kb)-fK_2(kb)\\
0, & Y_2(kb) & , I_2(kb) & , -K_2(kb)\\
a, & Y_1(ka) & , I_1(ka) & , K_1(ka)\\
\frac{2}{k}, & Y_0(ka) & , I_0(ka) & , -K_0(ka)
\end{vmatrix}}{\Delta},$$

$$A_2 = \frac{n^2}{k^4}\,\frac{\begin{vmatrix}
J_1(kb)-fJ_2(kb), & 0, & -I_1(kb)+fI_2(kb), & -K_1(kb)-fK_2(kb)\\
J_2(kb) & , 0, & I_2(kb) & , -K_2(kb)\\
J_1(ka) & , a, & I_1(ka) & , K_1(ka)\\
J_0(ka) & , \frac{2}{k}, & I_0(ka) & , -K_0(ka)
\end{vmatrix}}{\Delta},$$

$$A_3 = \frac{n^2}{k^4}\frac{\begin{vmatrix} J_1(kb)-fJ_2(kb), & Y_1(kb)-fY_2(kb), & 0, & -K_1(kb)-fK_2(kb) \\ J_2(kb) & , Y_2(kb) & , 0, & -K_2(kb) \\ J_1(ka) & , Y_1(ka) & , a, & K_1(ka) \\ J_0(ka) & , Y_0(ka) & , \frac{2}{k}, & -K_0(ka) \end{vmatrix}}{\Delta},$$

$$A_4 = \frac{n^2}{k^4}\frac{\begin{vmatrix} J_1(kb)-fJ_2(kb), & Y_1(kb)-fY_2(kb), & -I_1(kb)+fI_2(kb), & 0 \\ J_2(kb) & , Y_2(kb) & , I_2(kb) & , 0 \\ J_1(ka) & , Y_1(ka) & , I_1(ka) & , a \\ J_0(ka) & , Y_0(ka) & , I_0(ka) & , \frac{2}{k} \end{vmatrix}}{\Delta}.$$

We are now able to write down, from equation (124.7), the relation for u and in this way to estimate the stress in the material when the consequence of the centrifugal force is ignored. A similar substitution in the formula (124.8) next yields the corresponding bending moment $\mathfrak{T}$, in a form which may here be written briefly as

$$\mathfrak{T} = D\Big[A_1\Big\{-k^2J_1(kr) + \frac{(m-1)k}{mr}J_2(kr)\Big\} + A_2\Big\{-k^2Y_1(kr) + \frac{(m-1)k}{mr}Y_2(kr)\Big\} + A_3\Big\{k^2I_1(kr) - \frac{(m-1)k}{mr}I_2(kr)\Big\} + A_4\Big\{k^2K_1(kr) + \frac{(m-1)k}{mr}K_2(kr)\Big\}\Big]. \quad . \quad . \quad (124.9)$$

In order to complete the calculations, and at the same time to mention a point of some interest in connection with the physical properties of metals, we shall suppose that the effect of a rotation ω_s about the axis of the shaft is to change the direct modulus of elasticity of the material from E to E_s.

Thus if the foregoing analysis yields a natural frequency of magnitude $\frac{p_1}{2\pi}$, it is to be inferred from equation (119.1) that the relation

$$E_s = \frac{p^2{}_1 + \omega^2{}_s}{p^2{}_1}E$$

may be employed as a first approximation. Hence, replacing E in equation (124.5) by this expression for E_s, the rotation ω_s is taken into consideration by substituting

$$k^4 = \frac{3(m^2-1)\rho p^2{}_1\omega^2{}_s}{m^2Egh^2(p^2{}_1+\omega^2{}_s)}. \quad . \quad . \quad . \quad (124.10)$$

for $$k^4 = \frac{3(m^2 - 1)\rho\omega^2}{m^2 E g h^2}.$$

in the preceding equations.

Furthermore, the modulus of elasticity E_s varies with the temperature, and this second factor might be combined with the rotation if sufficiently strong experimental support is given to the view that E could be treated as a continuous function of the velocity of rotation and temperature, after the manner suggested here.

Of the suppositions employed to simplify this solution, the most important are the assumptions that the velocity of precession is constant and the thickness of the equivalent disc is uniform. But the errors introduced through these and the other suppositions are not always additive when actual wheels in marine turbines describe motion of the kind under consideration, so that the stress and frequency of vibration computed on the basis of equation (124.10) may not differ sensibly from the true values.

CHAPTER VII

GENERAL SURVEY

125. Introduction. We may profitably devote this chapter to a survey of the general subject and of some of the more salient principles and applications, since the foregoing theory has been exemplified by a wide variety of problems and necessarily adapted to the special circumstances of each case.

A problem may be considered as solved once we have evaluated the maximum state of vibration to which the structural system can be subjected, in that the dynamic forces, bending-moments and shearing forces may then be calculated by the methods described in the example of Art. 40, Ex. 3 of Art. 56, and Ex. 2 of Art. 124.

This implies a sufficient knowledge of the engineering practice and science concerned, as will be understood from previous references to the theory of structures, the strength of materials, aerodynamics, geophysics, hydraulics, and thermodynamics. Lagrange's method is, of course, also applicable to electrical systems, for we have described it in the language appropriate to material systems merely as a matter of convenience.

126. For a specified system of structural members and forces, the calculations include:

(i) The distribution of force due to the inertia of the structure.
(ii) The distribution of the dissipative forces resisting motion.
(iii) The extraneous forces and the reactions.

By combining these effects, paying due attention to any differences of phase which exist, we can formulate, in terms of the disturbing force and the amplitude of vibration, the dynamic bending-moments and shearing-forces to be allowed for in a specification.

In all cases the maximum dynamic effects are to be superposed on the maximum bending-moments and shearing-forces caused by the static loads. Although the maximum dynamic and static forces acting on a structural member do not, in general, occur simultaneously, it is evident that the process of direct superposition can be followed in finding the greatest stress in the member.

No unsurmountable obstacles will lie in the way of computing the natural modes of vibration when the structure is simple and its component parts are of geometrical form, but a practical problem is usually too complicated for exact solution and the best that can

be hoped for from theory is a rough approximation of the natural frequencies and a correspondingly approximate idea of the position of the nodes.

It is advisable whenever possible to check results so found by experimental methods. This part of the work consists, in essentials, in applying a periodic force to a heavy or concentrated mass of the structural system, and fixing to the same mass a sensitive instrument for detecting vibration. The resonant frequencies will be registered by this fixed detector when the periodicity of the exciting force is slowly varied over the range to be investigated. In electrical connection with this instrument, and forming part of the complete apparatus, is a similar detector that can be moved about the structure under examination. Thus the position of a node is indicated by the appropriate phase relation between the voltage responses of the fixed and movable detectors, and the nodal pattern can be constructed with ample accuracy because the phase will attain the value of 180 deg. at the instant when the movable detector traverses a nodal line.

Though the component displacements parallel to the axes of the system are not generally independent, an inspection of the shape of the structure usually suffices to make known which of the component displacements may be regarded as of the second order of small quantities.

It is here worth while to remark that the frequency with which a particular system vibrates is influenced by the disposition of the load, to an extent that sometimes becomes more and more significant as we approach the higher modes of vibration. To this is to be attributed, for example, the appreciable cyclic variation in displacement of the crankshaft of an engine as disclosed by a torsiograph record.

127. If, as is commonly the case, a given mechanical or structural system constitutes the disturbing agency, we employ the geometrical relations of the system to express its kinetic energy in terms of any one of the co-ordinates.

It follows from what was said in Art. 126 that all the forces must appear in the equations of motion, of which (21.4) is an example that exhibits the fluid pressure on the piston and the inertia of the moving parts of an engine. The attendant frictional forces cannot be ignored under certain conditions, as may be illustrated by reference to the mechanisms implied in equations (37.7) and (37.16); in the case of high-speed locomotives the most serious troubles are those likely to ensue from the high stress imposed by frictional resistances of the valves and the inertia forces due to their weight. This explains the increasing use of anti-friction bearings in the various pin-joints of locomotive valve-gears.

When the disturbing forces vary according to a complicated law, the problem is reduced to one of simple harmonic motion by first restricting the analysis to the primary harmonic component of the resultant force. Then the operation of successive approximation offers a means of improving the result, by taking account of the second harmonic component, and so on, to the required degree of accuracy. In this way the analysis of Chapter II may be employed to demonstrate that the increase in the running speed and in the number of revolutions a minute with modern locomotives has led, through the inertia of the working parts, to corresponding increases in the stresses, and notably to overloading of the crank-pins and crosshead bearings. From the resulting equations we conclude that the stress in the crank-pins, when running under steam, is to be ascribed chiefly to the pressure of the working fluid, the inertia of the reciprocating parts, and the centrifugal action of the rotating parts. Again, the highest stresses on the bearings will occur when, at maximum speed, the regulator valve is suddenly closed and the inertia effects are thereby deprived of the moderating counter-action of the steam pressure.

Certain pitfalls will be obviated in the work of deciding upon the best method of avoiding resonance for a specified system vibrating in a given mode if we bear in mind two simple results: first, the relative position of the nodes are not invariably the same in the free and the forced vibrations, and, second, resonant vibration cannot be set up by an exciting force that acts at a node, or by an exciting couple that acts at an anti-node. Hence the advantage to be gained by moving a mass is the greatest when the movement is from a node to an anti-node, or *vice versa*, according to the aim of the alteration.

128. A useful purpose will be served if we next review the more salient implications of the theory.

In this work the term 'dynamical system' relates to one or more *rigid* bodies which may or may not be independent. Further, such bodies may influence one another in various ways: by impact, through the action of gravitational or electro-magnetic forces, through frictionless constraints, and, with certain restrictions, by the operation of perfect fluids. In other words, Lagrange's method will apply in all circumstances provided we exclude internal work due to deformation, friction, etc.

A continuous (elastic) system does not comply with this specification of a dynamical system, though it may, for most practical purposes, be regarded as a dynamical system composed of an infinitely large number of particles and frictionless constraints. Moreover, it is often practicable to replace such a continuous system by a body having a finite number of degrees of freedom,

which we shall briefly refer to as a ' semi-rigid body '. Thus a beam may be imagined as bending in conformity with an assumed law that involves but one degree of freedom.

The motion of a semi-rigid body does not, as a rule, agree in all particulars with that of the actual system so represented, but our applications of Rayleigh's theorem clearly demonstrate that the discrepancy will become less and less significant as the assumed curve approaches the disturbed shape of the actual system.

The advantages offered by this approximate method can hardly be over-estimated, since we may thus arrive at useful solutions when an exact determination would be difficult or even impracticable. An example is presented by the complicated equations for the oscillation of the wing of an aeroplane moving through the air, because an insight into its *stability* can be gained by treating the structure as a semi-rigid body having only two degrees of freedom, involving flexural and torsional displacements.

129. As any change in the co-ordinates corresponds to a change in the position of a dynamical system, it is of some importance to notice that we have supposed an infinitesimal change in the co-ordinates to give a possible change in the configuration. The reason is that this supposition is not always true, because it occasionally happens that certain changes in the co-ordinates do not give a possible displacement of a specified system, owing to dynamical restrictions.

A system is said to be *holonomic* when this supposition holds good, and to be *non-holonomic* when it does not. Hence we can say that the preceding theory has been exemplified with reference to holonomic systems, in which the number of degrees of freedom is equal to the number of independent co-ordinates required to specify the configuration. But so far as vibrations about equilibrium are concerned, the difference between holonomic and non-holonomic systems is unimportant.

130. The conditions of equilibrium are most conveniently formulated with the help of the principle of virtual work.

If a particle subject to a system of forces is in equilibrium, no work will be done in any displacement of the particle so long as the forces remain constant in magnitude and direction. This is an ideal case in so far as in most actual systems the forces do vary as their points of application are moved. Nevertheless, it is always feasible to treat the forces as constant quantities in finding the *conditions of equilibrium*, since the *stability* alone is affected by any change brought about in a force due to movement of its point of application.

We can likewise determine the equilibrium of a system of particles by computing the work done in an arbitrary displacement of each particle. The labour of evaluating the work done by the

internal forces between the several particles will be avoided if we assume all the displacements to be infinitesimals and such as agree with the constraints, for the forces associated with the constraints do not then appear in the expression for the virtual work of the system as a whole.

The special characteristic of stability just mentioned enters into the consideration of a number of engineering problems, of which the 'hunting' of locomotives is one, but it is by no means easy, without recourse to tests, to ascertain the extent in the case of a gyroscopic system of this kind.

131. In view of the fact that, with the exception of certain systems, the equations of dynamics are non-linear and do not admit of exact solutions, a note may be called for to explain why the exemplification of the foregoing theory has been limited to motion governed by linear differential equations with constant coefficients.

The reason is that the procedure often leads to useful particular solutions, such as have reference to vibration about a position of equilibrium or one of steady motion. In proceeding thus we utilize the result that, for such states, the motion which ensues from a small initial displacement is expressed by a set of linear differential equations with coefficients that remain sensibly constant throughout the implied small vibrations.

132. By way of definition, we next notice that in the theory of dynamics a generalized co-ordinate is said to be *ignorable* when it does not appear explicitly in the Lagrangian function of a conservative system, only the corresponding velocity being present.

With this convention, the steady motion of a *conservative system having ignorable co-ordinates* is one in which all the non-ignorable co-ordinates and the generalized velocities corresponding to all the ignorable co-ordinates have constant values.

Again, with a system possessing ignorable co-ordinates, it can be proved [1] that the dynamical equations are reducible, by 'ignoration of co-ordinates', to a set containing only the non-ignorable co-ordinates. Then, however, the kinetic potential L is replaced by the 'modified' Lagrangian function R, and the latter may contain terms linear in the generalized velocities. This problem of vibration about a steady state of motion is distinguished from that of vibration about a position of equilibrium by the presence of such linear terms in R, as was observed in Art. 99.

133. In questions pertaining to the small motion of a system subject to forces of aerodynamic origin we usually have to consider a system of non-conservative type, and to take account of the fact

[1] E. T. Whittaker, *Analytical Dynamics*, page 193, third edition.

that such forces are influenced by the previous history of the motion. In an aeroplane, for instance, buffeting of the tail-unit may be produced by aerodynamic disturbances that originate at the front part of the machine, in the form of vortices that break away from both the trailing edge and root of the wing and subsequently impinge on the tail-unit. We may also have to consider the slip stream from the airscrew.

Nevertheless, small oscillations in, say, a state of equilibrium or one of steady motion, are customarily investigated on the supposition that the changes in value of the aerodynamic forces depend linearly on the quantities above denoted by q, $\dot{q}$, $\ddot{q}$, i.e. on the generalized forms of the co-ordinates, the velocities and the accelerations.

134. Use is made of the same supposition in the study of flutter in aircraft, notwithstanding the fact that, in a practical sense, the term 'flutter' relates to vibration that increases with the time until either the structure fails or limits are imposed on the amplitude by frictional forces that do not, in general, vary in a linear manner with the displacement.

The stated supposition is introduced by assuming the structure of an aeroplane to be free from friction, and only linear laws of variation to be involved throughout the motion.

If, for this ideal aeroplane, the critical speed for flutter of a given type be defined as the lowest forward flying speed at which free oscillation of that type is steady, the motion caused by a small initial displacement will tend to a simple harmonic form at the critical speed itself, and will ultimately die away at lower speeds. Furthermore, within a restricted range of speeds in excess of the critical value the amplitude of vibration will increase indefinitely with the time, even when the initial disturbance is small. It is to be inferred from the discussion on equation (52.7) that in the determinantal expression for the prescribed critical speed of flutter there will be at least one pair of conjugate purely imaginary roots.

For an actual aeroplane subject to frictional forces at speeds above the critical, a large disturbance produces flutter and a small disturbance vibration of the damped type. It is also possible for vibration with constant amplitude and frequency to be initiated by a disturbing agency that varies with the speed and is directly proportional to the limiting friction. Hence it appears that steady vibration for a particular speed is unstable and, therefore, either flutter or damped vibration will take place when the specified initial conditions as to the character of the disturbing agency are violated.

One of the important inferences to be drawn from this is that an aeroplane which is subject to the effect of solid friction may be free from flutter within a particular range of flying speeds, and yet

closely approach a state of instability when a certain speed is attained. Another is that the flutter-characteristics of such an aeroplane are to be interpreted in the light of two diagrams, showing in turn the variation of the frequency of vibration and of the damping factor with the speed of flying.

135. Stress-waves are involved in all problems of the kind discussed in this work, in a manner which is set forth in Chapter IV, provided the linear relation between stress and strain holds good throughout the motion.

A combination of that information and the theory of heat engines affords a theoretical basis for a study of the time-lag between injection and rise of pressure in the cylinder of an oil engine.

Experimental data on this and analogous questions are included in a paper by L. J. Le Mesurier and R. Stansfield,[1] who examined the vibration from records of the strains measured at a point situated between a cylinder head and the flange on the bedplate of an oil engine with five cylinders. When running on fuel having the shortest delay involved in these tests, the range of the displacements about the mean position amounted to 0·0027 in., with the peak values occurring a few degrees after the end of the delay period. It was further found that the shock of combustion caused the whole structure of the engine to execute forced vibrations, which quickly died away and left only natural vibrations having a frequency of about 400 cycles per second. At other places on the engine the disturbed motion was much more complicated, owing to the superposition of interference effects on the vibration characterized by the frequency of 400 cycles per second.

From similar tests with high-speed engines the same investigators concluded that the maximum pressure in cylinders supplied with long-delay fuels would be appreciably greater than with short-delay fuels unless the injection were retarded and the efficiency correspondingly reduced.

136. Stress-waves are, under the most general conditions as to shape and nature of the material, modified by the phenomena of dispersion and distortion, including reflection and refraction as particular cases.

These conclusions apply also to *supersonic waves* of the same type, which have frequencies that lie beyond the range of audible sound. In fact, no useful purpose would here be served by distinguishing too closely between ordinary sound waves and supersonic waves, since many of the consequences of the latter could be produced by sound waves of sufficiently high intensity.

Supersonic waves having frequencies up to 50,000 cycles per

[1] *Trans. N.-E. Coast Inst. Engrs. Shipb.*, vol. 53, page 223 (1937).

second are most conveniently generated by the mechanical distortion of a ferro-magnetic body when placed in the magnetic field of a magneto-striction oscillator.

But for still higher frequencies, up to 200×10^6 cycles per second, a piezo-electric oscillator must be employed, in which the vibrations of a suitably cut quartz plate are maintained by its inclusion in an appropriate oscillatory triode circuit.

Various metallurgical and chemical operations present fields of application for supersonic waves of great intensity, by reason of the energy thus placed at our disposal. A simple demonstration of this consists in directing a wave-train on to the edge of a steel knife for an interval of ten minutes, or so, when it will be found that the edge has been partly destroyed by the waves. Use may also be made of the *non-periodic* changes in temperature resulting from the absorption of wave-energy at the boundary of the specimen under test. Actually, it is easy to burn the surface of a piece of cork by placing it in contact with an emitter of such waves.

Since the energy of supersonic waves can be propagated several feet in iron or steel, we now realize that these waves may be employed to improve the structure of metals. But waves coming within the range of audible sound suffice for this purpose, as is proved by some experiments of G. Schmid and L. Ehret,[1] whose results suggest that such waves inhibit the formation of coarse dendritic structures. The consequent increase in fineness of the grain was, it is of interest to notice, accompanied by an alteration in the physical properties of the materials tested, the effect of the waves on the Brinell hardness being to raise it from 34 ± 1 kg. to 52 ± 1 kg. per square millimetre in the case of antimony, and from 78 ± 7 kg. to 96 ± 4 kg. per square millimetre in the case of duralumin. Again, the waves caused the needle-like structure of silumin to break up, though the Brinell hardness was not sensibly affected in this case.

Important research into metallurgy can evidently be carried out with simple devices for producing ordinary vibrations, and this brief reference to a practical aspect of the theory of Chapter IV may be supplemented by mention of other papers on the subject.[2]

137. Now, a knowledge of the methods and apparatus of experimental physics is all that is required to utilize the reflection and refraction of high-frequency waves in the testing of materials.

A device for detecting the presence of flaws in a metal is an obvious application of the theoretical result that the waves will be

[1] *Zeits. für. Elektrochem.*, vol. 43, page 869 (1937).

[2] S. Sokoloff, *Acta Physicochemica*, Moscow, vol. 3, page 930 (1935); G. Masing and J. Ritzau, *Zeits. für Metallkunde*, vol. 28, page 293 (1936); E. Hindemann, *Arch. für das Eisenhüttenwesen*, vol. 12, page 185 (1938).

deflected if their paths are intersected by a crack, an air hole or a similar defect in the metal. The position and size of such a flaw can be estimated from the variation in the amplitude of the pick-up registered by a piezo-electric detector when the supersonic waves traverse the flaw.

The photo-elastic method of exploring the distribution of stress is perhaps the best known application of double refraction. By the aid of this apparatus E. G. Coker and M. Salvadori [1] have studied the stress-waves in a small-scale model of a locomotive wheel, though obstacles of an experimental character must be surmounted before the ordinary form of the apparatus can be regarded as of general utility for periodic loads.

It is therefore worth while to point out that there is a direct method based also on Art. 73, in so far as high-frequency vibration in solids and fluids causes optical refraction that can be photographed. Here the work includes the mathematical analysis of spectrum phenomena produced by the attendant periodic changes in density, which may have a frequency of an order as high as 6×10^6 cycles per second. With the help of a valve-transmitting circuit and a piezo-quartz plate fixed to one face of the specimen we can obtain, on passing a parallel beam of light through a suitably prepared specimen, a refraction pattern that depends almost entirely on the elastic constants of the material, being practically independent of the shape of the specimen. Any changes brought about in these constants by continuous application of the load will accordingly be exhibited by a series of spectrum patterns.

As regards fluids, notable advances have been made in hydraulics through studies of refraction, in virtue of the close relation which exists between certain types of gravity-waves and a compressional wave moving with supersonic velocity in gaseous media. The degree of success attained in research along these lines is in no small measure attributable to the ease with which such compressional waves can be produced and examined in the laboratory. In air, for instance, the disturbance is confined to an extremely small fraction of a millimetre, of the order of 0·003 mm. under favourable conditions of testing, and from the consequent sharp change in density arises an equally sharp change in the refractive index of the medium, of a magnitude that can be photographed by transmitted light. The available apparatus is easily adapted to give an image of the wave-fronts set up by projectiles in flight, by jets emerging with supersonic velocities into fluids at rest, or by boundary roughness and change of cross-section in conduits through which fluids flow with high velocities. These photographs provide valuable information once we recognize the striking resemblance between

[1] *Proc. I. Mech. E.*, vol. 131, page 493 (1935).

such wave-fronts and phenomena that influence the flow of, say, water in open channels.

Another sphere of investigation is indicated by the fact that a magneto-striction appliance, vibrating with a sufficiently large amplitude, may be used to tear asunder a thin rod of ferro-magnetic material, with the aim of finding the tensile strength under alternating load. Such experiments might be extended to cover the connection between the tensile strengths under high-frequency loading on the one hand, and low-frequency on the other.

138. Mention has already been made of the transmission of waves through a material in which the value of the modulus of elasticity E is different for the states of tension and compression, as is sometimes found with concrete, masonry, and wood.

To throw more light on this question, suppose the bar represented in Fig. 100 to be made of such a material, and subjected to the disturbance specified in Art. 63. If the values in mind be distinguished by the suffixes t and c, at any instant we have, by equation (63.3),

$$\frac{\partial^2 u_t}{\partial t^2} = a^2_t \frac{\partial^2 u_t}{\partial x^2},$$

$$\frac{\partial^2 u_c}{\partial t^2} = a^2_c \frac{\partial^2 u_c}{\partial x^2},$$

where $a_t = \sqrt{\frac{E_t g}{\rho}}$, $a_c = \sqrt{\frac{E_c g}{\rho}}$ designate in succession the velocities of propagation for the states of tension and compression, and the density, ρ, of the material is treated as a constant quantity.

The vibratory motion at any point on the bar is readily formulated by combining equations (72.5) and (72.6), paying due regard to changes in the previous notation.

Of the inferences to be drawn from the resulting expression, the most important is that an element of the bar will execute *asymmetric* vibration provided our assumptions are valid. If experiment proves this to be so, the effect on the material will be a matter for further investigation with special reference to the vibration of structures and buildings due to traffic, and other causes.

139. Physiological Effect of Vibration. As we shall have occasion to speak of objectionable vibration, a few remarks may be made on what determines a nuisance in this respect.

It is manifest that the accuracy of experimental data on this point depends most of all on the recognition of unpleasantness by a person standing or sitting on a beam when it is vibrating with various amplitudes and increasing frequency. Any conclusions thus arrived at must be based on experiments relating to a large number of different persons, as tests give strong support to the

view that a specified degree of unpleasantness for any one person is influenced primarily by the magnitude of the acceleration, and secondarily by his posture and attitude with respect to the direction of motion. Association of ideas is perhaps another factor, for some people are much more sensitive to vibration in a quiet room than in the cabin of an aeroplane. Finally, investigations into what may be called the upper limit of human susceptibility to vibration must be brought into relation with the difficulty which even normal persons experience in drawing a distinction between what is merely perceptible and what causes annoyance.

The most reliable working rule is that supplied by A. Mallock,[1] who has defined a vibration as 'distinctly unpleasant' when it involves an acceleration greater than 5 per cent. that of gravity, assuming the motion to be simple harmonic in character.

140. Ships. The vibration met with in ships, and especially those propelled by oil engines or steam turbines, presents an instructive and important example of a complex system of structural members and disturbing agencies.

While any periodic force or couple applied to the structure of a ship is capable of initiating vibration, the disturbance will, in the main, only become objectionable when a state of resonance is maintained, since the possible disturbing agencies are not usually of sufficient magnitude to cause large displacements unless their effects are magnified in this way.

In order to prevent the more serious consequences of resonance, the possible natural frequencies of vibration for the complete structure should be computed, and the revolutions of the main and auxiliary engines arranged so that the frequencies of the primary and secondary components of their unbalanced effects shall not coincide with the calculated values for the structural system.

The possible modes of vibration for an existing ship are most conveniently traced from the records given by a set of vibrographs, provided the instruments are distributed over the structure in conformity with the result that the flexural and the torsional displacements are in general not independent. In the case of a proposed vessel, a first approximation to the natural frequencies in the lower modes of transverse vibration may be found by the method explained in Arts. 95 and 96, but it is to be remembered that the values so obtained may be substantially higher than the corresponding frequencies when the vessel is floating. It follows that the vibratory motion varies with the draught; for the extreme conditions of loading the frequencies may differ by 10 per cent., or more, with

[1] *Board of Trade Report on Vibration; Central London Railway*, page 5 (H.M. Stationery Office, 1902).

a vessel of average dimensions. This suggests, incidentally, that vibration can be mitigated by an alteration in the disposition of the ballast.

Another variable factor is the form of a ship. In the majority of vessels the extreme after end is not fully water-borne but is a cantilever supported by the more buoyant sections of the afterbody and, therefore, has an inherent liability to vibrate under conditions of varying immersion. The finer the afterbody lines, the more marked are likely to be the consequences of helm action, dynamic unbalance of the propellers, differing thrust of the individual blades, wave-impact in a quarter sea, or any other agencies tending to disturb a symmetrical distribution of stresses and reactions. Hence we may expect the greatest displacement to be near the stern, and, in a ship with several decks and a number of propellers, the displacement to increase somewhat at the after ends of the decks and at sections that roughly coincide with the wing propellers.

Moreover, localized vibration of high frequency and small displacement may occur, owing to the excitation of individual members of the structure. But, from an engineering point of view, these disturbances are of secondary importance, in that they often can be mitigated by the introduction of light stiffening at appropriate places in the system.

The remedial measures to be taken with the principal vibration in the hull are not so easily decided upon, even in the early stages of design. A change in the working speed of the most suitable size and type of engine may result in a substantial loss of efficiency, and prohibitive expense may be incurred in the alteration necessary to effect a sensible change in the natural frequencies of the hull. Consequently, a compromise must be made between the requirements of the engine builder and those of the naval architect, though the interests of both are always served by a rigidly constructed engine-frame that prevents the cylinders from vibrating like inverted pendulums.

Since the bending-moments and shearing-forces induced in the main structure by 'heaving' are matters for the consideration of naval architects, rather than engineers, the reader may be referred to a paper by A. M. Robb.[1]

Hence we are guided by the class of a ship in practical applications of the theory. In ordinary merchant ships, for instance, the vibration in the transverse plane of the hull will generally involve modes defined by two or three nodes, the next higher mode being somewhat exceptional.

High-powered vessels are distinguished by the fact that, independent of resonance, serious vibration may be produced by periodic

[1] *Trans. Inst. N.A.*, vol. 60, page 188 (1918).

irregularities in the propeller-resistance. The cause, in extreme conditions, is to be found in the relatively great difference between the wake-velocity of the layer of water next to the hull and that of the layer traversed by a blade in its outermost position. The blade, in passing through the wake, is doing more work than the others and the magnitude of the discrepancy may suffice to maintain considerable vibration in the system.

141. The remedial treatment for ships with more than one propeller naturally depends on the number of propellers. With twin-screws, it may only be necessary to install a synchronizing device for the purpose of securing a constant difference of phase between the engines. In the more general case, however, we have to consider the advantage offered by a change in the number of blades on one or more of the propellers, or a change in the relative direction of rotation, or a slight difference in the speeds of the several propellers.

It is pertinent to remark that failure to recognize the presence of interference on a vibrograph record not infrequently leads to unnecessary errors in the frequency derived from the trace, since the phenomenon tends to 'mask' vibrations having different frequencies and amplitudes.

142. The natural frequency and the position of the nodes for a given propeller vibrating in a prescribed mode is easily determined by the experimental method mentioned in Art. 126. These characteristics will, of course, be different when the tests are carried out in air on the one hand, and water on the other, the effect of submergence being to lower the frequency through increase in both the effective inertia of the blades and the dissipative forces.

On writing ν for the resonant frequency of a specified blade, c for its coefficient of stiffness, and M for its effective mass, by Art. 38 we have, in any mode of vibration,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{c}{M}}.$$

Now, a useful method of estimating the water-load on the blade can be based on the work of H. Lamb, cited on page 359, by assuming that the total change in frequency is caused by the increase in inertia alone. Thus we regard the external loading in air as negligibly small in comparison with that in water and, therefore, take c to be independent of the medium. Hence if M_m be the mass of the metal and ν_a the resonant frequency in air, then

$$\nu_a = \frac{1}{2\pi}\sqrt{\frac{c}{M_m}} \quad . \quad . \quad . \quad . \quad . \quad (142.1)$$

Similarly, if M_w designate the equivalent mass of the water-loading, and ν_w the resonant frequency in water, we also have

$$\nu_w = \frac{1}{2\pi}\sqrt{\frac{c}{M_m + M_w}} \quad . \quad . \quad . \quad . \quad (142.2)$$

Thus, after squaring and dividing (142.1) by (142.2), it appears that

$$\frac{\nu^2_a}{\nu^2_w} = \frac{M_m + M_w}{M_m},$$

i.e.

$$M_w = M_m\left\{\left(\frac{\nu_a}{\nu_w}\right)^2 - 1\right\} \quad . \quad . \quad . \quad . \quad (142.3)$$

Provided experimentally determined values of ν_a and ν_w for a given mode are known, from equation (142.3) we could ascertain, to the implied degree of approximation, the magnitude of the water-loading M_w in that mode.

Closely related to the matter under discussion, though at present beyond the range of mathematical analysis, is the peculiar problem of 'singing' propellers, which give out sound that may vary from a musical note to a most objectionable noise when heard in the after peak of a ship. This phenomenon may occur with any type of propelling machinery, as is to be gathered from a paper by H. Hunter,[1] who has collected and commented upon examples relating to fourteen vessels, of which four were propelled by single-screw Diesel machinery, one by twin-screw triple-expansion engines, one by twin-screw geared turbines, and the remaining eight by single-screw triple-expansion machinery. It would seem that the contributory factors comprise the disturbing force, the location of its impact, and the propeller. The region most responsive to the impacts appears to lie near the tip of a blade, though the disturbing force is likely to arise from eddies initiated by the hull, and geometrical accuracy of the propeller together with the physical properties of the metal may ultimately decide whether the vibration produced results in audible sound or not.

143. Damping Devices and Spring Couplings. Several types of appliances for the purpose of damping vibration caused by machinery in general, and engines in particular, are available. The requisite degree of stiffness is usually introduced through springs, dashpots, or elastic materials composed of rubber, cork, and other substances. From Chapter IV and the description of typical dampers given in the literature cited below,[2] to which may be added a reference to a hydraulic damper for colliery winding ropes,[3] it

[1] *Trans. N.-E. Coast Inst. Engrs. Shipb.*, vol. 53, page 189 (1937).

[2] *Engineering*, vol. 124, page 259 (1927); vol. 129, page 647 (1930); vol. 134, page 118 (1932); vol. 138, page 18 (1934); vol. 140, pages 100, 350, 416, 519, 688 (1935); vol. 143, page 480 (1937).

[3] *Glückauf*, vol. 64, page 365 (1928).

will be seen that these appliances may be divided into two main classes, those in which energy of vibration is dissipated in the form of heat, and those in which shocks are mollified and frequencies modified.

The most effective type of spring mechanism, for both machinery and buildings, is that which secures a non-linear relation between the load and displacement, combined with a sudden change in the frequency at a predetermined ‘ speed ’ of excitation. But a choice cannot well be made without regard to the nature of the foundation, for the springing of an automobile is so slow that it has practically no effect on the engine mounting, whereas in an aeroplane the elasticity of the frame exerts an appreciable influence. For this reason it is frequently necessary to strengthen the floors before a suitable form of damper can be installed in old buildings with machinery mounted on wooden floors.

The same principles are involved in the work of designing a spring coupling of the type examined in Ex. 7 of Art. 38, which may be used to lower the natural frequency in torsional vibration of a given shaft, or to change the relative position of a node, or to protect gearing from excessive variations in torque. A reader who has followed the previous chapters will recognize that in arriving at the permissible variation in torque for specified gearing we are guided by the elastic properties of the material, the shape of the teeth, and *a fortiori* by the amount of ‘ backlash ’ likely to be present.

Couplings of this kind should be inspected from time to time, as the development of ‘ play ’ adversely affects the uniformity of motion, through alteration in the characteristics, and especially those of small couplings, which are not easily designed to secure contact between the working parts under the greatest possible variation in load.

144. Torsional Vibration of Crankshafts. An obvious combination of results obtained in Arts. 38 (*b*) and 60 will provide a theoretical basis for a study of the torsional vibration of the crankshaft of a specified engine if the crankshaft is reduced to an equivalent length of uniform shaft, by the method explained in Art. 108.

When gearing is included in the system, as in that of Art. 61, the calculations become more complicated because we must then introduce the analysis of Art. 116.

It is clear that only one degree of freedom will be implicated in the simple case of a single shaft if we ignore the interdependence of torsional and flexural displacements disclosed in Art. 111.

A special significance is in this connection attached to the crankshaft of an internal combustion engine, owing to the periodic torque generated by the working fluid, which will be given by

equation (21.4) provided account is taken of the phase-angles of the cranks.

In such applications of the theory it is customary to assign an 'order' number to each of the harmonic terms in the Fourier series for the torque, being equal to the number of impulses in one revolution of the engine; and to divide the critical speeds into two groups, called 'major' and 'minor', the difference being that all the vectors in the torque diagram have the same phase in the former, and different phases in the latter. But it is not to be supposed that the major critical speeds are always the more significant, for equation (21.4) can be used to prove that, even with a fairly symmetrical distribution of load reckoned axially over the crankshaft, we may have the resultant torque zero in the major critical speeds, and yet comparatively great in the minor critical speeds of orders $1\frac{1}{2}$, $4\frac{1}{2}$, . . .

145. Several questions of design can be settled from considerations of the position of the nodes with respect to the loads in a marine-oil-engine. For example, in the mode defined by one node the various cranks vibrate with practically the same amplitude, and to this extent the effects of the minor critical speeds annul one another, leaving only the major critical speeds to be examined.

In the next mode, however, one of the two nodes is likely to be found near the flywheel, and serious vibration may then occur at either the major or the minor critical speeds. Again, the short length of shafting used in a ship with the engines situated aft renders vibration in this mode a possible source of trouble.

By a similar line of argument we arrive at conclusions concerning practicable changes in the possible modes of vibration of a vessel propelled by oil engines, which may be summarized in the following order.

In the mode with one node.

(i) The natural frequency will not be greatly affected by a change in the polar moment of inertia of the flywheel if its mass is but a fraction of the total mass in motion. The same may be said about a change in the stiffness of the crankshaft, since it is considerably stiffer than the propeller shaft.

(ii) The natural frequency may, on the contrary, be altered by a change in the polar moment of inertia of the propeller, owing to its position in relation to the node. This applies also to a change in the diameter of the propeller shaft.

In the mode with two nodes.

(i) In this case the propeller is close to one of the nodes and its amplitude of vibration is accordingly small, so that the natural

frequency is not substantially affected by a change in the polar moment of inertia of the propeller, or by a change in the diameter of its shaft.

(ii) But it is practicable to alter the natural frequency by a change in the disposition of the load, due to the presence of a node on the crankshaft proper. The same result can be attained by a change in the diameter of the crankshaft.

146. Airscrews. It is readily seen from Art. 102, having regard to the elastic connection between an airscrew and the crankshaft of its engine, that vibration of the blades can arise from a number of causes, even when the aeroplane is flying on a straight and level course and the airscrew is perfectly balanced. The disturbance may be produced by periodic torque-impulses in the drive, or by aerodynamic agencies. Further, both linear and angular vibration at the hub-centre may be brought about by whirling of the shaft, or by a periodic movement of the engine in relation to its mounting; and the crank-webs of some engines are apt to vibrate like tuning forks and to induce what is practically a 'fore-and-aft' vibration in the blades.

Though a blade can vibrate in a direction parallel to any one of its principal axes, the radial displacement will be irrelevant when the component displacements normal to and along the chord of the blade are under examination.

Here the question of resonance is extremely important, for the stresses in an airscrew might well increase almost fourfold, compared with the values when resonance is absent, if as a result of this phenomenon the aftermost end of the crankshaft undergoes a relative twist amounting to only 2 deg.

Hence we realize that alternating stress of very high value may act on rotating blades, remembering the wide range of frequencies involved in the unbalanced forces and couples, and the fact that the tip-speed of metal airscrews approaches the velocity of sound in air. Indeed, airscrews of aluminium have been known to fail when the stresses induced by the combined action of centrifugal and thrust effects amounted to 3,000 lb. per square inch, corresponding to 21 per cent. of the limiting stress in fatigue for the material. A steel blade of hollow section, to quote another case in point, failed under a computed stress of 10,000 lb. per square inch with material having a fatigue limit of 35,000 lb. per square inch.

It is, of course, feasible to design blades having a natural frequency which does not correspond with the resonant characteristics of a given engine, but in some instances this procedure leads to a substantial increase in the weight of the blades. This is possible in the case of geared engines having nine cylinders, which can be arranged to operate with particular airscrews in modes coming

within the range defined by two or three nodes. The practical solution to the problem might at first seem to consist in employing airscrews of minimum weight, and at the same time ensuring a fairly even torque on the crankshafts by increasing the number of cylinders, but this may only give rise to further trouble. With engines having fourteen cylinders, for example, the periodicity of firing is so high that it may produce torsional vibration of the blades, and as the angle of attack of a blade changes, the aerodynamic load increases and the damping may vary in a complicated manner. It is to be inferred from the examples of Art. 101 that under these conditions the magnitude and character of the dissipative forces may ultimately determine whether the blades remain stable or not.

Of equal importance is the fact that reduction in the weight of airscrews lessens the centrifugal and gyroscopic forces acting on the hub, and increases the useful load which can be air-borne. The first of these points merits special attention in the work of designing a variable-pitch airscrew which is inevitably heavier than an equivalent fixed-pitch airscrew.

Turning to the numerical aspect of the problem, we realize that the operation of computing the possible frequencies of vibration for a given blade will become troublesome when two or more of the influential factors above mentioned are included in the analysis. We therefore proceed by considering each factor separately on the basis of the most appropriate suppositions concerning the nature of the extraneous forces, and the degrees of fixity at the root of the blade and between its hub and shaft. There is, unfortunately, a very limited amount of reliable data on the degrees of fixity for actual systems of the kind in question.

In ordinary flight, however, the continuous supply of energy associated with the torque-impulses and the unbalanced inertia forces of the engine renders this combination the most significant of the factors to be considered. The effects of these disturbing agencies can be mitigated by inserting an elastic coupling between the engine and its airscrew, or by introducing in the system a 'dynamic damper' of the type described by A. N. Troshkin.[1]

We are now in a position to understand why the development of airscrews must proceed in close association with that of aero-engines.

147. The natural frequencies in the lower modes of transverse vibration may be estimated by the methods of Arts. 57 and 95–8, on the suppositions that the blade is equivalent to the slender beam specified by the relations (96.18), and the forces of gravity and damping are negligibly small quantities. In order to examine the airscrew alone, however, it is essential to regard the shaft as rigidly

[1] *Mech. Engineering*, vol. 59, page 668 (1937).

supported in the transverse and longitudinal directions and infinitely stiff in comparison with the blades. Though it is by no means easy to assign definite boundary conditions to actual blades, we may, as a first approximation, imagine the given blade as rigidly fixed at its root, and treat the components of transverse vibration as independent phenomena.

Finally, the prescribed frequencies are evaluated in the usual manner, by first omitting the rotation, and then bringing it in through equation (57.7).

It follows from what has been said that we should secure as great a difference as possible between the calculated natural frequencies of the blades and the frequency involved in the torque-impulses of the engine.

If an ideal blade of simple form is examined in this way, it will be found that the effect of rotation is to raise its natural frequencies by amounts that vary from 15 per cent. to 25 per cent. for the first two or three 'normal' modes of vibration. The degree of accuracy to be attached to these figures is a matter for experimental investigation, because the calculations are based on assumptions which do not readily lend themselves to theoretical verification.

148. Still more reliable values for the frequencies could be obtained by repeating the analysis with the quantities denoted by the I's in equations (96.18) modified so as to include the helix angle of actual blades, were it not difficult to discover series for the I's that converge rapidly.

But this is not the only obstacle in the way of progress along these lines, for we might well question more than ever the implied independence between the component displacements normal to and along the chord of such a blade, remembering that it is of helical form.

An instructive insight into what actually takes place can, nevertheless, be gained from the general theory of gyroscopic systems.

In as far as slight deviations from a perfectly straight and horizontal course are inevitable under ordinary conditions of flying, the disturbed motion of an airscrew will conform with the equations of Arts. 99–101, whence we learn that in the forced vibration there will usually be differences of phase, variable with the frequency, between the displacements and the force. In other words, the motion will be elliptic harmonic, a distorted 'figure of eight' being a possible form in the present circumstances. A consequence of this is that *opposite* signs may be attached to the components of the bending moment reckoned both parallel and normal to the chord at any section of a constituent blade. Moreover, the maximum variation in torque on the shaft may, in certain modes, be equal to the *difference* of those components.

It is further seen in a general way, having regard to the proportions of actual blades, that in the fundamental mode of vibration the component bending-moments will probably have opposite signs at the root, where the greatest bending moment acts, and the vibration will be in a direction nearly *normal to* the chord, which means that failure due to resonance is then likely to occur at the root of a blade. According to Arts. 97 and 98, failure at the same place may also ensue from whirling of the shaft, or what is the same thing in this sense, a periodic movement of the engine in relation to its mounting. But these conclusions do not necessarily hold in the second mode of transverse vibration, involving two nodes, for we may then have to consider a greater dynamic bending-moment which is applied with a frequency that approaches the natural frequency in a direction *parallel to* the chord of the blade.

Again, it appears from Ex. 1 of Art. 101 that, with small damping, severe vibration of a blade may arise from moderate variations of torque on the shaft. Though drawn from theory, this inference is remarkable as suggesting that important characteristics of the vibratory motion of airscrews in flight might be traced from records showing the variations in thrust and in torque on the shaft if suitable instruments were available. For a specified airscrew-crankshaft combination this practical method of solution should be arranged so as to facilitate separate and systematic investigations into the several variables mentioned above. This result may be expressed otherwise by saying that some of the observed irregularities in the (torsional) critical speeds of aero-engines are attributable to the shape of actual blades.

It is worth while to repeat that the foregoing conclusions relate to what may be called ordinary flight, in which the stresses in the airscrew-engine system are relatively low compared with those to be expected when the aeroplane is put into a fast ' spin '. Further, from Ex. 2 of Art. 124 we may infer that high stresses may be produced in the blades of an airscrew when the aeroplane yaws or pitches in a periodic manner.

149. Railway Bridges. It was pointed out in Chapter II that the vibration set up by a train as it crosses a bridge is caused mainly by the hammer-blow, introduced through balancing or partly balancing the horizontal inertia forces of the locomotive.

On the assumptions implied in equation (54.7), however, a train hauled by a perfectly balanced locomotive may initiate vibration in a bridge having joints that resemble pin-joints. Then the bridge is apt to vibrate vertically when the horizontal component of force denoted by X_8 varies periodically with a frequency approximately equal to the relevant natural frequency of the loaded bridge. The quantity designated by X_8 may represent the pull at the draw-bar.

The problem is complicated by the fact that the equation of motion for a railway bridge contains variable coefficients, as might be anticipated from a remark made in Art. 131, but the solution of the differential equation will present no unsurmountable difficulty if the frequency of the bridge is assumed constant.

In the present state of knowledge an estimate of bridge damping is still a matter of conjecture for new structures, yet it can hardly be doubted that this damping prescribes limits to the vibration when the displacement is large and synchronous in character.

We have to consider a determinate impulse applied at given instants to various points x measured from one of the abutments when a specified locomotive crosses the bridge at a known speed. This impulse is a function of x, and the attendant stress-waves in the bridge are modified by the phenomena disclosed in Art. 88. If, with the same conventions as before, the motion of the train is expressed by $x = K(t)$, the curve in the plane (τ, η) of Fig. 142 will be defined by the equation

$$\tau + \eta = 2K\left(\frac{\tau - \eta}{2a}\right),$$

and the initial conditions, corresponding to those on which (88.3) are based, will be

$$u = 0, \ \frac{\partial u}{\partial t} = \Phi(x)$$

along that curve.

Independent of this 'frictional' effect is the resistance to movement of the springs of the locomotive, by a force that sensibly agrees with the law of friction represented by Fig. 54, namely that of dry solid bodies. This factor cannot be ignored, for reasons which are fully explained in the treatises cited on page 91, and in papers by C. E. Inglis,[1] by R. H. Foxlee and E. H. Greet,[2] and by W. E. Gelson.[3]

150. Geophysical Surveying. A number of purposes will be served if we next describe how the principles of seismology are employed to detect the depth of an interface of different strata.

Imagine such an interface as lying below the point O in Fig. 185, and suppose three seismographs or recording instruments I_1, I_2, I_3 to be arranged in a line passing through O, at 'shooting distances' l_1, l_2, l_3 from that point. Also, suppose a blasting cap or charge of high explosive to be placed on, or a few feet below, the surface-soil at O, and connected electrically to the galvanometer in each of the three instruments by means of a thin wire.

[1] *Min. Proc. Inst. C.E.*, vol. 234, page 358 (1934).
[2] *Ibid.*, vol. 237, page 239 (1935).
[3] *Ibid.*, vol. 237, page 314 (1935).

If the powder is fired at what may be regarded as time $t = 0$, at that instant the electrical connection between the blasting cap and each instrument will be broken as a result of the explosion and, in consequence, the trace on the moving roll of paper or film will indicate the event by a 'kick'. It follows from the theory of Art. 78 that a system of compressional waves will simultaneously be generated in the earth, and some of the waves will travel through the surface-soil only, while others suffer either partial reflection or refraction at the interface. The wave-fronts are spherical or nearly spherical, and their paths are, on the average, concave upwards, but for all practical purposes these paths may be treated as straight lines.

With depths likely to be explored by engineers, however, the writer's personal experience is that the most accurate results are obtained on the assumption that the waves are refracted, and not

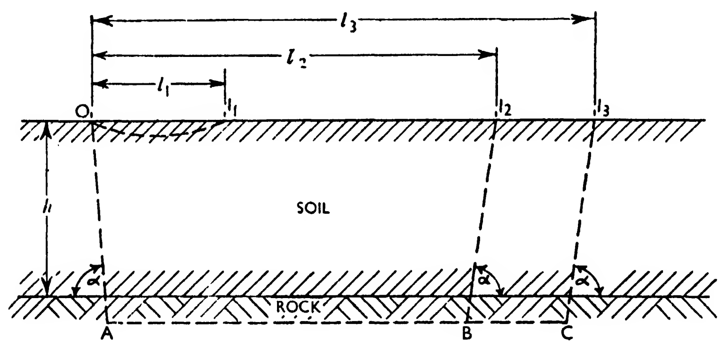


FIG. 185.

reflected, at the interface of surface-soil and rock illustrated by Fig. 185. Then, as the disturbance travels outwards from O, the time of arrival of the direct and the refracted waves at the instruments will be registered by 'kicks' in the trace on each of the three instruments.

The path of the first wave to reach a particular instrument will obviously be determined partly by the ratio of the shooting distance to the depth of rock h, and partly by the velocities of propagation in the strata under survey.

Let t_1 denote the interval of time thus recorded for the first shock to arrive at the instrument I_1. This pulse will pass through the surface-soil alone if, as we shall suppose, the shooting distance l_1 is comparatively short. Now the wave-velocity, v, in the surface-soil may be calculated from the formula

$$v = \frac{l_1}{t_1}$$

by measurement of the record.

At this stage of the investigation it is frequently possible to ascertain the general character of the surface-soil by comparing the numerical value of v with the velocities of propagation in typical strata, some of which are tabulated at the end of this article.

To fix ideas, we shall take the instrument I_2 as placed so that the second shock to reach it is the refracted wave associated with the path $OABI_2$ in the figure. The film of this instrument will therefore indicate, with a fair degree of certainty, the arrival of two waves, relating in turn to the direct and the refracted pulses.

On writing α for the inclination of OA to the horizontal, and V for the wave-velocity in the rock, it is readily proved that the refracted pulse will arrive in the shortest possible time if the condition $\cos\alpha = \dfrac{v}{V}$ is fulfilled.

As regards the next step, experience shows that the best results are attained by having the direct and the refracted shocks registered simultaneously on the same film. We are guided by this fact in selecting the most suitable trace from the records of the instruments I_2 and I_3. If, for definiteness, we assume this coincidence to be exhibited on the trace of I_2, then

$$\frac{l_2}{v} = \frac{2h}{v\sin\alpha} + \frac{l_2 - 2h\cot\alpha}{V}, \quad . \quad . \quad . \quad . \quad (150.1)$$

and the required depth is therefore given by the relation

$$h = \tfrac{1}{2}l_2\sqrt{\frac{V-v}{V+v}}, \quad . \quad . \quad . \quad . \quad . \quad (150.2)$$

with the velocities v, V evaluated from the records of I_1, I_2.

The same instruments may be employed to detect faults or to find the slope, if any, though in the latter of these applications it will be necessary to 'shoot' in both directions, up and down the slope of the rock-surface, unless the wave-velocities in the surface-soil and the rock are known. Putting this into symbols, if the interface is inclined at an angle φ to the horizontal, the apparent velocity in the rock will be greater or less than the true velocity according as the wave travels up or down the slope. Hence, if the true velocities in the surface-soil and rock be designated in succession by a_1 and a_2, on writing $\sin\theta$ for $\dfrac{a_1}{a_2}$, we have the apparent velocities V', up and down the slope, in the rock expressed by

$$V' = a_2\frac{\sin\theta}{\sin(\theta\pm\varphi)}.$$

This description would be incomplete without a brief reference to the practical advantages and disadvantages of this method. In the first place, the detection of anticlines and faults depends only

on the measurement of a time-difference, and not on a knowledge of the absolute velocities of propagation in the strata under examination. But, in the second place, it is not always easy to distinguish between the two pulses that matter on the same film, and it must also be remembered that the paths of these pulses are not, strictly speaking, straight, as is assumed in the theory. Again, there is no reason to suppose that the pulse recorded has followed the quickest route.

A fair degree of accuracy may, nevertheless, be attained on the assumption that all the angles marked α in Fig. 185 are similar and equal to 90 deg. Making this substitution in equation (150.1), we obtain, instead of (150.2),

$$h = \tfrac{1}{2}l_2\left(1 - \frac{v}{V}\right),$$

which means that the best results to be had from the instrument I_2 are secured by making the shooting distance

$$l_2 = \frac{2h}{1 - \dfrac{v}{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150.3)$$

Since applications of the foregoing relations implies some knowledge of the strata to be explored, a table of approximate values of the velocity of sound through different materials is appended.

VELOCITY OF A COMPRESSIONAL WAVE IN TYPICAL STRATA

Material.	Kilom. per second.	Feet per second.
'Made-up' Ground	0·2–0·4	660–1,300
Sand and Gravel (dry)	0·7–1·2	2,300–3,900
,, ,, ,, (wet)	1·3–1·8	4,300–5,900
Clay (loamy)	1·7–2·0	5,600–6,600
Sandstone (siliceous)	1·9–2·3	6,200–7,500
,, (clayey)	2·3–2·5	7,500–8,200
,, (chalky)	2·5–3·5	8,200–11,500
Marl (small chalk content)	2·5–3·0	8,200–9,800
,, (large chalk content)	3·5–4·5	11,500–14,700
Volcanic Rock	2·9	9,500
Sediments	3·5	11,500
Chalk (soft)	4·3–4·8	14,100–15,700
,, (hard)	5·0–6·0	16,400–19,700
'Salt-Dome'	4·7–5·3	15,400–17,400
Plutonic Rock	5·2	17,100
Archaen Rock	5·6	18,400
Schist	6·3	20,700
Granite	8·3	27,000

If, by way of illustrating the use of this tabulated information, $V = 2{\cdot}5$ km. per sec. and $v = 0{\cdot}2$ km. per sec., corresponding to

sandstone below 'made up' ground, the relation between the shooting distance l_2 and the depth h becomes

$$l_2 = 2{\cdot}2h,$$

after introducing in equation (150.3) the stated values.

151. This method has been applied with remarkable success in the work of prospecting for deposits of oil, though these usually occur at too great a depth for direct detection by their intrinsic properties. Fortunately, deposits of oil and, in certain places, deposits of sulphur are associated with 'salt domes', this being the name ascribed to peculiar upward intrusions of rock-salt crystallized from ancient land-locked seas. Such deposits are therefore detected by virtue of the fact that compressional waves travel faster through the mass of salt than the surface-soil, as is to be inferred from the above table.

It is thus particularly easy and inexpensive to locate deposits of stone for quarrying, because a single blasting cap often supplies sufficient energy for this purpose. Moreover, the same type of apparatus may be employed to explore for rock below rivers or in shallow waters, in connection with harbour and sea defence work.

In the general case, however, a decision on the quantity of explosive to use should be based on considerations of the nature of the exploration and the kind of instruments employed. For example, when surveying for oil in open country, such as is found on the Gulf of Mexico, charges of 250 lb., or more, of high explosive may be fired with safety and the instruments arranged to cover a radius of about 6 miles.

But an ounce of dynamite may fully suffice in densely populated areas, since a useful record of the shock could be registered a few thousand feet away provided the ground is free from 'microseisms', which will be discussed in Art. 162. It is essential to impose this condition because the amplitude of vibration involved in a microseism is then of the same order of magnitude as that generated by such a small quantity of powder.

Speaking generally, the shooting distance is made as short as possible, and the shot placed as deep as possible, up to 20 ft. in extreme cases, for the rock may be covered by weathered surfaces, through which sound travels much slower than in the solid rock. There is also a fair amount of evidence on which to base the working rule that the deepest point on the path of a shock wave is from one-quarter to one-third the shooting distance.

The graph exhibited on a film is naturally influenced by the nature of the surface-soil, as it forms one side of the interface traversed by the waves that reach the instrument.

If, on the one hand, the shot is fired in a dense material and the

seismograph is placed on sand, the disturbance will only be traced with certainty if the sand in contact with the base of the instrument behaves as an elastic material throughout the shock. This condition imposes a restriction on the permissible amplitude of the impulse and, incidentally, connotes practical limitations to the damping properties of sand placed under grillages and foundations in general.

If, on the other hand, the shot is fired in clayey or wet soil, the wave-energy will easily be transmitted to the instrument, whence we learn that a minimum quantity of explosive might be employed under these conditions.

But in all cases, it will be found, the highest possible degree of accuracy is attained by recording acceleration instead of amplitude.

This 'seismic' method of surveying may obviously be utilized in connection with foundations for buildings and machinery, and by it we can also acquire valuable information concerning certain kinds of damage done by explosions to neighbouring buildings.

152. The reader will no doubt have noticed in the table of Art. 150 that the presence of water in soil, and notably in clay, sand

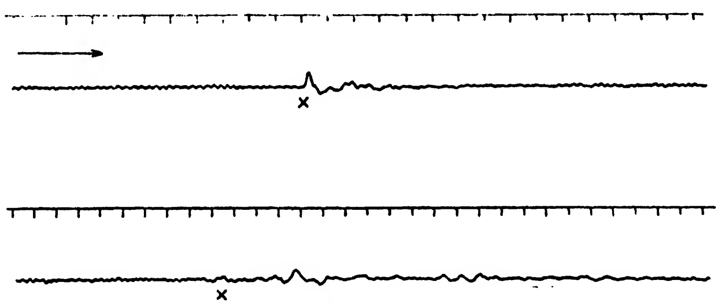


FIG. 186.

and gravel, is accompanied by a substantial increase in the velocity of propagation. Of much greater significance is the fact that water-logged soil is extremely sensitive to vibratory motion, so that good drainage should be included in the specification of foundations subject to vibration.

This extraordinary property of water-logged ground is well illustrated by Fig. 186, which is reproduced from a record showing the vertical displacement caused on the floor of a house by heavy looms in an adjacent factory. The distance between the looms and the instrument amounted to about 150 ft.; and an examination of the intervening strata revealed clay, gravel and running sand, at an average depth of 11 ft.

The upper graph refers to the disturbance produced by starting one of the looms, the shuttle being thrown at the instant marked by

the cross. When the looms were started at random, however, the floor of the house was subjected to the vibration represented by the lower graph, where the cross indicates the instant at which the first shuttle was thrown.

In this case, it may be noticed, the disturbance experienced in the house was due partly to the presence of the water-logged soil, and partly to faulty design of the floor that supported the looms, since other tests demonstrated that this floor, which consisted of a concrete slab, vibrated like a drum-skin.

Here the time-scale is marked at intervals of 0·1 second on the upper scale in each graph, and the vibration of high frequency and relatively low amplitude is to be attributed to the electric motor employed to drive the looms and other machinery in the factory. As measurement of several traces disclosed a periodic series of accelerations as high as 10 per cent. that of gravity, it follows from what was said earlier that this is an example of objectionable vibration.

Each problem under this heading requires separate consideration, for unexpected consequences may follow the usual remedy of lowering the level of ground-water, owing to the conspicuous sensitivity of certain kinds of water-logged soil. We may elucidate the point by reference to a particular turbo-generator which, after operating for some years without trouble in this regard, was more or less suddenly subjected to a serious state of resonance with the ground. This phenomenon was actually due to lowering the level of ground-water, and tests with a model further showed that the 'critical speed' for the ground was influenced by the shape and dimensions of the foundation, as well as by the level of ground-water.

153. Marine Surveying. In recent years highly successful acoustical methods have been developed to measure the depth of the ocean by emitting sound waves from a diaphragm fitted in the hull of a ship which may be travelling at speed. These waves, having a frequency of the order of 40,000 cycles per sec., travel in the water to the bottom of the sea, whence they are in part reflected back to a hydrophone receiver on the opposite side of the ship, where they are converted into electrical impulses which may be subsequently amplified. By this means we measure the time occupied in a to-and-fro journey of a pulse and thus calculate the depth of the ocean, since the velocity of propagation in sea water has been accurately determined. A fairly continuous record of the depth of the Atlantic Ocean between Europe and America has indeed been obtained in this way.

Information on the subject is required for different purposes. Depths ranging from 2 fathoms to 20 fathoms of water below the keel, as well as depths from 100 fathoms downwards, are of special

interest to navigators of large ships. Again, the success achieved in trawling depends on the ease and accuracy with which depths down to 350 fathoms, and more, can be measured, because the fish are mainly 'bottom feeders'. Very short intervals of time are implicated in this work, amounting to only 0·0024 sec. in the transmission of a pressure-wave through one fathom of water characterized by a velocity of propagation of 4,900 ft. per sec.

It will readily be understood that this method might be employed to detect the presence of icebergs provided they have sufficiently large underwater surfaces.

The time taken by a compressional wave to travel a given distance is necessarily affected by both the salinity and the temperature of the sea; the average velocity of propagation is 4,750 ft. per sec. in the Baltic Sea, 4,900 ft. per sec. in the Atlantic Ocean, and 5,100 ft. per sec. in the Red Sea. In a sea having a mean density of 1·02, this velocity is about 4,830 ft. per sec. for a temperature of 32 deg. F., and it increases to 5,045 ft. per sec. when the temperature becomes 90 deg. F.

It is also clear that the transmission of such waves will be affected adversely by air bubbles, at the boundaries of which the wave-energy suffers partial reflection, and by the phenomenon of dispersion when the disturbance is propagated in water that consists of layers having different temperatures and salinities. In this way the wave-energy is scattered, and experience shows that the amplitude of a compressional wave having a frequency of 35,000 cycles per sec. will be reduced to about two-thirds its original value after travelling 25 miles in an average sea.

The principal dimensions of the transmitting instrument may easily be found by the analysis of Art. 83. Thus, writing, as previously, p for the working pressure of the instrument, r and $\frac{\omega}{2\pi}$ in turn for its amplitude and frequency of oscillation, K for the bulk modulus of sea water, and a for the velocity of propagation, it appears, on taking maximum values, that

$$p = \frac{K\omega r}{a}$$

$$= a\omega r\rho,$$

where ρ represents the mean density of the sea; and the power, W, transmitted per unit area of the diaphragm is given by

$$W = \frac{K\omega^2 r^2}{2a}.$$

Therefore, combining these formulae,

$$W = \frac{p^2}{2a\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (153.1)$$

Consequently, if the effective area of the diaphragm be designated by A, it follows that its amplitude

$$r = \frac{p}{a\omega\rho}$$

$$= \frac{1}{a\omega\rho}\sqrt{\frac{2a\rho W}{A}}$$

$$= \sqrt{\frac{2W}{a\omega^2\rho A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (153.2)$$

As a numerical example, let it be required to find the working pressure and amplitude of vibration for a diaphragm to transmit 1 watt per square centimetre to sea water specified by a velocity of propagation of 1,490 metres per sec. and a density of 1·02 through an instrument operating at 40,000 cycles per second. Substituting these values in equations (153.1) and (153.2), we have, in approximate numbers, 10^6 dynes per square centimetre for the working pressure of the instrument, and 5×10^{-5} cm. for its amplitude of oscillation.

154. Aerial Surveying. The analogous process of measuring the height of an aeroplane, from its cabin, may likewise be effected by means of an echo reflected from the earth after a time-delay equal to twice the height divided by the velocity of propagation. It is, for obvious reasons, here essential to use 'wireless' waves, which give an echo almost instantaneously, in less than 0·000002 sec. for heights below 1,000 ft.

L. Espenschied and R. C. Newhouse [1] have given an account of the development and performance of an appliance designed for this purpose. In the cabin of the aeroplane is fitted an instrument for transmitting waves having a frequency that varies, about a mean value of 450,000 megacycles per sec., at a definite rate with respect to the time. Accordingly, for a particular signal, the frequency of the transmitter will have changed during the interval between transmission and reception in the cabin, the difference between the frequency of the transmitter and that of the echo being equal to the product of the rate of change and the time of transit. The equipment includes a receiver that combines the reflected pulse with some of the outgoing wave-energy and, at the same time, registers the frequency of the consequent 'beats'.

A study of this receiver presents an instructive and practical problem in interference. When flying over rough water, wooded country or cities the reflected signal is received almost simultaneously from surfaces situated at different distances away, with the result that the recorded interference undergoes periodic fluctuations. In extreme conditions this fluctuation may momentarily reduce the

[1] *Bell System Tech. Jour.*, vol. 18, page 222 (1939).

echo below the minimum required for accurate work, but it is claimed that the apparatus operates steadily over all kinds of country for heights less than 2,500 feet.

The character of the surface under survey is thus traced by variations in the reading of a meter, the contour of a city giving greater fluctuations in both the frequency and amplitude of oscillation than does cultivated land.

Reliable instruments of this kind will be of great value, not only for surveying, but also for navigation in fog and in airports surrounded by tall structures.

155. Traffic and Vibration of Roads. If a wheel of a vehicle strikes an object or irregularity on the surface of a road, a vibrograph placed on the ground would indicate two 'kicks', coinciding in succession with the instant of impact and the instant at which the wheel drops on to the road. According to the theory of Chapter IV, the first of these impacts will increase uniformly with the speed of the vehicle and be almost independent of the nature of the subsoil, since the stress-waves start from the upper surface of the road material. But the second impact might well vary between wide limits, with subsoils such as are found in this country, because its value is influenced by the amount of energy reflected back at the interface formed by the road material and subsoil.

Certain general ideas about these impacts may be acquired from the data contained in the table of Art. 150. For instance, in normal weather and with a subsoil of clay the magnitude of the impact will be intermediate between the values for sand and hard chalk. This characteristic figure might be expected to increase somewhat after periods of dry weather in the case of clay, and also in the case of gravel partly frozen as a result of cold weather. That is to say, with a specified vehicle, a record will register appreciable differences in the magnitudes of the impact for dry and wet conditions of clayey subsoils, and probably contain irregularities of the kind shown in Fig. 186 if the water-content is high.

156. Design of Buildings to Withstand Earthquakes. Since serious disturbances in the earth show, as few other experiences do, what slight difference of circumstance transforms grandeur to dust, it is not surprising that they are recorded in the most ancient history. There are a few scattered records of earthquakes in China dating back to 596 B.C., and the series is fairly complete from the year 1371 onwards. Information on the date, intensity and district of 1,898 earthquakes in Japan, between A.D. 416 and 1867, has been catalogued by S. Sekiya.[1] Further, E. Bertrand[2] has similarly

[1] *Coll. Sci. Jour.*, Tokyo, vol. II, page 315 (1899).

[2] *Mémoires Historiques et Physiques sur les Tremblemens de Terre*, page 22 (La Haye, 1757).

described 155 earthquakes which took place in Switzerland within the period A.D. 563 and 1754.

Seneca tells us that in his youth he had already written a treatise on earthquakes (*Aliquando de motu terrarum volumen iuvenis ediderim* [1]), whence it would seem that even in those days interest in this subject sometimes extended beyond the mere act of listing such events, and there is no doubt that this was so with the natives of South America who told the Spaniards that in constructing tall houses they were building their own sepulchres.[2] In view of this report, it may be that the design of the remarkable flat arch which once adorned the church of Santo Domingo in the old town of Panama was based on advice given to the Spaniards by the original inhabitants of South America, for even the ruins of the Inca civilization bear marks of great skill and knowledge in the art of constructing large buildings in regions subject to earthquakes.

It was not, however, until the latter half of the nineteenth century that scientific methods were applied to the study of earthquakes, by J. Milne. But the field of inquiry differs for seismologists and engineers, and it is only within recent years that adequate attention has been given to the engineering aspect of the problem.

In the first place, the broad aim of the seismologist is that of obtaining data such as will enable him to study the formation and constitution of the earth, for which purpose the recording of distant earthquakes affords an excellent means of investigation.

In the second place, the data which mainly interest engineers cannot well be derived from records taken with the delicate seismographs used in geophysical work on the transmission of stress-waves originating at a distant source, as the destructive effect on buildings is more or less localized near the origin of the disturbance. On this point H. Jeffreys[3] has expressed the opinion that most disturbances, including all large earthquakes, have focal depths not exceeding 35 km., and that the most violently shaken areas are roughly confined to a radius of 20 km. The same seismologist has further shown that the size of the foci may be quite small; his study of an earthquake in Jersey, which occurred in 1926, led to the conclusion that the whole of the energy could have been supplied by the failure of a cube of granite with an edge of 200 metres.

The increasing importance of this subject for engineers may be attributed to the fact that the habitable parts of the earth now include seismic regions which formerly were but sparsely populated. It is pertinent to remark that there is no evidence of any systematic

[1] *L. Annaei Senecae Naturalium Quaestionum Libri Septem*, VI, 4.

[2] *Trans. Roy. Soc.*, vol. 51, page 570 (1760).

[3] *Mon. Not. Roy. Astron. Soc., Geophy. Suppl.*, vol. I, pages 396, 483 and 500 (1928); vol. 3, page 131 (1933).

tendency in the sequence of great earthquakes, according to F. J. W. Whipple,[1] who has examined statistically 420 earthquakes that were recorded in Japan between the years 1900 and 1931.

157. An idea of the accelerations involved in earthquakes is best conveyed by reference to widely used scales of intensity. That associated with the names of M. S. de Rossi and F. A. Forel [2] is, with various modifications to suit local conditions, employed in the investigation of earthquakes in many countries in America and Europe, with the exception of Italy, where the scale put forward by G. Mercalli [3] is preferred. Both of these scales contain ten degrees of intensity, though their significance differs somewhat in the two tables, as may here be indicated by a description of the Rossi-Forel scale with the corresponding reference numbers of Mercalli signified by roman numerals.

ROSSI-FOREL SCALE OF INTENSITY

1. Microseismic shock; recorded by one or more seismographs of the *same* pattern; felt by an experienced observer. (I)
2. Extremely feeble shock; recorded by a number of seismographs of *different* patterns; felt by a small number of persons at rest. (II)
3. Very feeble shock; strong enough for the direction or duration to be appreciated; felt by several persons at rest. (III)
4. Feeble shock; creaking of floors and ceilings; disturbance of movable objects, doors, windows; felt by several persons in motion. (IV)
5. Shock of moderate intensity; ringing of swinging bells; disturbance of furniture and beds; felt generally by everyone. (IV)
6. Fairly strong shock; general ringing of house bells; stopping of pendulum clocks; oscillation of chandeliers; visible agitation of trees and shrubs; general awakening of those asleep; some startled persons leaving their dwellings. (V)
7. Strong shock; ringing of church bells; overthrow of movable objects; fall of plaster, but without damage to buildings; general panic. (VI)
8. Very strong shock; fall of chimneys; cracks in the walls of buildings. (VII)
9. Extremely strong shock; partial or total destruction of some buildings. (VIII)
10. Shock of extreme intensity; great disaster; buildings ruined; disturbance of strata, fissures in the earth's crust, fall of rocks from mountains. (IX and X)

[1] *Mon. Not. Roy. Astron. Soc., Geophy. Suppl.*, vol. 3, page 233 (1934).
[2] *Arch. des Sciences Physiques et Naturelles*, Geneva, vol. II, page 148 (1884).
[3] *I Terremoti della Liguria e del Piemonte*, page 19 (Naples, 1897).

The significance of these data may be interpreted in more precise terms through the work of E. S. Holden [1] on the equivalent maximum accelerations for degrees 1 to 9 of the Rossi-Forel scale; his results, taken in the above order and expressed in millimetres per second per second, are as follows: 20, 40, 60, 80, 110, 150, 300, 500 and 1,200. Within this range, then, the greatest acceleration is slightly more than 12 per cent. that of gravity.

In order to represent the highest of all intensities, A. Cancani [2] suggested a scale with twelve degrees, in which the equivalent maximum accelerations, with the same units as before, are respectively 2·5, 5, 10, 25, 50, 100, 250, 500, 1,000, 2,500, 5,000 and 10,000. Here the greatest acceleration is nearly equal to that of gravity.

158. Disturbances of the order specified by the foregoing scales of intensity would, of course, be transmitted to buildings through their foundations, and a mass of useful information on the consequences would doubtless be disclosed by inspection of the cracks and fissures in the foundations and walls of ancient buildings situated in seismic regions, and by tests with models of the buildings and foundations.

There are many such structures to be found in various parts of the world, as, for instance, the walls and, particularly, the towers of the cathedrals of Lima (Peru) and Strasbourg (France), having regard to their histories and the geological structures of those cities. In the case of the edifice at Lima, founded in 1535, its twin-towers were, in the course of construction, three times destroyed by earthquakes, and, during an earthquake in 1838, the lofty tower of the adjacent church of Santo Domingo is said to have leaned over like the well-known campanile at Pisa. Apart from the concrete 'corset' which now encircles the base of its towers, the church of Nôtre Dame in Strasbourg is noteworthy because there are records of serious shocks in the neighbourhood for the years 1289, 1356, 1357, 1669 and 1728.

159. In tests with models we may approximately exhibit the elastic properties of a prescribed kind of ground by means of springs placed at the base of the foundation, and study the stresses at various points on the system by subjecting the base to an assumed motion of the earth. A simple experiment with such models is to demonstrate how easily buildings with heavy roofs and foundations on soft or loose ground may be overthrown, as is most likely to happen with the slender structures of wood found in tropical countries.

A much more difficult question arises with large buildings of

[1] *Amer. Jour. Science*, vol. 35, page 428 (1888).

[2] *C.R. des Séances de la 2me Conf. Sismol. Intern.*, page 281 (1904).

masonry or concrete, namely relative movement between the subsoil and foundations when the ground is soft or wet, though this may, in favourable circumstances, be remedied by suitable systems of piling, assuming that the plates are securely fixed in the case of timber piles.

For this and other reasons careful attention should be paid to the disposition of reinforcements in piles and foundations of concrete. Stated briefly, all piers and piles should be well capped and tied together so as to form a continuous structural system, with the reinforcements carried across any unavoidable construction joints. In this way we ensure, as far as may be, a fairly even distribution of load over foundations during the most severe phase of an earthquake a point which deserves notice because the stress on foundations usually increases with repeated shocks. These requirements are, in many instances, fulfilled by reinforced mat- or block-foundations.

The present considerations also have reference to reinforced concrete columns. Faulty design of the ends of such columns is not an uncommon cause of failure, with which is probably associated the torsional action mentioned in Art. 161, and on this account the helical type of reinforcement is better than the straight-bar type. Here, as in all parts of reinforced concrete systems, the bars should be designed to withstand the full load in the principal transverse directions, and arranged to prevent the concrete outside the metal cracking and falling away in the early phase of an earthquake, thereby leaving the reinforcements unsupported in the later phase.

Another significant matter under the heading of foundations is the topography of a site. We often meet with considerable damage in buildings erected on river-banks, cuttings, sharps, and bluffs. This may partly be accounted for by reflection or refraction of the stress-waves at such abrupt changes of section in the earth, but there is need for experimental investigation into this extremely important aspect of the subject, as experience shows that the strata, level of any ground-water present, and height above sea-level influence the damage done on sites of this kind.

The seismic characteristics of a locality enter into the general problem of foundations, as well as of the structure proper, because at certain places situated some distance away from the epicentre of an earthquake the vibration felt is negligibly small compared with that of adjacent ground. Although the cause of such extraordinary variations of amplitude as occur within comparatively short distances is still unexplained, settling of sediments and coincidence of periods in the later phase of a shock are possible factors. The writer's personal experience near and on the Pacific coast between Trujillo (Peru) and Valparaiso (Chile) gives some support to his opinion that this peculiarity may also arise from faults or outcrops at which the stress-waves suffer refraction or partial reflection.

Such places would thus be, so to say, isolated during earthquakes of certain intensities. Something of this nature undoubtedly happened in an accidental explosion, on February 13, 1930, of about half a ton of powder at the Du Pont explosive factory, Gibbstown, N.J., since the damage there consisted mainly of broken windows, whereas the greatest effect of the shock was felt some 30 miles away. In view of this fact and a knowledge of the geological structure of that district, it is seen that the waves initiated by the explosion travelled below the Delaware River and emerged at an outcrop near the region most seriously affected.

The phenomena which attend the transmission of stress-waves across an interface of hard and marshy kinds of ground is also a matter of conjecture at the present time. Light might be thrown on this obscure problem by firing a fairly large quantity of high explosive near such regions, and tracing the interfacial phenomena from seismograms taken on both sides of the boundary.

160. Turning to the structural part of buildings, close attention must obviously be given to all questions connected with resonance.

But this is not the only factor in the matter of design, as inspection and study of the consequences of a number of earthquakes, of various intensities, suggests that the most important characteristics include the magnitude of the acceleration and of the predominant period, as well as the duration and number of shocks that cause substantial damage. This broad view must be taken, for apart from the fact that a condition corresponding approximately to synchronism is a remote possibility when considering the several types and sizes of buildings to be found in any one town, it should be borne in mind that the actual disturbance is generally very irregular, resembling an impulse in extreme cases.[1]

We may examine the effect of a sharp shock by making the interval of time τ in equation (79.2) correspondingly short. An intense shock does not, however, always cause more damage than a number of less intense disturbances. Furthermore, the mere addition of material to structures may only increase the damage done under conditions of resonance, as is to be inferred from Art. 92. Consequently, buildings designed to withstand earthquakes are distinguished by the most efficient disposition of the minimum amount of material for an assumed distribution of extraneous load.

Except in special circumstances, a value of about one second may be assigned to the predominant period of earthquakes. But the natural period of vibration of buildings should be made much shorter wherever possible because the cracks produced in the material by repeated shocks will, with buildings of average size and type, cause their natural periods to increase and to approach

[1] *Jour. Roy. Soc. of Arts*, vol. 88, pages 804 and 807 (1940).

the predominant period of earthquakes. This tendency is in agreement with conclusions arrived at in Art. 55.

We are naturally guided by the seismic history of a place in deciding on the acceleration to be assumed in the work of design; values defined by the limits of 10 per cent. and 25 per cent. that of gravity are stated in official regulations on the subject.

While it is customary to neglect all but the horizontal component of acceleration of the ground, the distribution of load over certain structures renders them particularly liable to failure under stresses produced by vertical movement. Moreover, the earth may describe relatively large vertical movements, as were observed by N. H. Heck and F. Neumann [1] at distances within the range of 17 miles and 37 miles from the epicentre of an earthquake in California; the records indicated a maximum acceleration of nearly 25 per cent. that of gravity, a period of 0·3 sec., and a computed maximum displacement of 6 cm. (2·4 in.).

161. A full examination should be so arranged as to enable the designer to recognize where the weakest part of a given structure is situated. From the practical point of view, it is by no means an easy task adequately to deal with details, since it is not always practicable to secure fixed-end conditions, which were assumed in Ex. 3 of Art. 56 and the theory of Art. 92; but this is an instance in which zeal for an ideal should be proportional to the difficulties surrounding it. This is especially so in anything relating to the design and construction of trusses, as both experience and the theory of Art. 79 show that appreciable additional loads may be transmitted through trusses when a building is vibrating.

A first approximation to the natural frequency for a given system is most conveniently obtained by assuming, as in Ex. 3 of Art. 56, the masses and loads to be concentrated at the corresponding floor-levels. The real problem is complicated by the fact that in actual buildings the amplitude of vibration tends to increase with the period and to exhibit other, but smaller, variations as the systems pass through a state of resonance. Furthermore, the variation of amplitude with damping commonly decreases with increase in the damping, though we sometimes meet with cases in which the reverse order of things holds good. It is, in fact, difficult to generalize on the degree of damping for complete structures, as a remarkably small amount is present in some systems, and especially those exemplified by bridges and elevated tanks.

Resonance may, of course, take place with respect to either of the principal transverse axes of a structure, and an earthquake may approach from any point of the compass. There is, indeed, no reason to suppose that the disturbed motion is in general of the simple

[1] *Engineering News-Record*, vol. 110, page 804 (1933).

linear type; the horizontal motion which actually occurred in a particular case, involving a single-storey building and an earthquake of intensity 7 on the Rossi-Forel scale, would be represented by Fig. 86 if that motion were moderately damped. In these circumstances, a building tends to rotate about a vertical axis which may or may not pass through its centre of mass, and the magnitude of the stresses may be much more serious than is suggested by the usual assumption of simple bending and shearing forces, owing to the torsional action involved here.

There is scarcely any doubt that the basis of design should rest on the supposition that the stress is determined by the combined longitudinal and transverse displacements taken, for convenience and safety, at their maximum values. From this we infer that the degrees of stiffness should be approximately the same with respect to the principal axes of symmetry in the horizontal plane. Consequently, if annexes or wings must be added to the main part of a building, the connecting members should be designed to fail in the early phase of a probable earthquake, and thus enable the chief parts of the system to vibrate independently.

This question of symmetry has an important bearing on systems having masonry towers, in which the damage will be augmented if the tower vibrates with a period that differs greatly from the period of the remainder of the building.

Previous remarks on trusses suffice to indicate the need for low-pitched roofs and flat arches having easy curves at the abutments; and for windows and doors in the walls of a building to be arranged at a common level, with continuous horizontal beams or lintels placed across the openings. Advantage may be taken of indispensable windows and doors as a means of securing a minimum height above ground-level of the centre of gravity of the main walls, this being essential because such walls frequently receive almost the full impact of earthquakes.

Wire mesh may accordingly be utilized in the bonding material of masonry at sections in a structure where we should expect high tensile stress, due partly to the reflection of stress-waves at the outer boundary of, say, a wall. To exemplify, we remark that, once in a while, the end wall of a house or terrace of houses collapses when the bonding material is not strengthened in this way, leaving the other walls standing after an earthquake.

The present line of argument would, if pursued, ultimately lead back to the problem of foundations, through the design of ties across the base, and between columns, beams, trusses and bracing generally. In this part of the work, eccentric loading, inadequate bearing surfaces and anchorages are, for obvious reasons, to be avoided as potential sources of danger.

Owing to the lengthy description of constructional details which would be required to present a full account of the effect of earthquakes on dams and the remedial measures which may be adopted, we refer the reader to the extremely valuable Final Reports on the Boulder Canyon Project, and especially the one cited below.[1] It may be pointed out that the treatment of this subject in those reports has reference to the design and construction of retaining walls subject to disturbances in the earth.

162. Microseisms. This is the name ascribed to small oscillations of the ground which are registered by sensitive seismographs when placed on most parts of the earth, and a brief description of the influential factors will serve to elucidate a point made in Art. 151.

Microseisms are distinguished by periods within the interval of 3 sec. and 10 sec., and amplitudes up to about 0·001 in. Observation of conditions at a given station show that the ratio of horizontal to vertical components depends on the geological structure, but not on the distance from the ocean; also, the amplitudes and periods vary greatly throughout the year, both being larger in the winter months. Records of the movement as registered at Hamburg and Copenhagen are included in a paper by J. A. Archer.[2]

F. J. W. Whipple and A. W. Lee [3] have expressed the view that microseismic movements are due to Rayleigh waves starting from storm centres over the oceans, not far from places where sea-waves break on rocky coasts or in shallow waters. A series of their observations taken at Kew and Eskdalemuir showed no indication of regular diurnal variation of amplitude, but a tendency for the period to be greater at midnight than at midday. The reverse order of things may, however, be experienced; at Graz there are well-marked diurnal variations in the north and east components, with the greatest amplitudes and periods taking place at midday.

It is possible to have equivalent microseismic activity at different stations, as for example at Kew, Potsdam and Pulkovo, where there is good agreement between the mean amplitudes with given periods recorded in the course of a year.

A. W. Lee [4] has further shown that the velocity of propagation of microseismic waves, and the ratio of horizontal to vertical components are both affected by the ratio of wave-length to thickness of layer. It appears that these components of amplitude vary

[1] *Technical Investigations, Part V, Bulletin No. I; Trial Load Method of Analysing Arch Dams,* page 163 (U.S. Bureau of Reclamation, Denver, Colorado, 1938).

[2] *Mon. Not. Roy. Astron. Soc., Geophy. Suppl.,* vol. 4, page 193 (1937).

[3] *Ibid.,* vol. 2, page 367 (1931); vol. 3, page 288 (1935).

[4] *Ibid.,* vol. 3, page 244 (1934).

considerably with changes in the thickness of layer; the effect of thin layers is chiefly on the horizontal component, whereas the vertical component becomes large when the wave-length is less than about eight times the thickness of layer.

163. Tidal Action. In hydraulic structures we may have to consider periodic tilting movements of the ground near tidal waters which follow from local loading of the earth's crust, first, by the tides, and, secondly, by changes in direction of the force due to the combined effect of the earth's gravitational force and the tide-generating forces.

Tilting of the earth on this account at Bidston, Liverpool, has been investigated by A. T. Doodson and R. H. Corkan,[1] who found large tilting movements in unison with the loading produced by the tidal waters that extend north of the observatory. The relative variations in tilt and tide on successive days, in 1933, are exhibited in

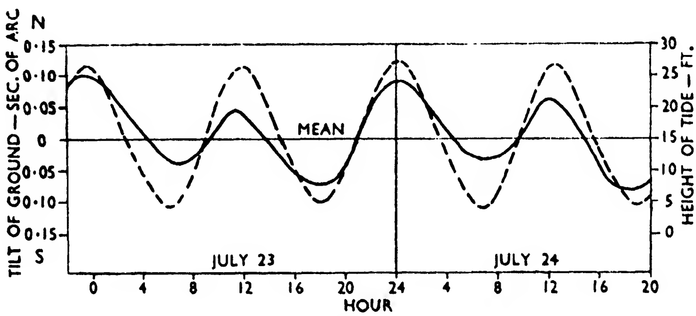


FIG. 187.

Fig. 187, which is reproduced from their paper; here the full line represents the tilt-curve, and the dotted line the tide-curve. It was discovered that the semi-diurnal constituents were in fair agreement, but the diurnal constituents of the tilt were much greater than those of the tides and in opposite phase. To explain this anomalous variation in the diurnal part of the tilt, the authors of this paper put forward the interesting suggestion that the tilt may, with a fair degree of accuracy, be separated from another tilt, which they call 'body-tilt'. It is mainly to this body-tilt that the large diurnal abnormality in tilting of the ground is attributed. Records of the tilt at Bidston have been further analysed by R. H. Corkan.[2]

The stresses which may thus be induced in the foundations of hydraulic structures are indicated by the work of H. Pinkwart,[3] whose measurement at points near the front of a steel sheet-piled quay-wall disclosed a maximum displacement of 3·6 mm. (0·14 in.)

[1] *Mon. Not. Roy. Astron. Soc., Geophy. Suppl.*, vol. 3, page 204 (1934).
[2] *Ibid.*, vol. 4, page 481 (1939).
[3] *Zeits. für Vermessungswesen*, vol. 65, page 570 (1936).

in a direction parallel to the wall, and one of 5·7 mm. (0·23 in.) at right-angles. As might be anticipated, the movement varied in a complicated manner with the state of the tide; on a falling tide, a point on the top of a bridge-pier first moved 1 mm. (0·04 in.) upstream, and then moved downstream, attaining a maximum displacement of 4·5 mm. (0·18 in.) at low water.

APPENDIX

SUCCESSIVE substitution of the values r, $s = 1, 2$ in equation (96.12) leads at once to the relations

$$f_1 = \frac{1}{\sqrt{2}},\ f_2 = \frac{\cos \alpha_2 x \cosh \alpha_2 l + \cosh \alpha_2 x \cos \alpha_2 l}{(\cos^2 \alpha_2 l + \cosh^2 \alpha_2 l)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad \text{(i)}$$

whence we obtain the second differential coefficients with regard to x in the form

$$f_1'' = 0,\ f_2'' = \alpha_2^2 \frac{-\cos \alpha_2 x \cosh \alpha_2 l + \cosh \alpha_2 x \cos \alpha_2 l}{(\cos^2 \alpha_2 l + \cosh^2 \alpha_2 l)^{\frac{1}{2}}} \quad . \quad \text{(ii)}$$

In the notation of equations (96.13),

$$D_{rs} = \int_{-l}^{l} (1 - bx^2) f_r'' f_s'' dx,$$

hence, by virtue of equations (ii),

$$D_{11} = \int_{-l}^{l} (1 - bx^2) f_1'' f_1'' dx = 0,$$

$$D_{12} = \int_{-l}^{l} (1 - bx^2) f_1'' f_2'' dx = 0,$$

$$D_{21} = \int_{-l}^{l} (1 - bx^2) f_2'' f_1'' dx = 0,$$

$$D_{22} = \int_{-l}^{l} (1 - bx^2) f_2'' f_2'' dx$$

$$= \frac{\alpha_2^4}{\cos^2 \alpha_2 l + \cosh^2 \alpha_2 l} \int_{-l}^{l} (1 - bx^2)(-\cos \alpha_2 x \cosh \alpha_2 l + \cosh \alpha_2 x \cos \alpha_2 l)^2 dx.$$

This relation for D_{22} contains several integrals, the indefinite forms of which, with the constants of integration omitted as irrelevant, may be arranged in the following order:

$$\int \cos^2 \alpha_2 x dx = \frac{x}{2} + \frac{\sin 2\alpha_2 x}{4\alpha_2},$$

$$\int \cos \alpha_2 x \cosh \alpha_2 x dx = \frac{1}{2\alpha_2}(\cos \alpha_2 x \sinh \alpha_2 x + \sin \alpha_2 x \cosh \alpha_2 x),$$

$$\int \cosh^2 \alpha_2 x dx = \frac{x}{2} + \frac{\sinh 2\alpha_2 x}{4\alpha_2},$$

$$\int x^2 \cos^2 \alpha_2 x dx = \frac{x^3}{6} + \left(\frac{x^2}{4\alpha_2} - \frac{1}{8\alpha_2^3}\right) \sin 2\alpha_2 x + \frac{x \cos 2\alpha_2 x}{4\alpha_2^2},$$

$$\int x^2 \cos \alpha_2 x \cosh \alpha_2 x dx = \frac{x^2}{2\alpha_2}(\cos \alpha_2 x \sinh \alpha_2 x + \sin \alpha_2 x \cosh \alpha_2 x)$$
$$- \frac{x}{\alpha_2^2}(\sin \alpha_2 x \sinh \alpha_2 x)$$
$$+ \frac{1}{2\alpha_2^3}(\sin \alpha_2 x \cosh \alpha_2 x - \cos \alpha_2 x \sinh \alpha_2 x),$$

$$\int x^2 \cosh^2 \alpha_2 x dx = \frac{x^3}{6} + \left(\frac{x^2}{4\alpha_2} + \frac{1}{8\alpha_2^3}\right)\sinh 2\alpha_2 x - \frac{x \cosh 2\alpha_2 x}{4\alpha_2^2}.$$

Evaluating these integrals between the prescribed limits, remembering that

$$\sin(-l) = -\sin l, \quad \cos(-l) = \cos l,$$
$$\sinh(-l) = -\sinh l, \quad \cosh(-l) = \cosh l,$$

we find

$$\int_{-l}^{l} \cos^2 \alpha_2 x \cosh^2 \alpha_2 l dx = \cosh^2 \alpha_2 l\left(l + \frac{\sin 2\alpha_2 l}{2\alpha_2}\right), \quad \text{(iii)}$$

$$2\int_{-l}^{l} \cosh \alpha_2 x \cosh \alpha_2 x \cos \alpha_2 l \cosh \alpha_2 l \, dx$$
$$= \frac{1}{\alpha_2}(\cos^2 \alpha_2 l \sinh 2\alpha_2 l + \cosh^2 \alpha_2 l \sin 2\alpha_2 l), \quad \text{(iv)}$$

$$\int_{-l}^{l} \cosh^2 \alpha_2 x \cos^2 \alpha_2 l dx = \cos^2 \alpha_2 l\left(l + \frac{\sinh 2\alpha_2 l}{2\alpha_2}\right), \quad \text{(v)}$$

$$b\int_{-l}^{l} x^2 \cos^2 \alpha_2 x \cosh^2 \alpha_2 l dx$$
$$= b \cosh^2 \alpha_2 l\left\{\frac{l^3}{3} + \left(\frac{l^2}{2\alpha_2} - \frac{1}{4\alpha_2^3}\right) \sin 2\alpha_2 l + \frac{l \cos 2\alpha_2 l}{2\alpha_2^2}\right\}, \quad \text{(vi)}$$

$$2b\int_{-l}^{l} x^2 \cos \alpha_2 x \cosh \alpha_2 x \cos \alpha_2 l \cosh \alpha_2 l dx$$
$$= b\left\{\frac{l^2}{\alpha_2}(\cos^2 \alpha_2 l \sinh 2\alpha_2 l + \cosh^2 \alpha_2 l \sin 2\alpha_2 l)\right.$$
$$- \frac{l}{\alpha_2^2}(\sin 2\alpha_2 l \sinh 2\alpha_2 l)$$
$$\left.+ \frac{1}{\alpha_2^3}(\cosh^2 \alpha_2 l \sin 2\alpha_2 l - \cos^2 \alpha_2 l \sinh 2\alpha_2 l)\right\}, \quad \text{(vii)}$$

$$b\int_{-l}^{l} x^2 \cosh^2 \alpha_2 x \cos^2 \alpha_2 l dx$$
$$= b \cos^2 \alpha_2 l\left\{\frac{l^3}{3} + \left(\frac{l^2}{2\alpha_2} + \frac{1}{4\alpha_2^3}\right) \sinh 2\alpha_2 l - \frac{l \cosh 2\alpha_2 l}{2\alpha_2^2}\right\} \quad \text{(viii)}$$

Further, equation (96.7) gives, in terms of $\alpha_2 l$,

$$\tan \alpha_2 l + \tanh \alpha_2 l = 0,$$

the first two roots of which have been shown to be zero and 2·3650, nearly. Taking $\alpha_2 l = 2{\cdot}3650$ as the relevant root, we thus see that the

values required in the foregoing integrals are, in approximate numbers,

$$\alpha_2 = \frac{2\cdot3650}{l},\ \alpha_2{}^2 = \frac{5\cdot593}{l^2},\ \alpha_2{}^3 = \frac{13\cdot23}{l^3},\ \alpha_2{}^4 = \frac{31\cdot28}{l^4},$$

$$\begin{array}{lll} \cos\alpha_2 l = -0\cdot7133, & \cos^2\alpha_2 l = 0\cdot5089, & \cos 2\alpha_2 l = 0\cdot0176, \\ \sin\alpha_2 l = 0\cdot7009, & \sin^2\alpha_2 l = 0\cdot4914, & \sin 2\alpha_2 l = -0\cdot9998, \\ \cosh\alpha_2 l = 5\cdot369, & \cosh^2\alpha_2 l = 28\cdot82, & \cosh 2\alpha_2 l = 56\cdot65, \\ \sinh\alpha_2 l = 5\cdot275, & \sinh^2\alpha_2 l = 27\cdot83, & \sinh 2\alpha_2 l = 56\cdot64, \end{array}$$

$$\frac{\alpha_2{}^4}{\cos^2\alpha_2 l + \cosh^2\alpha_2 l} = \frac{31\cdot28}{29\cdot33 l^4}.$$

On introducing these numerical values in equations (iii) to (viii), and adding the results in accordance with the relation for D_{22}, it appears that

$$D_{22} = \frac{31\cdot28}{l^3}(1 - 0\cdot0875bl^2).$$

Similarly, from equations (96.13), we can write

$$F_{rs} = \int_{-l}^{l}(1 - ax^2)f_r f_s dx,$$

and the evaluation of this involves the integrals

$$\int\cos\alpha_2 x dx = \frac{1}{\alpha_2}\sin\alpha_2 x,$$

$$\int\cosh\alpha_2 x dx = \frac{1}{\alpha_2}\sinh\alpha_2 x,$$

$$\int x^2\cos\alpha_2 x dx = \left(\frac{x^2}{\alpha_2} - \frac{2}{\alpha_2{}^3}\right)\sin\alpha_2 x + \frac{2x}{\alpha_2{}^2}\cos\alpha_2 x,$$

$$\int x^2\cosh\alpha_2 x dx = \left(\frac{x^2}{\alpha_2} + \frac{2}{\alpha_2{}^3}\right)\sinh\alpha_2 x - \frac{2x}{\alpha_2{}^2}\cosh\alpha_2 x.$$

Therefore, taking account of equations (i),

$$F_{11} = \int_{-l}^{l}(1 - ax^2)f_1{}^2 dx = \left(\frac{1}{\sqrt{2}}\right)^2\int_{-l}^{l}(1 - ax^2)dx = l\left(1 - \frac{al^2}{3}\right),$$

$$F_{12} = \int_{-l}^{l}(1 - ax^2)f_1 f_2 dx = \frac{1}{\sqrt{2}\,(\cos^2\alpha_2 l + \cosh^2\alpha_2 l)}\int_{-l}(1 - ax^2)$$

$$(\cos\alpha_2 x\cosh\alpha_2 l + \cosh\alpha_2 x\cos\alpha_2 l)dx$$

$$= 0\cdot2970al^3$$
$$= F_{21},$$

$$F_{22} = \int_{-l}^{l}(1 - ax^2)f_2{}^2 dx$$

$$= \frac{1}{\cos^2\alpha_2 l + \cosh^2\alpha_2 l}\int_{-l}^{l}(1 - ax^2)(\cos\alpha_2 x\cosh\alpha_2 l + \cosh\alpha_2 x\cos\alpha_2 l)^2 dx.$$

The magnitude of F_{22} is readily found by substituting a for b and making appropriate changes in the signs of the component integrals of D_{22}. On effecting this work and putting $\alpha_2 l = 2\cdot3650$, a repetition of the procedure used in calculating D_{22} leads, finally, to

$$F_{22} = l(1 - 0\cdot4590al^2),$$

in round numbers.

LIST OF AUTHORS CITED

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(*The numbers refer to the pages, and the term 'waves' refers to waves of stress*)